

A PROPOSED DOUBLE MOVING AVERAGE (DMA) CONTROL CHART

by

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CADANGAN CARTA KAWALAN PURATA BERGERAK BERGANDA DUA (DMA)

ABSTRAK

Teknik carta kawalan telah digunakan secara meluas dalam industri untuk mengawal kualiti proses pengeluaran. Carta kawalan dengan ingatan diperkenalkan sebagai alternatif kepada carta Shewhart untuk pengesanan anjakan tetap proses yang kecil. Antara carta kawalan dengan ingatan yang luas digunakan ialah carta purata bergerak (MA), carta purata bergerak berpemberat eksponen (EWMA) dan carta hasil tambah longgokan (CUSUM). Teknik carta kawalan purata bergerak berpemberat eksponen berganda dua (DEWMA) telah diselidiki beberapa tahun kebelakangan ini dan didapati lebih cekap daripada teknik EWMA untuk pengesanan anjakan kecil dalam min proses. Tesis ini memperkembangkan penggunaan teknik pelicinan bereksponen sebanyak dua kali pada carta EWMA, seperti dalam kes carta DEWMA kepada pengiraan purata bergerak sebanyak dua kali pada carta MA untuk min proses. Carta MA yang baru ini dikenali sebagai carta purata bergerak berganda dua (DMA). Kajian simulasi yang dijalankan menunjukkan bahawa carta DMA yang dicadangkan menambahbaikkan prestasi purata panjang larian (ARL) carta MA. Carta DMA dan DEWMA ialah carta kawalan dengan ingatan yang berdasarkan dua skema purata berpemberat yang berlainan. Semua kajian simulasi dalam tesis ini dijalankan dengan menggunakan program "Sistem Analisis Berstatistik (SAS)" versi 8.0. Dua

contoh diberikan untuk menunjukkan bagaimana carta kawalan DMA yang dicadangkan digunakan dalam situasi sebenar.

A PROPOSED DOUBLE MOVING AVERAGE (DMA) CONTROL CHART

ABSTRACT

Control chart techniques have been widely used in industries to monitor the quality of manufacturing processes. Memory control charts are introduced as alternatives to the Shewhart charts for quick detections of small sustaining process shifts. Among the widely used memory control charts are the moving average (MA) chart, exponentially weighted moving average (EWMA) chart and cumulative sum (CUSUM) chart. The double exponentially weighted moving average (DEWMA) technique has been investigated in recent years and is found to be more efficient than the EWMA technique for detecting small shifts in the process mean. This thesis extends the technique of performing exponential smoothing twice on an EWMA chart, as in the case of the DEWMA chart to computing the moving average twice on an MA chart for the process mean. The new MA chart is known as the double moving average (DMA) chart. The simulation study conducted shows that the proposed DMA chart improves upon the average run length (ARL) performance of the MA chart. The DMA and DEWMA are memory control charts that are based on two different weighted average schemes. All of the simulation studies in this thesis are conducted using the "Statistical Analysis System (SAS)" version 8.0 program. Two examples are given to show how the proposed DMA chart is used in a real situation.

CHAPTER 1

INTRODUCTION

1.1 An Overview of Statistical Process Control (SPC)

SPC is defined as the application of statistical methods in monitoring, controlling and ideally improving the quality of a process through statistical analysis. Its four basic steps include measuring the process, eliminating variances in the process to make it consistent, monitoring the process and improving the process to its best target value.

The use of SPC can be summarized as follows (Garrity, 1993):

- i) Reduce and eliminate errors, scrap and rework in the process.
- ii) Improve communication throughout the entire organization.
- iii) Encourage participation in quality improvement.
- iv) Increase involvement in the decision-making process.
- v) Simplify and improve work procedures.
- vi) Manage the process by facts and not opinion.

The key steps for implementing SPC are (Montgomery, 2001):

- i) Identify defined processes.
- ii) Identify measurable attributes of the process.
- iii) Characterize natural variation of attributes.
- iv) Monitor the process variation.
- v) If a process is in-control, continue to monitor the process variation.

vi) If a process is out-of-control, identify and remove the assignable causes.

Then resume with the monitoring of the process variation.

Seven well known SPC tools are the histogram, check sheet, Pareto diagram, cause-and-effect diagram, scatter plot, process flow diagram and control chart. The most important among these seven tools is the control chart, also referred to as a process-behaviour chart and it is a statistical tool intended to assess the nature of variation in a process and to facilitate forecasting and management. There are different types of control charts each of which performs best for a particular kind of data. A process that is not in statistical process control, also known as, out-of-control is caused by the presence of one or more assignable causes. The purpose of a control chart is to bring an out-of-control process back into an in-control state. An in-control historical sample data is used to estimate the process parameters such as the mean and the standard deviation and the purpose of control limits is to check whether a future process is in statistical control. If a process is out-of-control, investigations must be made to find and remove the assignable causes. The center line of a control chart can be determined in advance once a process is brought into statistical control and if it is operating at a satisfactory level. The Shewhart charts which are commonly used in SPC are the $\bar{X}-R$ and $\bar{X}-S$ charts for variable data and the c , u , p and np charts for attribute data.

1.2 Objectives of the Study

The EWMA and MA are memory control charts which are superior to the Shewhart chart in the detection of small process shifts. To date, numerous extensions of the EWMA and MA charts have been proposed. Attempts have also been made by Zhang and Chen (2005) to increase the detection speed of an EWMA chart by proposing the DEWMA chart. However, until now no attempt is made to increase the speed of an MA chart in responding to an out-of-control condition. Thus, the objective of this thesis is to propose a DMA control chart as a superior alternative to the MA chart. The DMA chart is quicker to detect out-of-control signals, especially those involving small shifts. Through a simulation study, we also show that the DMA chart has a better performance than the MA chart. We give two examples of applications to show how the DMA chart is put to work in a real situation.

