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# The Coefficients of Functions with Positive Real Part and Some Special Classes of Univalent Functions

Rosihan M. Ali
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
**e-mail:** rosihan@cs.usm.my

# PREFACE

The SEAMS-Gadjah Mada University International Conference 2003 on Mathematics and Its Applications was held on 14 – 17 July 2003 at Gadjah Mada University, Yogyakarta, Indonesia. The Conference is the forth conference held by Gadjah Mada University and SEAMS. The former was held in 1989, 1995 and 1999.

The Conference has achieved its main purposes of promoting the exchange of ideas and presentation of recent development, particularly in the areas of pure and applied mathematics, which are represented in South East Asian Countries. The Conference has also provided a forum of researchers, developers, and practitioners to exchange ideas and to discuss future direction of research. Moreover, it has enhanced collaboration between researchers from countries in the region and those from outside.

During the 4-day conference there were 13 plenary lectures and 117 contributed papers communications. The plenary lectures were delivered by Prof. Chew Tuan Seng (Singapore), Prof. Edy Soewono (Indonesia), Prof. D. K. Ganguly (India), Prof. F. Kappel (Austria), Prof G. Desch (Austria), Prof. G. Peichl (Austria), Prof. J. A. M. van der Weide (the Netherland), Prof. K. Denecke (Germany), Prof. Lee Peng Yee (Singapore), Prof. Soeparna Darmawijaya (Indonesia), Prof. Suthep Suantai (Thailand), Prof. V. Dlab (Canada) and Dr. Widodo (Indonesia). Most of the contributed papers were delivered by Mathematicians from Asia.

The proceedings consists of 5 invited lectures and 64 refereed contributed papers.

In this occasion, we would like to express our gratitute and appreciation to the following sponsors:

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Rosihan M. Ali
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
e-mail: rosihan@cs.usm.my

**Abstract**

Several special classes of univalent functions $f$ in the unit disk $U$ are characterized by the quantity $zf'(z)/f(z)$ lies in a given region in the right-half plane. Amongst these are the classes $SS^*(\alpha)$ of strongly starlike functions of order $\alpha$ and $PS^*(\rho)$ consisting of parabolic starlike functions of order $\rho$. Both classes are closely related to the class $P$ of normalized analytic functions in $U$ with positive real part.

We derive some sharp non-linear coefficient estimates for functions in the class $P$. Using these estimates, we determine sharp bounds for the first four coefficients over the classes $SS^*(\alpha)$ and $PS^*(\rho)$, and their inverses. All possible extremal functions are found. In many of these problems, there cannot be a sole extremal function. The Fekete-Szegö coefficient functional is also treated.

***Keywords:*** *Univalent functions, analytic functions with positive real part, parabolic starlike functions, strongly starlike functions, coefficient bound, Fekete-Szegö coefficient functional.*

## 1. Introduction

Let $A$ denote the class of analytic functions $f$ in the open unit disk $U = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Some special classes of univalent functions are defined by natural geometric conditions. A well-known example is the class $S^*$ of starlike functions consisting of analytic functions $f \in A$ that map $U$ conformally onto domains starlike with respect to the origin $O$. Geometrically, this means that the linear segment joining $O$ to every other point $w \in f(U)$ lies entirely in $f(U)$.

Closely related to the class $S^*$ is the class $P$ of normalized analytic functions $p$ in the unit disk $U$ with positive real part such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in U$. It is known [11, p. 42] that a function $f \in A$ belongs to $S^*$ if and only if $zf'(z)/f(z) \in P$.

There are several subclasses of univalent starlike functions that are characterized by the quantity $zf'(z)/f(z)$ lies in a given region in the right-half plane. The region is often convex and symmetric with respect to the real axis. Ma and Minda [8] have given a very good unified treatment of such a study under a weaker condition that the region is starlike with respect to 1. We shall be interested in the following two subclasses.

