EXACT MULTIMONOPOLE SOLUTIONS OF THE YANG-MILLS-HIGGS THEORY*

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Abstract

We found some general exact static multimonopole solutions that satisfy the first order Bogomol'nyi equations and possess infinite energy. These multimonopole solutions can be categorized into two classes, namely the A2 and B2 solutions. The A2 solution is a multimonopole solution with all the magnetic charges superimposed at the origin. The B2 solution corresponds to a configuration of even number of isolated 1-monopoles located on a circle, symmetrically about the z-axis.

1 INTRODUCTION

The SU(2) Yang-Mills-Higgs theory, with the Higgs field in the adjoint representation, can possess both the magnetic monopole and multimonopole solutions. The 't Hooft-Polyakov magnetic monopole was discovered [2] in the mid-seventies. Solutions of a unit magnetic charge are spherically symmetric [2, 3]. However, multimonopole solutions possess at most axial symmetry [4]. In the limit of vanishing Higgs potential, monopole and multimonopole solutions had been shown to exist. Solutions that satisfy the Bogomol'nyi condition or the Bogomol'nyi-Prasad-Sommerfield (BPS) limit have minimal energies.

In this paper, we used the extended and generalized ansatz of ref.[1]. We work on the SU(2) Yang-Mills-Higgs model with a vanishing Higgs potential. The scalar Higgs field in our work is then taken to have no mass or self-interaction.

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We found that the SU(2) Yang-Mills-Higgs theory do possess some exact static multimonopole solutions. They satisfy the first order Bogomol'nyi equations and possess infinite energies at the origin. These multimonopole solutions are categorized into two classes, namely the A2 and B2 solutions. The two classes possess very different characteristic which we shall discuss in later section. The A2 solution represents configuration with all the magnetic charges superimposed at the origin. There are no zeros of Higgs field at finite r. The B2 solution corresponds to even number of equally spaced 1-monopoles located on a circle in the equatorial plane.

The SU(2) Yang-Mills-Higgs model consist of the Yang-Mills vector fields A^a_μ and the Higgs scalar field Φ^a in 3+1 dimensions. The index a is the SU(2) internal space index. For a given a, Φ^a is a scalar whereas A^a_μ is a vector under the Lorentz transformation. The SU(2) Yang-Mills-Higgs Lagrangian in 3+1 dimensions is

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi_a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2, \tag{1}$$

where μ is the mass of the Higgs field, and β is the strength of the Higgs potential. The vacuum expectation value of the Higgs field is then $\mu/\sqrt{\lambda}$. The Lagrangian from Eq.(1) is invariant under the set of independent SU(2) gauge transformations at each space-time point.

The covariant derivative of the Higgs field is

$$D_\mu\Phi^a = \partial_\mu\Phi^a + \epsilon^{abc}A^b_\mu\Phi^c, \tag{2}$$

where A^a_μ is the gauge potential and the gauge field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu. \tag{3}$$

The gauge field coupling constant g is set to one here. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) group indices a, b, c run from 1 to 3 whereas the spatial indices μ, ν, α run from 0 to 3 in Minkowski space.

The equations of motion obtained from Eq.(1) are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + \epsilon^{abc}A^{b\mu}F^c_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu\Phi^c, \tag{4}$$

$$D^\mu D_\mu\Phi^a = 0. \tag{5}$$

We will examine only the static solutions with $A^a_0 = 0$. The conserved energy of the static system which is obtained from the Lagrangian is

$$E = \int d^3x\left(\frac{1}{2}B^a_i B^a_i + \frac{1}{2}D_i\Phi^a D_i\Phi^a + \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2\right). \tag{6}$$

The indices i, j and k are purely spatial indices where i, j, k run from 1 to 3. This energy vanishes when the gauge potential, A^a_i is zero or a pure gauge, $\Phi^a\Phi_a = \mu^2/\lambda$ and $D_i\Phi^a = 0$. The tensor

$$F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c. \tag{7}$$

introduced by 't Hooft [2] can be identified with the electromagnetic field tensor. This tensor can also be written in a more transparent form [4]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c, \tag{8}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, the unit vector $\hat{\Phi}^a = \Phi^a/|\Phi|$ and the Higgs field magnitude $|\Phi| = \sqrt{\Phi^a\Phi^a}$. The Abelian electric field is $E_i = F_{0i}$ and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$. The topological magnetic current k_μ [4] is defined to be

$$k_\mu = \frac{1}{8\pi}\ \epsilon_{\mu\nu\rho\sigma}\ \epsilon_{abc}\ \partial^\nu\hat{\Phi}^a\ \partial^\rho\hat{\Phi}^b\ \partial^\sigma\hat{\Phi}^c, \tag{9}$$

and the corresponding conserved topological magnetic charge is

$$\begin{aligned} M &= \int d^3x\ k_0 = \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon^{abc}\partial_i\left(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) d^3x \\ &= \frac{1}{8\pi}\oint d^2\sigma_i \left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) \\ &= \frac{1}{4\pi}\oint d^2\sigma_i\ B_i. \end{aligned} \tag{10}$$