1.3 Organization of the Thesis

This section discusses the organization of the thesis. The first three chapters will be laid out for literature review. These chapters form the basis of the proposed DMA chart which will be discussed in Chapter 4.

Chapter 1 discusses an overview of statistical process control (SPC), the objectives of this study and the organization of this thesis.

Chapter 2 gives a brief explanation on some important concepts required in the understanding of control charts, such as the normal, binomial and Poisson distributions and the average run length (ARL). A short discussion on Shewhart control charts and a review on the different types of memory control charts are also provided.

Chapter 3 deals with a review on the various extensions of the MA chart such as the optimal and economic designs of the MA chart, the weighted MA chart, the MA chart for a joint monitoring of the process mean and variance, the MA chart for monitoring the fraction nonconforming and the Poisson MA chart.

Chapter 4 is the most important part of this thesis. It discusses a proposed DMA chart for individual measurements and subgrouped data. A simulation study is conducted in this chapter to evaluate the performance of the proposed DMA chart in comparison to that of the MA chart. Two examples are also given in this chapter to show how the proposed DMA chart is put to work in a real situation.

Chapter 5 concludes and summarizes the findings of this research. The contributions of the thesis will be highlighted. Topics and suggestions for further research are also identified in this chapter.

CHAPTER 2

SOME PRELIMINARIES AND REVIEW ON VARIABLES CONTROL CHARTS

2.1 INTRODUCTION

In this chapter, some commonly used continuous and discrete distributions in statistical quality control, such as the normal, binomial and Poisson distributions will be reviewed. A definition and some explanation on the average run length (ARL), which is used as a measure of a control chart's performance will also be given. A brief discussion on Shewhart control charts for variables and attributes data and a detailed review on the various types of memory control charts such as the moving average (MA), exponentially weighted moving average (EWMA), double exponentially weighted moving average (DEWMA) and cumulative sum (CUSUM) charts will be provided.

2.2 THE NORMAL DISTRIBUTION

The normal distribution, also called the Gaussian distribution, is the most important and widely used distribution in both the theory and applications of statistics. A normally distributed random variable X with mean, μ and variance, σ^2 can be denoted by $X \sim N(\mu, \sigma^2)$. Its probability density function is given by (Montgomery, 2001)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (2.1)$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. The normal density function is a symmetric, unimodal and bell shaped curve and is shown in Figure 2.1.

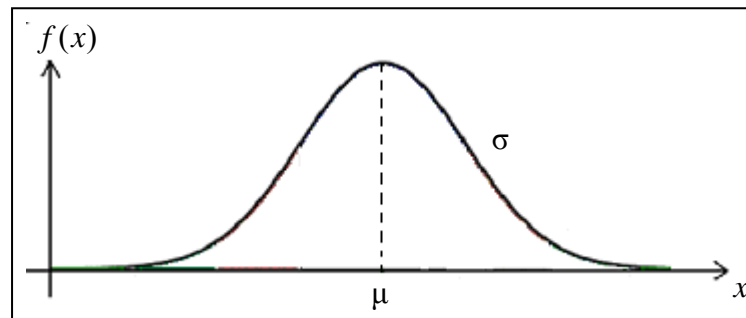


Figure 2.1. The density function of a normal distribution
(Source: Montgomery, 2001)

For a normal distribution, 68.26% of the population values fall between the $\mu \pm 1\sigma$ limits, 95.46% of the population values fall between the $\mu \pm 2\sigma$ limits and 99.73% of the population values fall between the $\mu \pm 3\sigma$ limits.

2.3 THE BINOMIAL DISTRIBUTION

The binomial distribution describes the behavior of a count variable X if the number of observations, n is fixed where each of the observations is independent and represents one of the two outcomes (success or failure) with the probability of a success as p . The binomial probability mass function with parameters $n > 0$ and $0 < p < 1$ is defined as follows (Montgomery, 2001):

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n \quad (2.2)$$

The mean and variance of the binomial distribution are $\mu = np$ and $\sigma^2 = np(1-p)$ respectively. The binomial distribution is an appropriate probability model for

sampling from an infinitely large population, where p represents the fraction of defective or nonconforming items in the population.

2.4 THE POISSON DISTRIBUTION

The Poisson distribution is used to model the number of events occurring within a given time interval. The binomial distribution with large n and small p is well approximated to the Poisson distribution with parameter, $\mu = np$. The Poisson probability mass function with parameter $\lambda > 0$ is defined as follows (Montgomery, 2001):

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \dots \quad (2.3)$$

The mean and variance of a Poisson distribution are $\mu = \lambda$ and $\sigma^2 = \lambda$ respectively. A common application of the Poisson distribution in statistical quality control is that it serves as a model for the number of defects or nonconformities that occur in an inspection unit of a product.