An analytic function $f \in A$ is said to be strongly starlike of order $\alpha$, $0 < \alpha \leq 1$, if it satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\alpha}{2} \quad (z \in U).$$

The set of all such functions is denoted by $SS^*(\alpha)$. This class has been studied by several authors [2,3,7,13,14]. More recently, Nunokawa and Owa [10] obtained a sufficient condition for functions $f \in A$ to belong to $SS^*(\alpha)$. With $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, the class $SS^*(\alpha)$ consists of functions $f$ such that $zf'(z)/f(z) \in \varphi(U)$, $z \in U$.

For $0 \leq \rho < 1$, let $\Omega_\rho$ be the parabolic region in the right-half plane

$$\Omega_\rho = \left\{w = u + iv : v^2 \leq 4(1-\rho)(u-\rho)\right\} = \{w : |w-1| \leq 1 - 2\rho + \operatorname{Re} w\}.$$

The class of *parabolic starlike functions of order* $\rho$ is the subclass $PS^*(\rho)$ of $A$ consisting of functions $f$ such that $zf'(z)/f(z) \in \Omega_\rho$, $z \in U$. This class is a natural extension of the class of normalized uniformly convex functions $UCV$ introduced by Goodman [4]. We recall that a convex function $f$ belongs to the class $UCV$ if it has the additional property that for every circular arc $\gamma$ contained in $U$ with centre also in $U$, the image arc $f(\gamma)$ is convex. It is known [9, 12] that $f \in UCV$ if and only if $zf'(z) \in PS^*\left(\frac{1}{2}\right)$

If

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

is in the class $SS^*(\alpha)$ (or $PS^*(\rho)$), then the inverse of $f$ admits an expansion

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots \tag{2}$$

near $w = 0$. In this paper, we derive some sharp non-linear coefficient estimates for functions in the class $P$. From these bounds, we determine sharp bounds for the first four coefficients of $|a_n|$ over both classes $SS^*(\alpha)$ and $PS^*(\rho)$, the first four coefficients of $|\gamma_n|$ over $SS^*(\alpha)$, and find all possible extremal functions. Although the natural choice for an extremal function would arise from $p(z) = \frac{1+z}{1-z} \in P$, we show that it cannot be the sole extremal function for these problems. Additionally, we obtain sharp estimate for the Fekete-Szegö coefficient functionals $\left|a_3 - t a_2^2\right|$ or $\left|\gamma_3 - t\gamma_2^2\right|$.

## 2. Preliminary results

The classes $SS^*(\alpha)$ and $PS^*(\rho)$ are closely related to the class $P$. It is clear that $f \in SS^*(\alpha)$ if and only if there exists a function $p \in P$ so that $zf'(z)/f(z) = p^{\alpha}(z)$. By equating coefficients, each coefficient of $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ can be expressed in terms of coefficients of a function $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ in the class $P$. For example,

$$\begin{aligned} a_2 &= \alpha c_1 \\ a_3 &= \frac{\alpha}{2}\left[c_2 - \frac{1-3\alpha}{2} c_1^2\right] \\ a_4 &= \frac{\alpha}{3}\left[c_3 + \frac{5\alpha-2}{2} c_1 c_2 + \frac{17\alpha^2 - 15\alpha + 4}{12} c_1^3\right] \end{aligned} \tag{3}$$

Using representations (1) and (2) together with $f(f^{-1}(w)) = w$ or

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots,$$

we obtain the relationships

$$\begin{aligned} \gamma_2 &= -a_2 \\ \gamma_3 &= -a_3 + 2a_2^2 \\ \gamma_4 &= -a_4 + 5a_2 a_3 - 5a_2^3 \end{aligned} \tag{4}$$

Thus coefficient estimates for the class $SS^*(\alpha)$ and its inverses may be considered as non-linear coefficient problems for the class $P$.