2 THE MULTIMONOPOLE SOLUTIONS

In this paper, the Higgs field is taken to have no mass and hence the self-interaction vanishes. The magnitude of the Higgs field vanishes as $1/r$ at infinity. It is in this limit that we are able to obtain explicit and exact multimonopole solutions. These solutions are solved from the Yang-Mills-Higgs equations by using both the second order Euler-Lagrange equations and the first order Bogomol'nyi equations $B^a_i \pm D_i\Phi^a = 0$ with the positive sign. There is no solution to the Bogomol'nyi equations with the negative sign. A multimonopole of magnetic charge M with all its magnetic charges superimposed at one point in space is denoted by a M-monopole.

We used the ansatz of Ref.[1] with the gauge fields and the Higgs field given by

$$\begin{aligned} A^a_\mu &= -\frac{1}{r}\psi(r)\left(\hat{\theta}^a\hat{\phi}_\mu + \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}R(\theta)\left(\hat{\phi}^a\hat{r}_\mu + \hat{r}^a\hat{\phi}_\mu\right) \\ &+ \frac{1}{r}G(\theta,\phi)\left(\hat{r}^a\hat{\theta}_\mu - \hat{\theta}^a\hat{r}_\mu\right), \end{aligned} \tag{11}$$

$$\Phi^a = \Phi_1\ \hat{r}^a + \Phi_2\ \hat{\theta}^a + \Phi_3\ \hat{\phi}^a. \tag{12}$$

with $\Phi_1 = \frac{1}{r}\psi(r)$, $\Phi_2 = \frac{1}{r}R(\theta)$, $\Phi_3 = \frac{1}{r}G(\theta,\phi)$. The spherical coordinate orthonormal unit vectors are defined by

$$\begin{aligned} \hat{r}^a &= \sin\theta\ \cos\phi\ \delta^a_1 + \sin\theta\ \sin\phi\ \delta^a_2 + \cos\theta\delta^a_3, \\ \hat{\theta}^a &= \cos\theta\ \cos\phi\ \delta^a_1 + \cos\theta\ \sin\phi\ \delta^a_2 - \sin\theta\ \delta^a_3, \\ \hat{\phi}^a &= -\sin\phi\ \delta^a_1 + \cos\phi\ \delta^a_2, \end{aligned} \tag{13}$$

With the ansatz, Eqs.(11) and (12), the equations of motion (4) and (5) can be simplified and reduced to four ordinary differential equations of first order:

$$\dot{G} + G\cot\theta = 0, G^{\phi}\csc\theta + G^2 = -b^2\csc^2\theta, \tag{14}$$

$$r\psi' + \psi - \psi^2 = -p, \tag{15}$$

$$\dot{R} + R\cot\theta - R^2 = p - b^2\csc^2\theta, \tag{16}$$

where p and b^2 are arbitrary constants. Prime means the partial derivative $\frac{\partial}{\partial r}$, dot means the partial derivative $\frac{\partial}{\partial\theta}$, superscript ϕ means the partial derivative $\frac{\partial}{\partial\phi}$.

The two classes of solutions obtained from Eq.(13) to Eq.(15) are the A2 and B2 solutions. The A2 solution is

$$R = \tan\theta + (m+2)\cos\theta, G = (m+2)\csc\theta\tan(m+2)\theta,$$
$$\psi = \frac{(m+1) - mr^{2m+1}}{1 + r^{2m+1}}, \quad p = m(m+1), b = -(m+2). \tag{17}$$

The B2 solution is

$$R = (m+1)\cot\theta, G = (m+1)\csc\theta\tan(m+1)\theta,$$
$$\psi = \frac{(m+1) - mr^{2m+1}}{1 + r^{2m+1}}, \quad p = m(m+1), b = m+1. \tag{18}$$

The parameter m in Eq.(17) and (18) is a positive integer.