2.5 AVERAGE RUN LENGTH (ARL)

The performance of a control chart is usually characterized by its run length (RL) value which is defined as the number of sample points that must be plotted on a chart until the first out-of-control point is obtained. In general, the run length of a Shewhart chart can be calculated using Equation (2.4) (Ledolter and Burrill, 1999).

$$P(\text{RL} = k) = (1 - w)^{k-1} w, \quad k = 1, 2, \dots, \quad (2.4)$$

where w refers to the probability that a sample point falls outside the control limits. The ARL is defined as the average number of sample points that must be plotted before a point indicates an out-of-control condition. The ARL formula is obtained as follows (Ledolter and Burrill, 1999):

$$\begin{aligned}
 \text{ARL} &= E(RL) \\
 &= \sum_{k=1}^{\infty} k \cdot P(RL = k) \\
 &= w + 2w(1-w) + 3w(1-w)^2 + 4w(1-w)^3 + \dots \\
 &= w [1 + 2(1-w) + 3(1-w)^2 + 4(1-w)^3 + \dots] \\
 &= w / [1 - (1-w)]^2 = 1/w
 \end{aligned} \tag{2.5}$$

2.6 SHEWHART CONTROL CHARTS

The Shewhart control chart plots the observations of variables or attributes for multiple parts on the same chart. It was developed to address the requirement that several dozen measurements of a process must be collected before control limits are calculated (Ledolter and Burrill, 1999). In general, Shewhart charts can be divided into two groups, i.e., Shewhart charts for variables and attributes.

Variables involve numerical measurements such as length, volume or weight which are taken on a continuous scale, while attributes refer to measurements that are classified as either conforming or nonconforming. The $\bar{X} - R$ or $\bar{X} - S$ charts are used to monitor a process that is based on continuous data. The \bar{X} chart monitors the process mean while the R and S charts monitor the process

variability. The $\bar{X} - R$ charts are used for sample sizes, $n \leq 10$ while the $\bar{X} - S$ charts for $n > 10$ (Montgomery, 2001).

The p , np , c and u charts are used to monitor attribute data. A p chart is used to monitor the process fraction nonconforming while a np chart monitors the number of nonconforming units. The number of nonconformities in an inspection unit of a product is monitored using a c chart while the monitoring of the average number of nonconformities per unit of a product is made using the u chart (Montgomery, 2001). Attribute control charts have the advantage of allowing for quick summaries of various aspects of the quality of a product.

Variable charts are more sensitive to process shifts than attribute charts. Therefore, a variable control chart may alert us to quality problems first before any actual "unacceptables" is detected by an attribute chart (Montgomery, 2001).

2.7 MEMORY CONTROL CHARTS

The moving average (MA), exponentially weighted moving average (EWMA), double exponentially weighted moving average (DEWMA) and cumulative sum (CUSUM) charts are memory control charts which are superior alternatives to the Shewhart chart when the detection of small process shifts are of interest because they use information about a process contained in the entire sequence of points as opposed to the Shewhart chart which uses only the information given in the last sample. This feature makes memory control charts more

sensitive to small process shifts, say, with a magnitude of about 1.5σ or less, in the process mean, compared to the Shewhart chart (Montgomery, 2001).

2.7.1 THE MOVING AVERAGE (MA) CHART

The MA chart is a time weighted control chart based on a simple, unweighted moving average. Assume that individual measurements X_1, X_2, \dots , where $X_i \sim N(\mu, \sigma^2)$, for $i = 1, 2, \dots$, are obtained from a process. The MA statistic of span w at time i is defined as (Montgomery, 2001)

$$MA_i = \frac{X_i + X_{i-1} + \dots + X_{i-w+1}}{w} \quad (2.6)$$

for $i \geq w$. For periods $i < w$, we do not have w measurements to compute a moving average of width w . For these periods, the average of all measurements up to period i defines the MA. The mean and variance of the moving average statistic, MA_i are

$$E(MA_i) = E(X_i) = \mu \quad (2.7a)$$

and

$$Var(MA_i) = \frac{1}{w^2} \sum_{j=i-w+1}^i V(X_j) = \frac{1}{w^2} \sum_{j=i-w+1}^i \sigma^2 = \frac{\sigma^2}{w} \quad (2.7b)$$

respectively. The control limits of the MA chart are

$$UCL/LCL = \mu_0 \pm \frac{3\sigma}{\sqrt{w}} \quad (2.8)$$

for $i \geq w$, where μ_0 is the target value of the mean. The limits for periods $i < w$

are obtained by replacing $\frac{\sigma}{\sqrt{w}}$ with $\frac{\sigma}{\sqrt{i}}$ in Equation (2.8).

2.7.2 THE EXPONENTIALLY WEIGHTED MOVING AVERAGE (EWMA) CHART

The EWMA chart was introduced by Roberts (1959) and its statistic is defined as (Montgomery, 2001)

$$Z_i = \lambda X_i + (1 - \lambda)Z_{i-1}, \quad i = 1, 2, \dots \quad (2.9)$$

where $0 < \lambda \leq 1$ is the smoothing constant. The EWMA statistic can be expressed as a weighted linear combination of current and past observations. The smaller the value of the smoothing constant, λ the greater the influence of the past data (Montgomery, 2001). Continuing to substitute recursively for Z_{i-j} , $j = 2, 3, \dots, t$, in Equation (2.9), it can be shown that (Montgomery, 2001):

$$Z_i = \lambda \sum_{j=0}^{i-1} (1 - \lambda)^j X_{i-j} + (1 - \lambda)^i Z_0 \quad (2.10)$$

The weights $\lambda(1 - \lambda)^j$ in Equation (2.10) decrease geometrically with the age of the observation and these weights sum to unity because

$$\lambda \sum_{j=0}^{i-1} (1 - \lambda)^j = \lambda \left[\frac{1 - (1 - \lambda)^i}{1 - (1 - \lambda)} \right] = 1 - (1 - \lambda)^i \quad (2.11)$$

The exact control limits for the EWMA chart are

$$\text{UCL/LCL} = \mu_0 \pm k\sigma \sqrt{\frac{\lambda}{(2 - \lambda)} [1 - (1 - \lambda)^{2i}]} \quad (2.12a)$$

where μ_0 is the center line and k is the width of the control limits. The asymptotic limits for the EWMA chart based on the exact limits in Equation (2.12a) are

$$\text{UCL/LCL} = \mu_0 \pm k\sigma \sqrt{\frac{\lambda}{(2 - \lambda)}} \quad (2.12b)$$

since the term $[1 - (1 - \lambda)^{2i}]$ approaches unity as i becomes larger.

