UNIVERSITI SAINS MALAYSIA

Peperiksaan Semester Pertama
Sidang 1994/95

Oktober/November 1994

MSG 442 - Kaedah Unsur Terhingga

Masa : [3 jam]

Jawab **SEMUA** soalan.

1. (a) Selesaikan masalah berikut:

$$\frac{d^2\phi}{dx^2} - \phi + 2 = 0, \qquad 0 < x < 1$$

$$\phi(0) = 0\,, \quad \phi'(1) = 0$$

dengan kaedah unsur terhingga dan dengan menggunakan tiga unsur linear. Cari penyelesaian tepat dan bandingkan penyelesaian hampiran anda dengan penyelesaian tepat itu.

(50/100)

(b) Selesaikan masalah aliran haba berikut:

$$\frac{\partial^2\phi}{\partial x^2} = 4\,\frac{\partial\phi}{\partial t}\,, \qquad 0 < x < 8\,, \qquad t > 0$$

$$\phi(0,t) = 10\,, \quad \frac{\partial\phi}{\partial x}(8,t) = 0\,, \quad t > 0$$

$$\phi(x,0) = \begin{cases} 20x + 10\,, & 0 \le x < 2 \\ 50\,, & 2 \le x \le 8 \end{cases}$$

dengan kaedah bergumpal dan dengan menggunakan empat unsur, $\theta = 0$ dan $\Delta t = 1$. Cari suhu nod bagi dua langkah masa pertama.

(50/100)

.../2

2. (a) Katakan Ω ialah rantau A(0, 0), B(2, 0), C(1, 1), D(0, 1). Selesaikan masalah berikut:

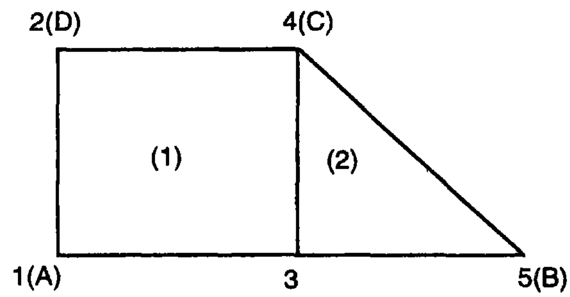
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{di dalam } \Omega ,$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{pada sisi AD dan BC} ,$$

$$\phi = 50 \quad \text{pada sisi AB} ,$$

$$\frac{\partial \phi}{\partial y} = -\phi + 2 \quad \text{pada sisi DC.}$$

dengan menggunakan satu unsur segiempat bilinear dan satu unsur segitiga linear seperti ditunjukkan di Rajah 1.



Rajah 1

(50/100)

(b) Katakan $N_1(\xi)$, $N_2(\xi)$, $N_3(\xi)$ ialah fungsi bentuk kuadratik 1-D. Cari

(i) $\int_{-1}^{1} N_1^2 \, d\xi$

(ii) $\int_{-1}^{1} N_2 N_3 \, d\xi$

(iii) $\int_{-1}^{1} \frac{dN_1}{dx} \cdot \frac{dN_2}{dx} \, d\xi$.

(30/100)

(c) Nilaikan

$$\int_A xy \, dA$$

dengan kuadratur Gauss dan dengan menggunakan tiga titik pensampelan, di mana A ialah segitiga (0, 0), (6, 2), (4, 4).

(20/100)

3. (a) Pertimbangkan masalah berikut:

$$D\frac{\partial^2\phi}{\partial x^2} = \lambda\frac{\partial\phi}{\partial t}, \quad a < x < b, \quad t > 0$$

$$\phi(x, 0) = f(x), \quad a \le x \le b$$

$$\phi(a, t) = 0, \quad \phi(b, t) = 0, \quad t > 0$$

Jika kaedah bergumpal dan kaedah beza ke depan digunakan untuk menyelesaikan masalah ini, cari syarat atas Δt supaya kaedah berangka ini adalah stabil.

(30/100)

(b) Pertimbangkan masalah:

$$D\frac{\partial^2\phi}{\partial x^2} + D\frac{\partial^2\phi}{\partial y^2} = \lambda\frac{\partial\phi}{\partial t}, \quad (x, y) \in \Omega, \quad t > 0$$

dengan syarat sempadan dan syarat awal yang sesuai. Jika kaedah bergumpal digunakan untuk mendapatkan $\left[c^{(e)}\right]$, cari syarat atas Δt bagi suatu segitiga bersisi sama dengan panjang sisi b supaya ayunan berangka dapat dielakkan.