The ansatz, Eqs.(11) and (12), has vanishing gauge potential, $A_\mu = \hat{\Phi}^a A^a_\mu$. Hence the Abelian electric field is zero and the Abelian magnetic field is independent of the gauge field A^a_μ. To calculate the Abelian magnetic field B_i, we rewrite the Higgs field of Eq.(12) from the spherical coordinate system to the Cartesian coordinate system,

$$\begin{aligned}\Phi^a &= \Phi_1\,\hat{r}^a + \Phi_2\,\hat{\theta}^a + \Phi_3\,\hat{\phi}^a \\ &= \tilde{\Phi}_1\,\delta^{a1} + \tilde{\Phi}_2\,\delta^{a2} + \tilde{\Phi}_3\,\delta^{a3}\end{aligned} \tag{19}$$

where

$$\begin{aligned}\tilde{\Phi}_1 &= \sin\theta\cos\phi\;\Phi_1 + \cos\theta\cos\phi\;\Phi_2 - \sin\phi\;\Phi_3 = |\Phi|\cos\alpha\sin\beta \\ \tilde{\Phi}_2 &= \sin\theta\sin\phi\;\Phi_1 + \cos\theta\sin\phi\;\Phi_2 + \cos\phi\;\Phi_3 = |\Phi|\cos\alpha\cos\beta \\ \tilde{\Phi}_3 &= \cos\theta\;\Phi_1 - \sin\theta\;\Phi_2 = |\Phi|\sin\alpha.\end{aligned} \tag{20}$$

The Higgs field unit vector is then simplified to

$$\hat{\Phi}^a = \cos\alpha\sin\beta\;\delta^{a1} + \cos\alpha\cos\beta\;\delta^{a2} + \sin\alpha\;\delta^{a3} \tag{21}$$

The Abelian magnetic field is found to be

$$\begin{aligned}B_i = &\frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right\}\hat{r}_i \\ &+\frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial r} - \frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\phi}\right\}\hat{\theta}_i \\ &+\frac{1}{r}\left\{\frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\theta} - \frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial r}\right\}\hat{\phi}_i\end{aligned} \tag{22}$$

where

$$\sin\alpha = \frac{\psi\cos\theta - R\sin\theta}{\sqrt{\psi^2+R^2+G^2}}, \beta = \gamma - \phi, \gamma = \tan^{-1}\left(\frac{\psi\sin\theta + R\cos\theta}{G}\right). \quad (23)$$

The Abelian field magnetic flux is

$$\begin{aligned} \Omega &= 4\pi M = \oint d^2\sigma_i B_i \\ &= \int B_i(r^2\sin\theta)\hat{r}_i \, d\theta \, d\phi, \end{aligned} \quad (24)$$

The magnetic charge M is given by

$$M = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right) d\theta d\phi \quad (25)$$

The magnetic charges of the A2 solution at the origin can be exactly integrated to be $m+3$. The net magnetic charges of the configuration when r tends to infinity can be obtained by using approximate integration and it is found also to be $(m+3)$. Hence the A2 solution consists of a $(m+3)$-monopole sitting at the origin of the coordinate axes. Therefore, when $m = 0$, we have a 3-monopole at the origin and when $m = 1$, we obtain a configuration of 4-monopole at the origin.

The B2 solution is different from the A2 solution in that the positive magnetic charges are not superimposed at one point in space. The B2 solution has isolated 1-monopoles located on a circle in the equatorial plane with rotational symmetry about the z-axis. The 1-monopoles here are finitely separated and these 1-monopoles are located at the zeros of the Higgs field. There is no multimonopole at the origin.

The case when $m = 0$ is a special case of the B2 solution, the zeros of Higgs field are located at infinity and there are no zeros of Higgs field at finite r. The configuration has no monopole at finite r. For $m = 1$, there are $2(m+1)$ equally separated 1-monopoles. Hence when $m = 1$, we have a configuration of four isolated 1-monopoles, each being located at the four zeros of the Higgs field. When $m = 2$, we have six isolated 1-monopoles at the six zeros of the Higgs field, and so on.

3 COMMENTS AND OUTLOOKS

The B2 solution with $m = 0$ does not have any zero of Higgs field at finite r. Hence instead of the two 1-monopole solution, we only obtain a solution that has no monopoles at finite r. The B2 solution contains only even number of finitely separated 1-monopoles. Hence it will be interesting if one can find solutions with odd number of isolated 1-monopoles to complement the B2 solution.

The two classes of solutions in this paper possess singularity at the origin. This singularity either corresponds to multimonopole as in the A2 solution or no monopole at all as in the B2 solution. The zeros of the Higgs field correspond to a 1-monopole in the B2 solution. In the A2 solution, there are no zeros of Higgs field at finite r.

A third multimonopole solution is the C solution which will be reported separately. Work can still be done to find different solutions to the Riccati equation of Eq.(14).

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