2.7.3 THE DOUBLE EXPONENTIALLY WEIGHTED MOVING AVERAGE (DEWMA) CHART

The DEWMA chart was proposed by Shamma *et al.* (1991) and extended by Shamma and Shamma (1992) and Zhang and Chen (2005). They showed that the DEWMA chart outperforms the EWMA chart for shifts in the mean ranging from 0.1σ to 0.5σ , where σ is the process standard deviation and that the two approaches have similar performances for mean shifts of larger than 0.5σ . The DEWMA statistic, W_t is computed from the EWMA statistic $Z_t = \lambda_1 \bar{X}_t + \lambda_2 Z_{t-1}$, $t = 1, 2, \dots$, to be (Zhang and Chen, 2005)

$$W_t = \lambda_3 Z_t + \lambda_4 W_{t-1}, \quad t = 1, 2, \dots, \quad (2.13)$$

where $Z_0 = c$, $W_0 = c$, $0 < \lambda_i < 1$, for $i = 1, 3$, $\lambda_1 + \lambda_2 = 1$, $\lambda_3 + \lambda_4 = 1$ and c is the starting value which is usually set as the target mean value, μ_0 . Note that

$\bar{X}_t = \sum_{k=1}^n X_{tk} / n$ is the sample mean of subgroup t . It was shown by Zhang and

Chen (2005) that

$$W_t = \lambda_1 \lambda_3 \sum_{j=1}^t \left\{ \lambda_2^{t-j} \sum_{k=0}^{t-j} \left(\frac{\lambda_4}{\lambda_2} \right)^k \right\} \bar{X}_j + \lambda_3 \sum_{j=0}^{t-1} \lambda_2^{t-j} \lambda_4^j Z_0 + \lambda_4^t W_0, \quad t = 1, 2, \dots \quad (2.14)$$

If $\lambda_1 = \lambda_3$, then

$$W_t = \lambda_1^2 \sum_{j=1}^t (t-j+1) \lambda_2^{t-j} \bar{X}_j + t \lambda_1 \lambda_2^t Z_0 + \lambda_2^t W_0, \quad t = 1, 2, \dots \quad (2.15)$$

For this case, the variance of W_t is

$$Var(W_t) = \frac{\sigma_0^2}{n} \lambda_1^4 \frac{1 + \lambda_2^2 - (t^2 + 2t + 1)\lambda_2^{2t} + (2t^2 + 2t - 1)\lambda_2^{2t+2} - t^2 \lambda_2^{2t+4}}{(1 - \lambda_2^2)^3} \quad (2.16)$$

If $\lambda_1 \neq \lambda_3$, then

$$W_t = \lambda_1 \lambda_3 \sum_{j=1}^t \frac{1 - (\lambda_4/\lambda_2)^{t-j+1}}{1 - (\lambda_4/\lambda_2)} \lambda_2^{t-j} \bar{X}_j + \lambda_2 \lambda_3 \frac{\lambda_2^t - \lambda_4^t}{\lambda_2 - \lambda_4} Z_0 + \lambda_4^t W_0, \quad t = 1, 2, \dots \quad (2.17)$$

and the variance of W_t is

$$Var(W_t) = \frac{\sigma_0^2}{n} \frac{\lambda_1^2 \lambda_3^2}{(\lambda_4 - \lambda_2)^2} \left\{ \frac{\lambda_4^2 (1 - \lambda_4^{2t})}{1 - \lambda_4^2} + \frac{\lambda_2^2 (1 - \lambda_2^{2t})}{1 - \lambda_2^2} - 2 \frac{\lambda_2 \lambda_4 [1 - (\lambda_2 \lambda_4)^t]}{1 - \lambda_2 \lambda_4} \right\} \quad (2.18)$$

The control limits for the DEWMA chart are

$$UCL_t / LCL_t = \mu_0 \pm L \sqrt{Var(W_t)} \quad (2.19)$$

where $Var(W_t)$ is given in Equation (2.16) for $\lambda_1 = \lambda_3$ or Equation (2.18) for $\lambda_1 \neq \lambda_3$ and L is a factor that controls the width of the DEWMA chart that together with λ_1 and λ_3 controls the performance of the chart. Zhang and Chen (2005) provide guidelines on the selection of the values of λ_1 , λ_3 and L . The DEWMA chart is constructed by plotting W_t against t for $t = 1, 2, \dots$, based on the limits in Equation (2.19).

2.7.4 THE CUMULATIVE SUM (CUSUM) CHART

The CUSUM chart was proposed by Page (1954). The performance of the CUSUM chart is approximately equivalent to that of the EWMA chart. There are several versions of the CUSUM chart but only the tabular CUSUM will be discussed here as it is the most commonly used. Let X_i be the i^{th} observation

of a process which is normally distributed with mean, μ and standard deviation, σ . The tabular CUSUM works by accumulating deviations from μ_0 that are above the target with statistic C^+ and accumulating deviations from μ_0 that are below the target with statistic C^- . The C^+ and C^- statistics are computed as follows (Montgomery, 2001):

$$C_i^+ = \max\left[0, X_i - (\mu_0 + K) + C_{i-1}^+\right] \quad (2.20a)$$

and

$$C_i^- = \max\left[0, (\mu_0 - K) - X_i + C_{i-1}^-\right] \quad (2.20b)$$

Here, $K = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$, where δ is the magnitude of a shift in terms of the number of standard deviations where a quick detection is desired, μ_0 is the target mean value and μ_1 is the out-of-control value of the mean that we wish to detect quickly. A process is considered out-of-control if either C^+ or C^- exceeds the decision interval, H . The value of H is usually chosen to be five times the process standard deviation, σ so that the CUSUM chart performs reasonably well. Montgomery (2001) provides recommendations for choosing the value, H in constructing a CUSUM chart with good ARL properties.

