NUMERICAL SOLUTION OF A LINEAR GOURSAT PROBLEM: STABILITY, CONSISTENCY AND CONVERGENCE

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Abstract:-The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of applications. These include mathematical modeling of reacting gas flows and supersonic flow. Recent numerical studies of the problem have focused on the implementation and accuracy aspects of various finite difference schemes. Theoretical considerations such as stability, consistency and convergence have not received much attention. In this paper we consider the theoretical aspects of a numerical scheme widely used to solve the problem by considering its application to model linear problem . We obtain results relating to the stability (using Von Neumann stability analysis), consistency and convergence of the scheme. We verify these theoretical results with data from computational experiments.

Key-Words:- Finite difference schemes; partial differential equations; means; stability; Consistency; convergence

1 Introduction

The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of physics and engineering. An and Hua (1981) and Chen and Li (2000) describes in detail how this equation arises and is used in the mathematical modeling of reacting gas flows and supersonic flows respectively. To investigate these mathematical models in greater detail, we need to resort to numerical techniques and in this regard the finite difference method is often used. The standard finite difference scheme for the Goursat problem is a method based on arithmetic averaging of function values and a recent study (Ismail et. al., 2004) has reaffirmed the advantages of this method (" the AM scheme").

What assurance is there that the solution obtained by a numerical method is close to the exact solution? One way would be to check that the computed results converge when the grid sizes are reduced. However, clearly it would be better if convergence can be guaranteed beforehand. It is well-known in numerical analysis that for linear problems this guarantee can be given if the numerical method is stable and consistent. The concept of stability is concerned with the growth, or decay, of errors (produced because the computer cannot give answers to an infinite number of decimal places) at any stage of the computation (Fletcher, 1990). The system of algebraic equations generated by the discretisation process is said to be consistent with the original partial differential equation if, in the limit that the grid spacing tends to zero, the system of algebraic equations is equivalent to the partial differential equation at each grid point (Fletcher, 1990). Consistency is concerned with how well the algebraic equations approximates the partial differential equation.

Recent studies of numerical methods for the Goursat problem (Ismail et.al., 2004 ; Wazwaz, 1993) have focused on implementation and accuracy of various finite difference schemes. In this paper we will investigate the stability, consistency and convergence of the AM scheme when applied to a model linear Goursat problem.

2 The Goursat Problem and the AM Scheme

The Goursat problem is of the form (Wazwaz, 1993):

$$u_{xy} = f(x, y, u, u_x, u_y)$$
$$u(x,0) = \phi(x), u(0,y) = \psi(y), \phi(0) = \psi(0)$$
$$0 \le x \le a, 0 \le y \le b \qquad \ldots(1)$$

The established finite difference scheme is based on arithmetic mean averaging of function values and is given by (Ismail et.al 2004):

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}\right) \qquad \ldots(2)$$

h denotes the grid size. As was mentioned earlier, we shall refer to the finite difference scheme (2) as the AM scheme. For the AM scheme, the function value at grid location $(i+1/2, j+1/2)$ is given by :

$$\frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}\right) \qquad \ldots(3)$$

We shall investigate the stability consistency and convergence of the AM scheme for linear Goursat problems by considering the model linear Goursat problem:

$$u_{xy} = u$$
$$u(x,0) = e^x$$
$$u(0,y) = e^y$$
$$0 \le x \le 2, 0 \le y \le 2 \qquad \ldots(4)$$

The analytical solution of (4) is e^{x+y} (Wazwaz, 1995)

The AM scheme for the partial differential equation in (4 is:

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}\left(u_{i+1,j+1} + u_{i,j} + u_{i+1,j} + u_{i,j+1}\right) \qquad \ldots(5)$$

Equation (5) can be rewritten as:

$$u_{i+1,j+1} = A\left(u_{i+1,j} + u_{i,j+1}\right) - u_{i,j} \qquad \ldots(6)$$

where $A = \frac{1+r}{1-r}$ with $r = \frac{h^2}{4} > 0$.

3 Stability

The stability of a finite difference scheme can be investigated using the Von Neumann method (Fletcher, 1990). In this method, the errors distributed along grid lines at one time level are expanded as a finite Fourier series. If the separate Fourier components of the error distribution amplify in progressing to the next time level, then the scheme is unstable.

The error equation for equation (6) is:

$$\varsigma_{i+1,j+1} = A(\varsigma_{i+1,j} + \varsigma_{i,j+1}) - \varsigma_{i,j} \qquad \ldots(7)$$

where $\varsigma_{i,j}$ is the error at the (i,j) grid point. We write $\varsigma_{i,j}$ as $\lambda^j e^{\sqrt{-1}\theta_m i}$ where λ is the amplification factor for the mth Fourier mode of the error distribution as it propagates one step forward in time and $\theta_m = m\pi h$. For linear schemes it is sufficient to consider the propagation of the error due to just a single term of the Fourier series representation i.e. the subscript m can be dropped.