Turning to the class $PS^*(\rho)$, Ali and Singh [1] has shown that the normalized Riemann mapping function $q$ from $U$ onto $\Omega_\rho$ is given by

$$q(z) = 1 + \frac{4(1-\rho)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2.$$

If $f(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in PS^*(\rho)$, and $h(z) = zf'(z)/f(z)$, then there exists a Schwarzian function $w$ in $U$ with $w(0) = 0$, $|w(z)| < 1$, and satisfying

$$h(z) = \frac{zf'(z)}{f(z)} = q(w(z)). \tag{5}$$

Hence the function

$$p(z)=\frac{1+q^{-1}(h(z))}{1-q^{-1}(h(z))}=1+c_1z+c_2z^2+\cdots$$

is analytic and has positive real part in $U$, that is, $p\in P$. It is now easily established that

$$\begin{aligned}
b_2&=\frac{8(1-\rho)}{\pi^2}c_1\\
b_3&=\frac{8(1-\rho)}{2\pi^2}\left[c_2-\left(\frac{1}{6}-\frac{8(1-\rho)}{\pi^2}\right)c_1{}^2\right]\\
b_4&=\frac{8(1-\rho)}{3\pi^2}\left[c_3-\left(\frac{1}{3}-\frac{12(1-\rho)}{\pi^2}\right)c_1c_2+\left(\frac{2}{45}-\frac{2(1-\rho)}{\pi^2}+\frac{32(1-\rho)^2}{\pi^4}\right)c_1{}^3\right]
\end{aligned}\tag{6}$$

Thus once again we see that coefficient estimates for $PS^*(\rho)$ may be viewed in terms of non-linear coefficient problems for the class $P$. Our principal tool in these problems is given in the following lemma.

**Lemma 1 [5].** *A function* $p(z)=1+\sum_{k=1}^{\infty}c_kz^k$ *belongs to* $P$ *if and only if*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j+\sum_{k=1}^{\infty}c_kz_{k+j}\right|^2-\left|\sum_{k=0}^{\infty}c_{k+1}z_{k+j}\right|^2\right\}\ge 0$$

*for every sequence* $\{z_k\}$ *of complex numbers which satisfy* $\lim_{k\to\infty}\sup|z_k|^{1/k}<1$.

**Lemma 2.** *If* $p(z)=1+\sum_{k=1}^{\infty}c_kz^k\in P$, *then*

$$\left|c_2-\frac{\mu}{2}c_1{}^2\right|\le\max\{2,2|\mu-1|\}=\begin{cases}2, & 0\le\mu\le 2\\ 2|\mu-1|, & \textit{elsewhere}\end{cases}$$

*If* $\mu<0$ *or* $\mu>2$, *equality holds if and only if* $p(z)=(1+\varepsilon z)/(1-\varepsilon z)$, $|\varepsilon|=1$. *If* $0<\mu<2$, *then equality holds if and only if* $p(z)=(1+\varepsilon z^2)/(1-\varepsilon z^2)$, $|\varepsilon|=1$. *For* $\mu=0$, *equality holds if and only if*

$$p(z):=p_2(z)=\lambda\frac{1+\varepsilon z}{1-\varepsilon z}+(1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z},\quad 0\le\lambda\le 1,\ |\varepsilon|=1.$$

*For* $\mu=2$, *equality holds if and only if* $p$ *is the reciprocal of* $p_2$.

Remark. Ma and Minda [8] had earlier proved the above result. We give a different proof.

*Proof.* Choose the sequence $\{z_k\}$ of complex numbers in Lemma 1 to be $z_0 = -\mu c_1/2$, $z_1 = 1$, and $z_k = 0$ if $k > 1$. This yields

$$\left| c_2 - \frac{\mu}{2} c_1{}^2 \right|^2 + |c_1|^2 \le |(1-\mu)c_1|^2 + 4,$$

that is,

$$\left| c_2 - \frac{\mu}{2} c_1{}^2 \right|^2 \le 4 + \mu(\mu - 2)|c_1|^2. \tag{7}$$

If $\mu < 0$ or $\mu > 2$, the expression on the right of inequality (7) is bounded above by $4(\mu-1)^2$. Equality holds if and only if $|c_1| = 2$, i.e., $p(z) = (1+z)/(1-z)$ or its rotations. If $0 < \mu < 2$, then the right expression of inequality (7) is bounded above by 4. In this case, equality holds if and only if $|c_1| = 0$ and $|c_2| = 2$, i.e., $p(z) = (1+z^2)/(1-z^2)$ or its rotations. Equality holds when $\mu = 0$ if and only if $|c_2| = 2$, i.e., [11, p. 41]

$$p(z) := p_2(z) = \lambda \frac{1+\varepsilon z}{1-\varepsilon z} + (1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}, \quad 0 \le \lambda \le 1, \ |\varepsilon| = 1.$$