CHAPTER 3

EXTENSIONS OF THE MOVING AVERAGE CONTROL CHART

3.1 Introduction

Till now, research on the moving average chart is quite limited in comparison to that of the exponentially weighted moving average (EWMA) chart. In this chapter, some extensions of the moving average chart such as the optimal and economic designs of the MA chart, the weighted MA chart, the MA chart for a joint monitoring of the process mean and variance, the MA chart for monitoring the fraction nonconforming and the Poisson MA chart will be reviewed.

3.2 A Review on the Extensions of the Moving Average Control Chart

Recently, extensions of the MA chart have been made to enhance the chart's ability in the detection of process shifts. Chen and Yu (2003) suggest an approach in the optimal design of the MA chart. Wong *et al.* (2004) develop a simple procedure for the optimal designs of the MA chart and the combined MA-Shewhart scheme for an easy selection of the chart's parameters to enable a quick detection of a desired magnitude of a shift.

Chen and Yang (2002), Yu and Wu (2004), and Yu and Chen (2005) propose economic designs of the MA chart which enable a determination of the optimal parameters of the chart in minimizing the total cost.

Sparks (2004) proposes the weighted moving average (WMA) chart and illustrates the advantages of using this chart as an efficient plan for monitoring a specific location shift compared to the EWMA and CUSUM charts when the size of a shift is known in advance.

Khoo and Yap (2004) propose the use of a single MA chart for a joint monitoring of the process mean and variance. The design of an MA chart for fraction non-conforming as a superior alternative to the standard p chart is suggested by Khoo (2004a). The new MA chart enhances the detection speed of the p chart. Khoo (2004b) also suggests the Poisson moving average chart for the number of nonconformities and shows that the new chart has a superior ARL performance to the standard c chart.

3.2.1 Optimal Designs of the MA and MA-Shewhart Charts

Wong *et al.* (2004) proposes an approach for the optimal designs of the MA and the combined Shewhart-MA charts using the simulation method, integral equation approach and curve fitting techniques to improve the sensitivity of the MA chart. The MA chart is used to detect small and moderate process shifts while the MA-Shewhart chart is used to detect large process shifts. The zero state and steady state ARLs are considered in the designs of these charts. The zero state run length refers to the run length at start up whereas a steady state run length refers to the run length from a certain point onwards, after the MA chart has reached a steady state.

3.2.1.1 The Optimal MA Chart

The following four step procedure is recommended for the design of this chart (Wong *et al.*, 2004).

Step 1: Specify the in-control ARL value.

Step 2: Decide on the smallest shift $\Delta\sigma_0/\sqrt{n}$ in the process mean to be detected quickly, where Δ denotes the magnitude of a shift in multiples of standard deviation in the sample mean.

Step 3: Determine the span, w that produces the smallest out-of-control ARL at the selected shift.

Step 4: Based on the value of the span w from Step 3, determine the constant, k such that the MA chart produces the in-control ARL specified in Step 1.

The MA chart obtained using the procedure described here is optimal for the shift $\Delta\sigma_0/\sqrt{n}$ among all MA charts with a similar in-control ARL. The regression models and plots given in Wong *et al.* (2004) and reproduced here in Table 3.1 and Figures 3.1a - 3.1d respectively, are used to simplify the procedure in Steps 3 and 4. Note that the span, w in Step 3 is determined using the formulae (Wong *et al.*, 2004)

$$\log_{10} w = a - b \log_{10} \Delta, \quad (3.1)$$

where a and b are coefficients given in Table 3.1 based on the in-control ARL specified in Step 1. Based on the value of w obtained in Step 3, Figures 3.1a - 3.1d are then used to determine the optimal value of k .

The control limits for the optimal MA chart are (Wong *et al.*, 2004)

$$UCL = \mu_0 + k \frac{\sqrt{w}}{\sqrt{\min(w,t)}} \frac{\sigma_0}{\sqrt{n}}, \quad t = 1, 2, \dots \quad (3.2a)$$

and

$$LCL = \mu_0 - k \frac{\sqrt{w}}{\sqrt{\min(w,t)}} \frac{\sigma_0}{\sqrt{n}}, \quad t = 1, 2, \dots \quad (3.2b)$$

where n is the sample size, while μ_0 and σ_0 are the in-control process mean and standard deviation respectively.

The statistic plotted on the optimal MA chart is defined as

$$M_t = \begin{cases} \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_t}{t}, & t = 1, 2, \dots, w-1 \\ \frac{\bar{X}_{t-w+1} + \bar{X}_{t-w+2} + \dots + \bar{X}_t}{w}, & t = w, w+1, \dots \end{cases} \quad (3.3)$$

where w is the span of the moving average and \bar{X}_t is the sample mean of the n observations collected at sample, t . A process is said to be out-of-control if M_t plots outside the limits given by UCL and LCL in Equations (3.2a) and (3.2b) respectively.