Substituting $\varsigma_{i,j} = \lambda^j e^{\sqrt{-1}\theta i}$ into equation (7) gives:

$$\lambda^{j+1} e^{\sqrt{-1}\theta(i+1)} = A\left(\lambda^j e^{\sqrt{-1}\,\theta(i+1)} + \lambda^{(j+1)} e^{\sqrt{-1}\theta i}\right)$$

i.e. $- \lambda^j e^{\sqrt{-1}\theta i}$

$$\lambda = \frac{Ae^{\sqrt{-1}\theta} - 1}{e^{\sqrt{-1}\theta} - A} \qquad \ldots(8)$$

For stability it is required that $|\lambda| \le 1 \forall\theta$ i.e. $\left|\frac{Ae^{\sqrt{-1}\theta} - 1}{e^{\sqrt{-1}\theta} - A}\right| \le 1\forall\theta$ which implies $\left|e^{\sqrt{-1}\theta} - A\right| \ge \left|Ae^{\sqrt{-1}\theta} - 1\right|$

Squaring both sides and after some manipulation we obtain that for $|\lambda| \le 1\forall\theta, A$ must satisfy $A^2 \ge 1$. Since $A = \frac{1+r}{1-r}$ and $r > 0$, we obtain that the scheme is stable $\forall r$ (except $r = 1$)

4 Consistency

To test for consistency, the exact solution of the partial differential equation is substituted into the finite difference scheme and values at grid points expanded as a Taylor series. For consistency, the expression obtained should tend to the partial differential equation as the grid size tends to zero (Twizell, 1984).

Substituting the exact solution into (5) leads to:

$$\frac{1}{h^2}\begin{Bmatrix} u(x_{i+1}, y_{j+1}) + u(x_i, y_j) - \\ u(x_{i+1}, y_j) - u(x_i, y_{j+1}) \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} u(x_{i+1}, y_{j+1}) + u(x_i, y_j) \\ + u(x_{i+1}, y_j) + u(x_i, y_{j+1}) \end{Bmatrix} \qquad .(9)$$

Expanding as a Taylor series about (x_i, y_j) gives:

$$\frac{1}{h^2}\begin{Bmatrix} u + hu_x + hu_y \\ + \frac{1}{2}(h^2u_{xx} + 2h^2u_{xy} + h^2u_{yy}) + \ldots \\ + u - (u + hu_x + \frac{h^2}{2}u_{xx} + \ldots) \\ - (u + hu_y + \frac{h^2}{2}u_{yy} + \ldots) \end{Bmatrix}$$

$$= \frac{1}{4}\begin{Bmatrix} u + hu_x + hu_y \\ + \frac{1}{2}(h^2u_{xx} + 2h^2u_{xy} + h^2u_{yy}) + \ldots \end{Bmatrix} + \frac{1}{4}\left\{u + hu_x + \frac{h^2}{2}u_{xx} + \ldots\right\} + \frac{1}{4}\left\{u + hu_y + \frac{h^2}{2}u_{yy} + \ldots\right\} + \frac{1}{4}u \qquad \ldots(10)$$

[note: all terms involving u in equation (10) are evaluated at (x_i, y_j)]

As $h \to 0$, equation (10) becomes $u_{xy} = u$. Thus the condition for consistency is satisfied.

5 Convergence

A solution of the algebraic equations which approximate a given partial differential equation is said to be convergent if the approximate solution approaches the exact solution for each value of the independent variables as the grid spacing tends to zero. The Lax Equivalence Theorem (Richtmyer and Morton, 1967) states that given a properly (well) posed linear initial value problem and a finite difference equation that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

That the Goursat problem is well posed was established by Garabedian (1964) by transforming it into an integrodifferential equation and then solving by the technique of successive approximations. We have established that the AM scheme for the linear Goursat problem (4) is both stable and consistent. Thus from the Lax Equivalence Theorem we can conclude it is convergent.

6 Computational Experiments

A computer program using the AM scheme to solve (4) was developed and the computed results are as follows:

	absolute errors at grid points			
h	(0.25,0.25)	(0.5,0.5)	(0.75,0.75)	(1.0,1.0)
0.500	6.9233875e-003	6.9233875e-003	2.0437596e-002	2.0437596e-002
0.250	5.1857520e-004	1.7070573e-003	3.2493354e-003	4.9962137e-003
0.100	5.5301566e-005	2.7212902e-004	5.7076620e-004	7.9460331e-004
0.050	2.0684065e-005	6.7997099e-005	1.2925661e-004	1.9848210e-004
0.025	5.170565e-006	1.6997083e-005	3.2308635e-005	4.9610004e-005

	absolute errors at grid points			
h	(1.25,1.25)	(1.5,1.5)	(1.75,1.75)	(2.0,2.0)
0.500	3.6461034e-002	3.6461304e-002	5.3882494e-002	5.3882494e-002
0.250	6.8729164e-003	8.8393041e-003	1.0872209e-002	1.2957280e-002
0.100	1.1530440e-003	1.4026219e-003	1.7883943e-003	2.0515447e-003
0.050	2.7267983e-004	3.5024469e-004	4.3025104e-004	5.1212559e-004
0.025	6.8152732e-005	8.7535555e-005	1.0752707e-004	1.2798397e-004

Table 1: h values and absolute errors at various grid points.

Although results at only eight points are displayed, numerical experiments indicate that the absolute error becomes smaller as h is decreased for all grid points tested.

7 Conclusions

Previous studies of the finite difference solution of the Goursat problem have focused on the accuracy and implementation aspects. In this paper we have studied the theoretical aspects of the finite difference solution of a linear Goursat problem using the AM scheme. Using the Von Neumann method we have shown that it is unconditionally stable and we have also shown it is consistent. Invoking the Lax Equivalence Theorem we deduce that the scheme is convergent. Numerical experiments that we have conducted verify that the scheme is convergent

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