Finally, when $\mu = 2$, then $|c_2 - c_1{}^2| = 2$ if and only if $p$ is the reciprocal of $p_2$. □

Another interesting application of Lemma 1 occurs by choosing the sequence $\{z_k\}$ to be $z_0 = \delta c_1{}^2 - \beta c_2$, $z_1 = -\gamma c_1$, $z_2 = 1$, and $z_k = 0$ if $k > 2$. In this case, we find that

$$\begin{aligned}\left|c_3 - (\beta+\gamma)c_1c_2 + \delta c_1{}^3\right|^2 &\le 4 + 4\gamma(\gamma-1)|c_1|^2 + \left|(2\delta-\gamma)c_1{}^2 - (2\beta-1)c_2\right|^2 - \left|c_2 - \gamma c_1{}^2\right|^2 \\ &= 4 + 4\gamma(\gamma-1)|c_1|^2 + 4\beta(\beta-1)\left|c_2 - \frac{\nu}{2}c_1{}^2\right|^2 - \frac{(\delta-\beta\gamma)^2}{\beta(\beta-1)}|c_1|^4 \end{aligned} \tag{8}$$

where $\nu := \dfrac{\delta(\beta-1) + \beta(\delta-\gamma)}{\beta(\beta-1)}$.

**Lemma 3.** *Let* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$. *If* $0 \le \beta \le 1$ *and* $\beta(2\beta-1) \le \delta \le \beta$, *then*

$$\left|c_3 - 2\beta c_1 c_2 + \delta c_1{}^3\right| \le 2.$$

*Proof.* If $\beta = 0$, then $\delta = 0$ and the result follows since $|c_3| \le 2$. If $\beta = 1$, then $\delta = 1$ and the inequality follows from a result of [6].

Now assume that $0 < \beta < 1$ so that $\beta(\beta-1) < 0$. With $\gamma = \beta$, we find from (8) that

$$\left|c_3 - 2\beta c_1 c_2 + \delta c_1{}^3\right|^2 \le 4 + 4\beta(\beta-1)|c_1|^2 + 4\beta(\beta-1)\left|c_2 - \frac{\nu}{2}c_1{}^2\right|^2 - \frac{(\delta-\beta^2)^2}{\beta(\beta-1)}|c_1|^4$$

$$\leq 4 + bx + cx^2 := h(x)$$

with $x = |c_1|^2 \in [0,4]$, $b = 4\beta(\beta - 1)$, and $c = -(\delta - \beta^2)^2 / \beta(\beta - 1)$. Since $c \geq 0$, it follows that $h(x) \leq h(0)$ provided $h(0) - h(4) \geq 0$, i.e., $b + 4c \leq 0$. This condition is equivalent to $|\delta - \beta^2| \leq \beta(1 - \beta)$, which completes the proof. □

With $\delta = \beta$ in Lemma 3, we obtain an extension of Libera and Zlotkiewicz result [6] that $\left|c_3 - 2c_1c_2 + c_1^3\right| \leq 2$.

**Corollary 1.** *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, *and* $0 \leq \beta \leq 1$, *then*

$$\left|c_3 - 2\beta c_1 c_2 + \beta c_1^3\right| \leq 2.$$

*When* $\beta = 0$, *equality holds if and only if*

$$p(z) := p_3(z) = \sum_{k=1}^{3} \lambda_k \frac{1 + \varepsilon e^{-2\pi i k/3} z}{1 - \varepsilon e^{-2\pi i k/3} z}, \quad (|\varepsilon| = 1)$$

$\lambda_k \geq 0$, *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *If* $\beta = 1$, *equality holds if and only if* $p$ *is the reciprocal of* $p_3$. *If* $0 < \beta < 1$, *equality holds if and only if* $p(z) = (1 + \varepsilon z)/1 - \varepsilon z)$, $|\varepsilon| = 1$, *or* $p(z) = (1 + \varepsilon z^3)/(1 - \varepsilon z^3)$, $|\varepsilon| = 1$.