3.2.1.2 The Optimal MA-Shewhart Chart

Since the MA chart is only sensitive in detecting small and moderate shifts, a combined MA-Shewhart scheme is also suggested by Wong *at al.* (2004) to increase the sensitivity of the MA chart in detecting large shifts. The MA chart

obtained is optimal in detecting a specified shift of $\Delta\sigma_0/\sqrt{n}$. Wong *et al.* (2004) recommend the following six-step procedure in the design of a combined MA-Shewhart chart. These steps are recommended such that the combined MA-Shewhart scheme has an in-control ARL, ARL_0 and the corresponding individual MA chart is optimal in detecting a shift of $\Delta\sigma_0/\sqrt{n}$ and has a resulting in-control ARL, $ARL_{0,MA}$. The resulting Shewhart chart will have a similar in-control ARL as the MA chart.

Step 1: Specify an in-control ARL, ARL_0 of the combined MA-Shewhart scheme.

Step 2: Calculate the corresponding in-control ARL, $ARL_{0,MA}$ of the MA chart for $w = 2, 3, \dots, 19, 20$ and 30 based on the ARL_0 specified in Step 1 using the table provided by Wong *et al.* (2004) which is given here as Table 3.2.

Step 3: Decide on the smallest shift $\Delta\sigma_0/\sqrt{n}$ where a quick detection is important for the MA chart.

Step 4: For each pair of $(w, ARL_{0,MA})$ obtained in Step 2, determine the shift $\Delta\sigma_0/\sqrt{n}$ for which the MA chart is optimal in detecting. This is done using the regression models in Table 3.1. Choose the pair $(w, ARL_{0,MA})$ with a Δ that is closest to the smallest shift $\Delta\sigma_0/\sqrt{n}$ decided in step 3.

Step 5: Based on the pair of $(w, ARL_{0,MA})$ determined in Step 4, find the constant k_M of the MA chart using Figures 3.1a – 3.1d.

Step 6: Using the same in-control ARL as that of the MA chart considered in Step 5, the constant k_s of the Shewhart chart can also be determined using Figure 3.1a.

If samples of size n are taken from a process that follows a $N(\mu_0, \sigma^2)$ distribution, the control limits for the combined MA-Shewhart scheme can be determined as

$$\text{UCL/LCL} = \mu_0 \pm k_s \frac{\sigma_0}{\sqrt{n}}, \quad (3.4)$$

for the Shewhart \bar{X} chart and

$$\text{UCL/LCL} = \mu_0 \pm k_M \frac{\sqrt{w}}{\sqrt{\min(w, t)}} \frac{\sigma_0}{\sqrt{n}}, \quad t = 1, 2, \dots \quad (3.5)$$

for the individual MA chart, respectively.

The combined MA-Shewhart scheme is implemented by plotting the sample mean, \bar{X}_t , for $t = 1, 2, \dots$, on the Shewhart \bar{X} chart and the statistic, M_t on the individual MA chart. Note that M_t is computed using Equation (3.3). A process is deemed out-of-control if a point plots outside the limits of either the Shewhart \bar{X} chart or the MA chart.

Table 3.1 Least-squares regression model relating the optimal span w of a MA chart to the shift Δ for which the chart is optimal in detecting:
 $\log_{10} w = a - b \log_{10} \Delta$

In-control ARL, ARL_0	Coefficient, a	Coefficient, b
50	0.724	1.39
100	0.802	1.42
200	0.914	1.64
250	0.940	1.67
300	0.948	1.66
370	0.973	1.64
400	0.993	1.66
500	1.000	1.67
600	1.010	1.68
700	1.020	1.67
800	1.040	1.70
900	1.040	1.67
1000	1.070	1.78
1500	1.100	1.77
2000	1.090	1.63

(Source: Wong *et al.*, 2004)

Table 3.2 Least-squares regression model relating the in-control ARL of a combined MA-Shewhart scheme to the in-control ARL of the individual MA chart: $ARL_{0,MA}$ of MA chart = $a + b \times$ (ARL₀ of the combined scheme)

Chart parameter, w	Coefficient, a	Coefficient, b
2	-20.2	1.78
3	-18.2	1.87
4	-18.2	1.91
5	-16.1	1.93
6	-13.8	1.94
7	-13.4	1.94
8	-12.8	1.95
9	-11.3	1.95
10	-11.5	1.96
11	-10.1	1.96
12	-9.33	1.96
13	-8.32	1.96
14	-7.69	1.97
15	-7.09	1.97
16	-6.53	1.97
17	-5.88	1.97
18	-5.55	1.97
19	-4.92	1.97
20	-3.80	1.97
30	-2.12	1.98

(Source: Wong *et al.*, 2004)