(30/100)

(c) Pertimbangkan masalah berikut:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - G\phi + Q = 0 \qquad \text{di dalam } \Omega$$

$$\phi = f \qquad \text{pada } \Gamma$$

di mana Γ ialah sempadan bagi suatu rantau Ω.
Dengan menggunakan carta aliran, terangkan langkah demi langkah bagaimana masalah tersebut di atas itu dapat diselesaikan jika unsur-unsur kuadratik 8-nod digunakan.

(40/100)

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Unsur Linear 1-D

$$\left[k^{(e)}\right] = \frac{D}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \left\{f^{(e)}\right\} = \frac{QL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Unsur Segitiga Linear

$$N_i = [a_i + b_i x + c_i y]/(2A), \quad N_j = [a_j + b_j x + c_j y]/(2A)$$
$$N_k = [a_k + b_k x + c_k y]/(2A)$$

dengan

$$2A = \begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix}$$

dan

$$a_i = X_j Y_k - X_k Y_j, \quad b_i = Y_j - Y_k, \quad c_i = X_k - X_j$$
$$a_j = X_k Y_i - X_i Y_k, \quad b_j = Y_k - Y_i, \quad c_j = X_i - X_k$$
$$a_k = X_i Y_j - X_j Y_i, \quad b_k = Y_i - Y_j, \quad c_k = X_j - X_i$$

$$\left[k_D^{(e)}\right] = \frac{D_x}{4A}\begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{bmatrix} + \frac{D_y}{4A}\begin{bmatrix} c_i^2 & c_i c_j & c_i c_k \\ c_i c_j & c_j^2 & c_j c_k \\ c_i c_k & c_j c_k & c_k^2 \end{bmatrix}$$

$$\left[k_G^{(e)}\right] = \frac{GA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad \left\{f^{(e)}\right\} = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\left[k_M^{(e)}\right] = \frac{ML_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{dll.}$$

$$\int_A L_1^a \, L_2^b \, L_3^c \, dA = \frac{a!\,b!\,c!}{(a+b+c+2)!}\,2A$$

Unsur Segiempat Tepat Bilinear

$$N_i = \frac{1}{4}(1-\xi)(1-\eta), \quad N_j = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_k = \frac{1}{4}(1+\xi)(1+\eta), \quad N_m = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_i = \left(1 - \frac{s}{2b}\right)\left(1 - \frac{t}{2a}\right), \quad N_j = \frac{s}{2b}\left(1 - \frac{t}{2a}\right)$$

$$N_k = \frac{st}{4ab}, \quad N_m = \frac{t}{2a}\left(1 - \frac{s}{2b}\right)$$

$$\left[k_D^{(e)}\right] = \frac{D_x a}{6b}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{D_y b}{6a}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

$$\left[k_G^{(e)}\right] = \frac{GA}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}; \qquad \left\{f^{(e)}\right\} = \frac{QA}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\left[k_M^{(e)}\right] = \frac{ML_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{dll.}$$

Unsur Kuadratik 1-D

$$N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = -(\xi+1)(\xi-1), \quad N_3 = \frac{1}{2}\xi(\xi+1)$$

Unsur Segitiga Kuadratik 6-Nod

$$N_1 = L_1(2L_1-1), \quad N_2 = 4L_1L_2,$$

$$N_3 = L_2(2L_2-1), \quad N_4 = 4L_2(1-L_1-L_2)$$

$$N_5 = 1 - 3(L_1+L_2) + 2(L_1+L_2)^2, \quad N_6 = 4L_1(1-L_1-L_2)$$

Unsur Segiempat Kuadratik 8-Nod

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), \qquad N_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1), \qquad N_4 = \frac{1}{2}(1-\eta^2)(1+\xi)$$

$$N_5 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1), \qquad N_6 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_7 = -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1), \qquad N_8 = \frac{1}{2}(1-\eta^2)(1-\xi)$$

Kuadratur Gauss-Legendre

$n=1$	$\xi_i = 0.0$	$W_i = 2.0$
$n=2$	$\xi_i = \pm 0.577350$	$W_i = 1.0$
$n=3$	$\xi_i = 0.0$	$W_i = 8/9$
	$\xi_i = \pm 0.774597$	$W_i = 5/9$
$n=4$	$\xi_i = \pm 0.861136$	$W_i = 0.347855$
	$\xi_i = \pm 0.339981$	$W_i = 0.652145$