*Proof.* We only need to find the extremal functions. If $\beta = 0$, then equality holds if and only if $|c_3| = 2$, i.e., $p$ is the function $p_3$ [11, p. 41]. If $\beta = 1$, then equality holds if and only if $p$ is the reciprocal of $p_3$. When $0 < \beta < 1$, we deduce from (8) that

$$\begin{aligned}\left|c_3 - 2\beta c_1 c_2 + \beta c_1^3\right|^2 &\leq 4 + 4\beta(\beta - 1)|c_1|^2 + 4\beta(\beta - 1)\left|c_2 - \tfrac{1}{2}c_1^2\right|^2 - \beta(\beta - 1)|c_1|^4 \\ &\leq 4 + 4\beta(\beta - 1)|c_1|^2 - \beta(\beta - 1)|c_1|^4 \leq 4.\end{aligned}$$

The bound 4 in the last inequality is obtained from simple calculus computations. Equality occurs in the last inequality if and only if either $|c_1| = 0$ or $|c_1| = 2$. If $|c_1| = 0$, then $|c_2| = 0$, i.e., $p(z) = (1 + \varepsilon z^3)/(1 - \varepsilon z^3)$, $|\varepsilon| = 1$. If $|c_1| = 2$, then $p(z) = (1 + \varepsilon z)/(1 - \varepsilon z)$, $|\varepsilon| = 1$. □

**Lemma 4.** *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, *then*

$$\left|c_3 - (\mu + 1)c_1 c_2 + \mu c_1^3\right| \leq \max\{2, 2|2\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \textit{elsewhere} \end{cases}$$

*Proof.* For $0 \leq \mu \leq 1$, the estimate follows from Lemma 3 with $\delta = \mu$, and $2\beta = \mu + 1$. For the second estimate, choose $\beta = \mu$, $\gamma = 1$, and $\delta = \mu$ in (8). Since $\mu(\mu - 1) > 0$, we conclude from (7) and (8) that

$$\left|c_3-(\mu+1)c_1c_2+\mu c_1{}^3\right|^2 \le 4+4\mu(\mu-1)\left|c_2-c_1{}^2\right|^2 \le 4(2\mu-1)^2. \quad \square$$

## 3. Coefficient bounds

For the larger class $S^*$ of starlike functions, R. Nevanlinna in 1920 [11, p. 46] proved that the coefficient of each function $f\in S^*$ satisfy $|a_n|\le n$ for $n=2,3,\cdots$. Brannan *et al.* [2] obtained a sharp bound for the third coefficient of functions in $SS^*(\alpha)$. We shall give an alternate proof, and additionally, derive a sharp estimate for the fourth coefficient in the result below. The general coefficient problem for the classes $SS^*(\alpha)$ and $PS^*(\rho)$ remains an open problem.

**Theorem 1.** *Let* $f(z)=z+a_2z^2+a_3z^3+\cdots\in SS^*(\alpha)$. *Then*

$$|a_2|\le 2\alpha,$$

*with equality if and only if*

$$\frac{zf'(z)}{f(z)}=\left(\frac{1+\varepsilon z}{1-\varepsilon z}\right)^{\alpha}, \quad |\varepsilon|=1. \tag{9}$$

*Further*

$$|a_3|\le\begin{cases}\alpha, & 0<\alpha\le\frac{1}{3}\\ 3\alpha^2, & \frac{1}{3}\le\alpha\le 1\end{cases}$$

*For* $\alpha>1/3$, *extremal functions are given by (9). If* $0<\alpha<1/3$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)}=\left(\frac{1+\varepsilon z^2}{1-\varepsilon z^2}\right)^{\alpha}, \quad |\varepsilon|=1, \tag{10}$$

*while if* $\alpha=1/3$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)}=p_2(z)^{-\alpha}=\left(\lambda\frac{1+\varepsilon z}{1-\varepsilon z}+(1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}\right)^{-\alpha}, \quad |\varepsilon|=1, \quad 0\le\lambda\le 1.$$