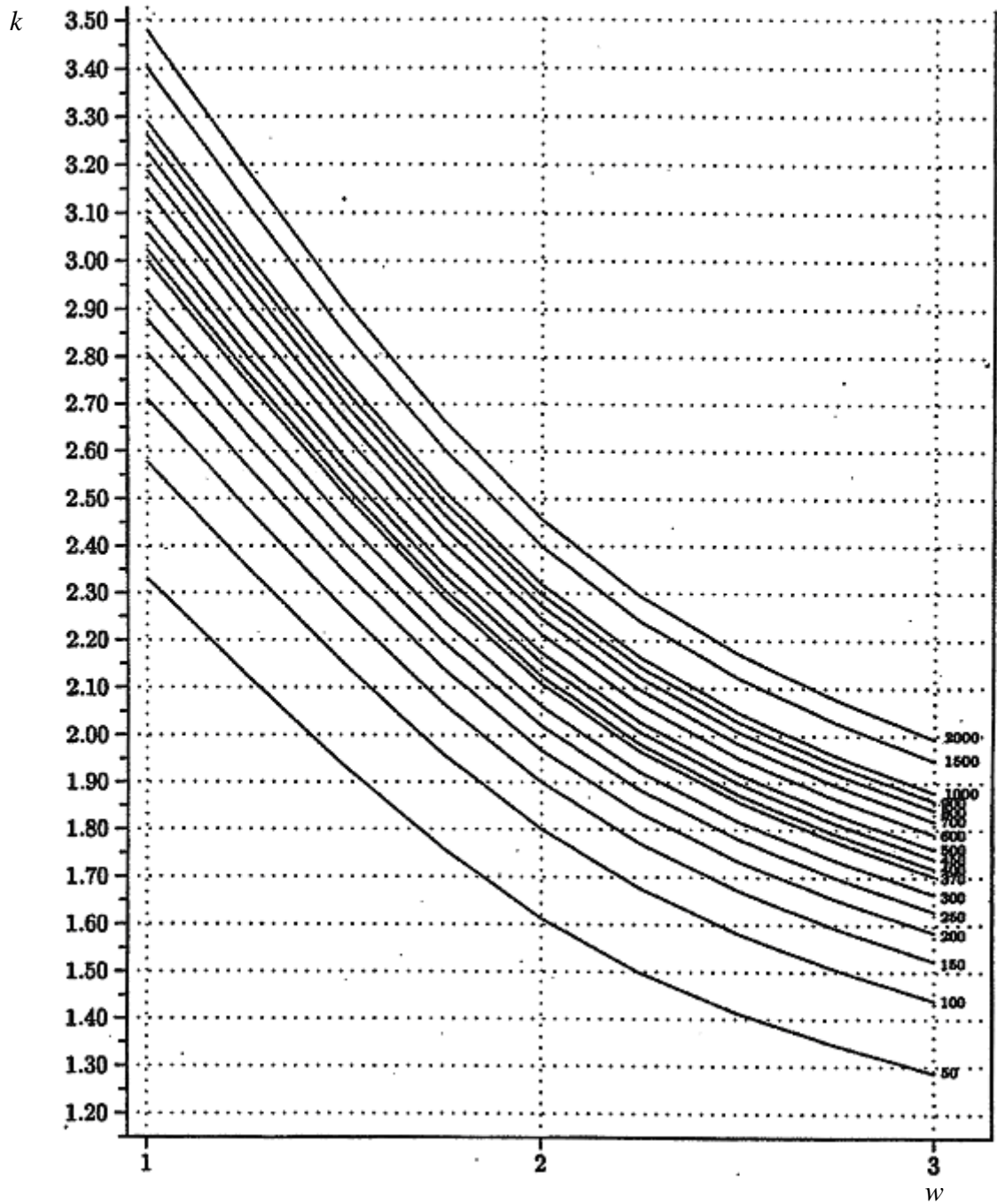


Figure 3.1a Combinations of optimal span, w and constant, k of the moving average chart with respect to various in-control ARL's, for $w = 1$, 2 and 3

(Source: Wong *et al.*, 2004)