*Moreover,*

$$|a_4|\le\begin{cases}\frac{2\alpha}{3}, & 0<\alpha\le\sqrt{\frac{2}{17}}\\ \frac{2\alpha}{9}\left(17\alpha^2+1\right), & \sqrt{\frac{2}{17}}\le\alpha\le 1\end{cases}$$

*For* $\alpha\ge\sqrt{2/17}$, *extremal functions are given by (9), while for* $0<\alpha\le\sqrt{2/17}$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)}=\left(\frac{1+\varepsilon z^3}{1-\varepsilon z^3}\right)^{\alpha}, \quad |\varepsilon|=1.$$

*Proof.* The following relations are obtained from (3):

$$a_2 = \alpha c_1$$
$$a_3 = \frac{\alpha}{2}\left[c_2 - \frac{1-3\alpha}{2}c_1^2\right]$$
$$a_4 = \frac{\alpha}{3}\left[c_3 + \frac{5\alpha-2}{2}c_1c_2 + \frac{17\alpha^2-15\alpha+4}{12}c_1^3\right] := \frac{\alpha}{3}E$$

The bound on $|a_2|$ follows immediately from the well-known inequality $|c_1| \le 2$. Lemma 2 with $\mu = 1-3\alpha$ yields the bound on $|a_3|$ and the description of the extremal functions.

For the fourth coefficient, we shall apply Lemma 3 with $2\beta = (2-5\alpha)/2$ and $\delta = (17\alpha^2 - 15\alpha + 4)/12$. The conditions on $\beta$ and $\delta$ are satisfied if $\alpha \le \sqrt{2/17}$. Thus $|a_4| \le 2\alpha/3$, with equality if and only if $zf'(z)/f(z) = \left[(1+\varepsilon z^3)/(1-\varepsilon z^3)\right]^{\alpha}$.

In view of the fact that $0 < \delta < 1$, and $\delta - \beta \ge 0$ provided $\alpha \ge \sqrt{2/17}$, Corollary 1 yields

$$|E| \le \left|c_3 - \frac{17\alpha^2-15\alpha+4}{6}c_1c_2 + \frac{17\alpha^2-15\alpha+4}{12}c_1^3\right| + \frac{17\alpha^2-2}{6}|c_1||c_2| \le \frac{2}{3}\left(17\alpha^2+1\right)$$

This completes the proof. □

**Theorem 2.** *Let $f \in SS^*(\alpha)$ and $f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$. Then*

$$|\gamma_2| \le 2\alpha,$$

*with equality if and only if*

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\varepsilon z}{1-\varepsilon z}\right)^{\alpha}, \quad |\varepsilon| = 1.$$

*Further*

$$|\gamma_3| \le \begin{cases} \alpha, & 0 < \alpha \le \frac{1}{5} \\ 5\alpha^2, & \frac{1}{5} \le \alpha \le 1 \end{cases}$$

*For $\alpha > 1/5$, extremal functions are given by (9). If $0 < \alpha < 1/5$, equality holds if and only if*

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\varepsilon z^2}{1-\varepsilon z^2}\right)^{\alpha}, \quad |\varepsilon| = 1,$$

*while if $\alpha = 1/5$, equality holds if and only if*

$$\frac{zf'(z)}{f(z)} = p_2(z)^{-\alpha} = \left(\lambda\frac{1+\varepsilon z}{1-\varepsilon z} + (1-\lambda)\frac{1-\varepsilon z}{1+\varepsilon z}\right)^{-\alpha}, \quad |\varepsilon| = 1, \ 0 \le \lambda \le 1.$$

*Moreover,*

$$|\gamma_4| \le \begin{cases} \frac{2\alpha}{3}, & 0<\alpha\le\frac{1}{\sqrt{31}} \\ \frac{2\alpha}{9}\left(62\alpha^2+1\right), & \frac{1}{\sqrt{31}}\le\alpha\le 1 \end{cases}$$

*For* $\alpha \ge 1/\sqrt{31}$, *extremal functions are given by (9), while for* $0<\alpha\le 1/\sqrt{31}$, *equality holds if and only if*

$$\frac{zf'(z)}{f(z)}=\left(\frac{1+\varepsilon z^3}{1-\varepsilon z^3}\right)^{\alpha}, \quad |\varepsilon|=1.$$

*Proof.* The following relations are obtained from (3) and (4):

$$\begin{aligned} \gamma_2 &= -\alpha c_1 \\ \gamma_3 &= -\frac{\alpha}{2}\left[c_2-\frac{1+5\alpha}{2}c_1{}^2\right] \\ \gamma_4 &= -\frac{\alpha}{3}\left[c_3-(1+5\alpha)c_1c_2+\frac{31\alpha^2+15\alpha+2}{6}c_1{}^3\right]:=-\frac{\alpha}{3}E \end{aligned} \tag{11}$$

As in the previous proof, the bounds on $|\gamma_2|$ and $|\gamma_3|$ are obtained from the well-known inequality $|c_1|\le 2$, and from Lemma 2.

For the fourth coefficient, we shall apply Lemma 3 with $2\beta=1+5\alpha$ and $\delta=(31\alpha^2+15\alpha+2)/6$. The conditions on $\beta$ and $\delta$ are satisfied if $\alpha\le 1/\sqrt{31}$. Thus $|\gamma_4|\le 2\alpha/3$, with equality if and only if $zf'(z)/f(z)=\left[(1+\varepsilon z^3)/(1-\varepsilon z^3)\right]^{\alpha}$.

For $1/\sqrt{31}<\alpha\le 1/5$, Corollary 1 yields

$$|E|\le\left|c_3-(1+5\alpha)c_1c_2+\frac{1+5\alpha}{2}c_1{}^3\right|+\frac{31\alpha^2-1}{6}|c_1|^3\le\frac{2}{3}\left(62\alpha^2+1\right)$$

It remains to determine the estimate for $1/5<\alpha\le 1$. Appealing to Lemma 4 with $\mu=5\alpha$, and because $31\alpha^2-15\alpha+2>0$ in $(0,1]$, we conclude that

$$\begin{aligned} |E| &\le\left|c_3-(1+5\alpha)c_1c_2+5\alpha c_1{}^3\right|+\frac{31\alpha^2-15\alpha+2}{6}|c_1|^3\le 2(10\alpha-1)+\frac{4}{3}(31\alpha^2-15\alpha+2) \\ &=\frac{2}{3}\left(62\alpha^2+1\right) \quad \square \end{aligned}$$

Now we introduce the following functions in $PS^*(\rho)$. Define $G_n, H, J\in A$ respectively by

$$\frac{zG_n'(z)}{G_n(z)}=q(z^{n-1}),\quad \frac{zH'(z)}{H(z)}=q\left(\frac{z(z-r)}{1-rz}\right),\quad \frac{zJ'(z)}{J(z)}=q\left(-\frac{z(z-r)}{1-rz}\right),\quad 0\le r\le 1.$$

It is clear from (5) that $G_n, H, J \in PS^*(\rho)$. Using (6), the following result can be established in a similar fashion to Theorem 1.

**Theorem 3.** *Let* $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in PS^*(\rho)$. *Then*

$$|b_2| \le \frac{16(1-\rho)}{\pi^2},$$

*with equality if and only if* $g = G_2$ *or its rotations. Further*

$$|b_3| \le \begin{cases} \frac{8(1-\rho)}{\pi^2}\left(\frac{2}{3} + \frac{16(1-\rho)}{\pi^2}\right), & 0 \le \rho \le 1 - \frac{\pi^2}{48} \\ \frac{8(1-\rho)}{\pi^2}, & 1 - \frac{\pi^2}{48} \le \rho < 1 \end{cases}$$

*For* $0 \le \rho < 1 - \frac{\pi^2}{48}$, *equality holds if and only if* $g = G_2$ *or its rotations. For* $1 - \frac{\pi^2}{48} < \rho < 1$, *equality holds if and only if* $g = G_3$ *or its rotations. If* $\rho = 1 - \frac{\pi^2}{48}$, *equality holds if and only if* $g = H$ *or its rotations. Additionally,*

$$|b_4| \le \begin{cases} \frac{16(1-\rho)}{3\pi^2}\left[\frac{128(1-\rho)^2}{\pi^4} + \frac{16(1-\rho)}{\pi^2} + \frac{23}{45}\right], & 0 \le \rho \le 1 + \frac{\pi^2}{16}\left(1 - \sqrt{\frac{89}{45}}\right) \\ \frac{16(1-\rho)}{3\pi^2}, & 1 + \frac{\pi^2}{16}\left(1 - \sqrt{\frac{89}{45}}\right) \le \rho < 1 \end{cases}$$