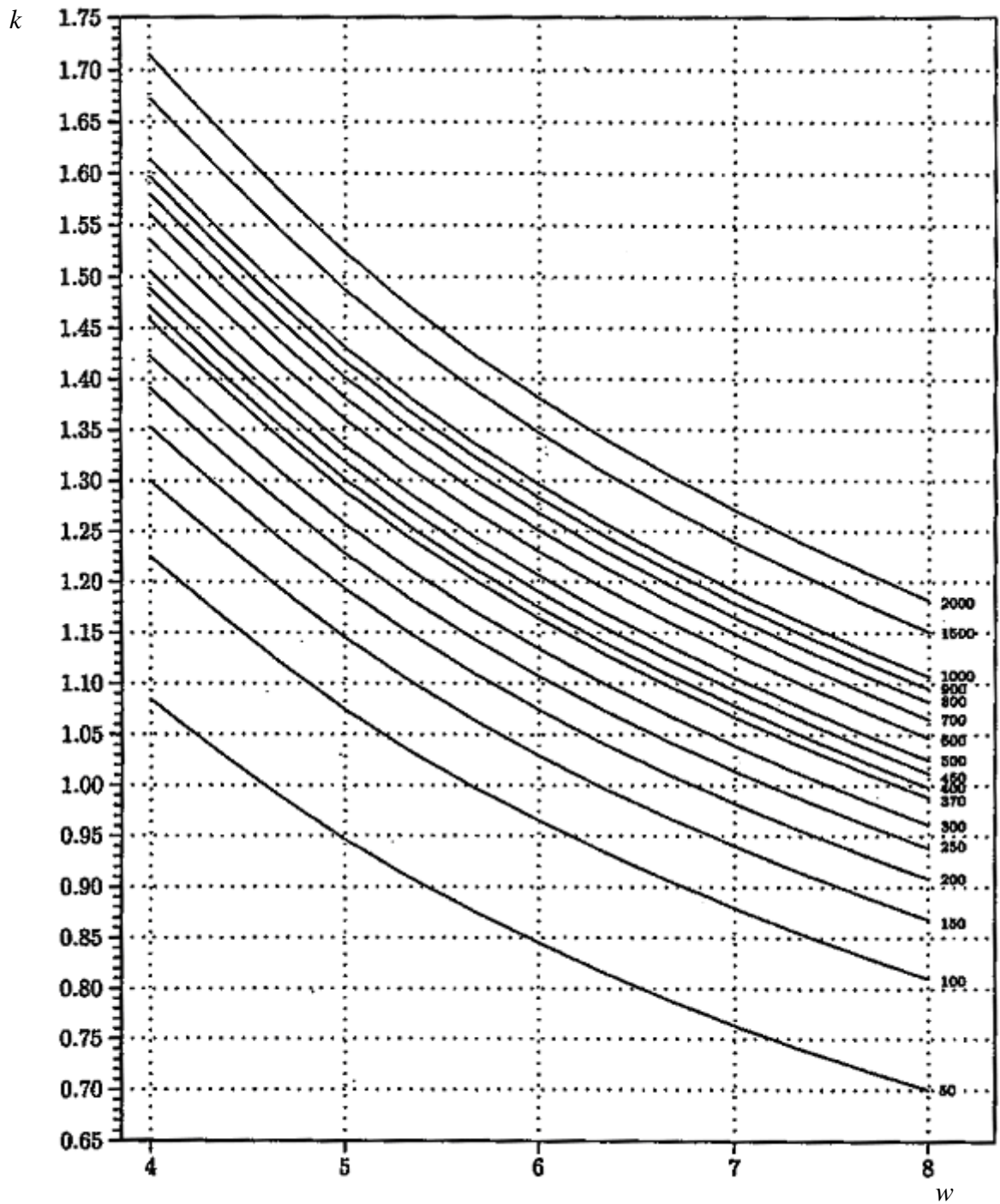


Figure 3.1b Combinations of optimal span, w and constant, k of the moving average chart with respect to various in-control ARL's, for $w = 4, 5, \dots, 8$

(Source: Wong *et al.*, 2004)

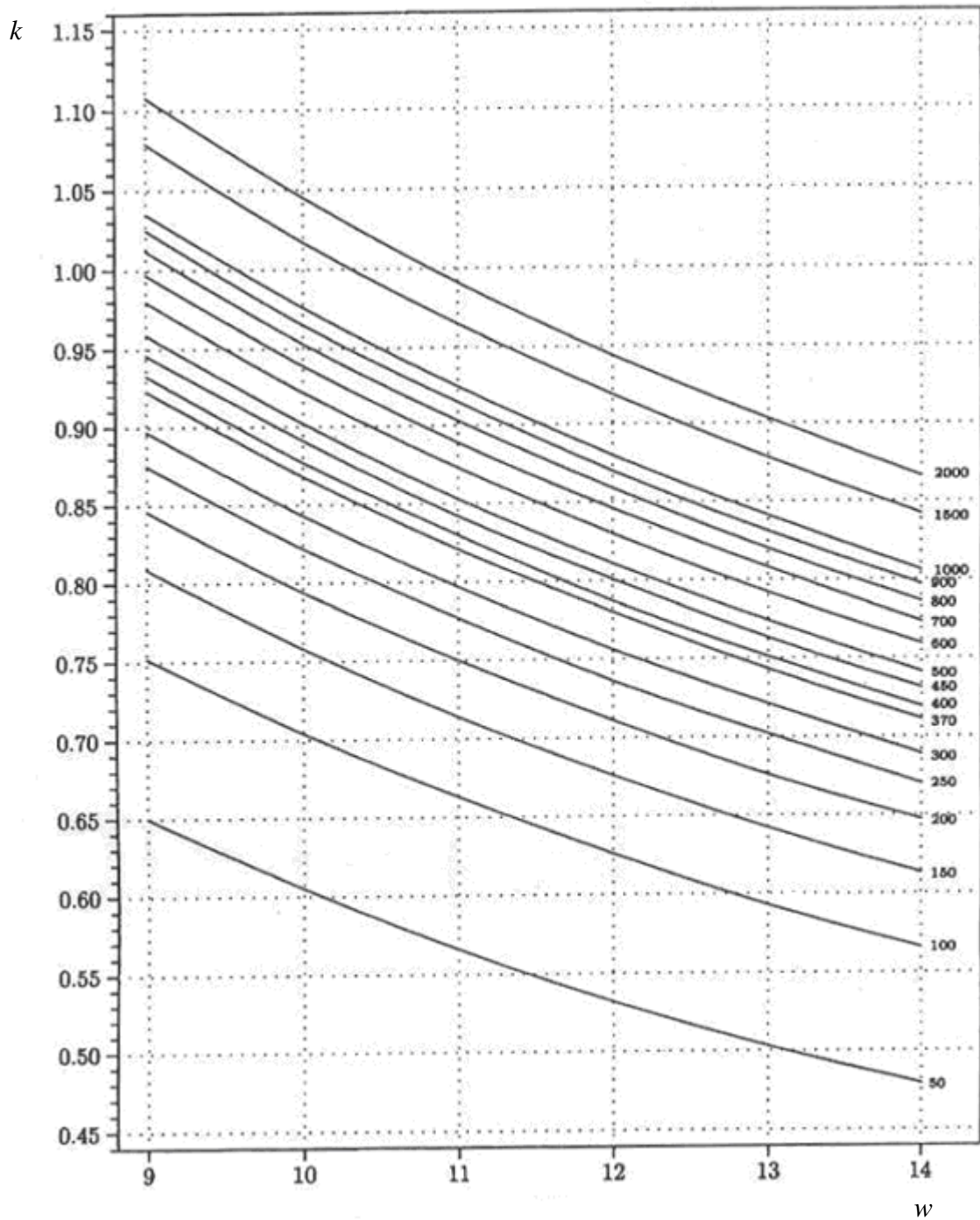


Figure 3.1c Combinations of optimal span, w and constant, k of the moving average chart with respect to various in-control ARL's, for $w = 9, 10, \dots, 14$

(Source: Wong *et al.*, 2004)