*Equality holds in the upper expression of the right inequality if and only if* $g = G_2$ *or its rotations, while equality holds in the lower expression of the right inequality if and only if* $g = G_4$ *or its rotations.*

## 4. Fekete-Szegö Coefficient Functional

**Theorem 4.** *Let* $f \in SS^*(\alpha)$ *and* $f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$. *Then*

$$\left|\gamma_3 - t\gamma_2^2\right| \le \begin{cases} (5-4t)\alpha^2, & t \le \frac{5-1/\alpha}{4} \\ \alpha, & \frac{5-1/\alpha}{4} \le t \le \frac{5+1/\alpha}{4} \\ (4t-5)\alpha^2, & t \ge \frac{5+1/\alpha}{4} \end{cases}$$

*If* $\frac{5-1/\alpha}{4} < t < \frac{5+1/\alpha}{4}$, *equality holds if and only if* $f$ *is given by (10). If* $t < \frac{5-1/\alpha}{4}$ *or* $t > \frac{5+1/\alpha}{4}$, *equality holds if and only if* $f$ *is given by (9). If* $t = \frac{5+1/\alpha}{4}$, *equality holds if and* *only if* $\frac{zf'(z)}{f(z)} = p_2(z)^{\alpha}$, *while if* $t = \frac{5-1/\alpha}{4}$, *then equality holds if and only if* $\frac{zf'(z)}{f(z)} = p_2(z)^{-\alpha}$.

*Proof.* From (11), we obtain

$$\gamma_3 - t\gamma_2^2 = -\frac{\alpha}{2}\left[c_2 - \frac{1+(5-4t)\alpha}{2}c_1^2\right].$$

The result now follows from Lemma 2. □

Remark. An equivalent result for the Fekete-Szegö coefficient functional over the class $SS^*(\alpha)$ was also given by Ma and Minda [7].

**Theorem 5.** *Let* $g(z) = z + b_2z^2 + b_3z^3 + \cdots \in PS^*(\rho)$. *Then*

$$\left|b_3 - tb_2^2\right| \le \begin{cases} \frac{16(1-\rho)}{3\pi^4}\left[24(1-\rho)(1-2t)+\pi^2\right] & t \le \frac{1}{2} - \frac{\pi^2}{96(1-\rho)} \\ \frac{8(1-\rho)}{\pi^2}, & \frac{1}{2} - \frac{\pi^2}{96(1-\rho)} \le t \le \frac{1}{2} + \frac{5\pi^2}{96(1-\rho)} \\ \frac{16(1-\rho)}{3\pi^4}\left[24(1-\rho)(2t-1)-\pi^2\right] & t \ge \frac{1}{2} + \frac{5\pi^2}{96(1-\rho)} \end{cases}$$

*If* $\frac{1}{2} - \frac{\pi^2}{96(1-\rho)} < t < \frac{1}{2} + \frac{5\pi^2}{96(1-\rho)}$, *equality holds if and only if* $g = G_3$ *or one of its rotations. If* $t < \frac{1}{2} - \frac{\pi^2}{96(1-\rho)}$ *or* $t > \frac{1}{2} + \frac{5\pi^2}{96(1-\rho)}$, *equality holds if and only if* $g = G_2$ *or one of its rotations. If* $t = \frac{1}{2} - \frac{\pi^2}{96(1-\rho)}$, *equality holds if and only if* $g = H$ *or one of its rotations, while if* $t = \frac{1}{2} + \frac{5\pi^2}{96(1-\rho)}$, *then equality holds if and only if* $g = J$ *or one of its rotations.*

Finally, we note that the estimates above can be used to determine sharp upper bounds on the second and third coefficients respectively.

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