Untuk Domain Segitiga

n	Titik	L_1	L_2	W_i
2	a	1/3	1/3	1/2
3	a	1/2	0	1/6
	b	1/2	1/2	1/6
	c	0	1/2	1/6

Masalah Bersandarkan Masa

$$\left([C] + \theta\Delta t[K]\right)\{\Phi\}_b = \left([C] - (1-\theta)\Delta t[K]\right)\{\Phi\}_a + \Delta t\left((1-\theta)\{F\}_a + \theta\{F\}_b\right)$$

Perumusan Konsisten

$$\left[c^{(e)}\right] = \frac{\lambda L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \left[c^{(e)}\right] = \frac{\lambda A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\left[c^{(e)}\right] = \frac{\lambda A}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix}$$

$$\Delta t > \frac{\lambda L^2}{6D\theta}, \qquad \Delta t < \frac{\lambda L^2}{12D(1-\theta)}$$

Perumusan Tergumpal

$$\left[c^{(e)}\right] = \frac{\lambda L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \left[c^{(e)}\right] = \frac{\lambda A}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[c^{(e)}\right] = \frac{\lambda A}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta t < \frac{\lambda L^2}{4D(1-\theta)}$$

[MSG473] **LAMPIRAN**

Rumus-rumus bagi Teorem Giliran:

1. M/M/1 :

$$\rho = \lambda/\mu$$

$$P_n = (1 - \rho)\rho_n \qquad \text{untuk } n = 0, 1, 2, \ldots$$

$$L = \frac{\lambda}{\mu - \lambda}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$W = \frac{1}{\mu - \lambda} \quad , \quad W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$P[W > t] = e^{-t/W}$$

$$P[W_q > t] = \rho e^{-t/W}$$

2. M/M/s:

$$\rho = \frac{\lambda}{s\mu}$$

$$P_0 = \left[\frac{(\lambda/\mu)^s}{s!}\frac{1}{(1-\rho)} + \sum_{n=0}^{s-1}\frac{(\lambda/\mu)^n}{n!}\right]^{-1}$$

$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & , \text{ jika } 0 \le n \le s \\ \dfrac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0 & , \text{ jika } n \ge s \end{cases}$$

$$L_q = \frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} P_0$$

$$W_q = \frac{L_q}{\lambda} \quad , \quad W = W_q + 1/\mu$$

$$L = L_q + \lambda/\mu$$

$$P[W_q > t] = \frac{P_0 s\mu(\lambda/\mu)^s}{s!(s\mu - \lambda)} e^{-(s\mu-\lambda)t}$$

[MSG473]

3. M/M/s dengan saiz sumber input terhad sebanyak M.

$$P_0 = \left[\sum_{n=0}^{s-1} \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{M} \binom{M}{n} \frac{n!}{s^{n-2}\, s!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$P_n = \begin{cases} P_0 \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n & , \text{ jika } 0 \le n \le s \\ P_0 \binom{M}{n} \frac{n!}{s^{n-s}\, s!} \left(\frac{\lambda}{\mu}\right)^n & , \text{ jika } s \le n \le M \\ 0 & , \text{ jika } n > M \end{cases}$$

$$L = P_0 \left[\sum_{n=0}^{s-1} n\binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{m} n\binom{M}{n} \frac{n!}{s^{n-s}\, s!} \left(\frac{\lambda}{\mu}\right)^n \right]$$

$$L_q = L - s + P_0 \sum_{n=0}^{s-1} (s-n) \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^n$$

$$W = \frac{L}{\lambda(M-L)} \quad , \quad W_q = \frac{L_q}{\lambda(M-L)}$$

4. M/G/1:

$$P_0 = 1 - \rho$$

$$L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

$$L = \rho + L_q$$

$$W_q = \frac{L_q}{\lambda} \quad , \quad W = W_q + \frac{1}{\mu}$$

5. $M/E_k/1$:

$$L_q = \frac{1+k}{2k} \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$W_q = \frac{1+k}{2k} \frac{\lambda}{\mu(\mu-\lambda)}$$

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$$W = W_q + 1/\mu$$

$$L = \lambda W$$

6. Model M/M/1/k

$$P_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{k+1}} & (\rho \neq 1) \\ \dfrac{1}{k+1} & (\rho = 1) \end{cases}$$

Untuk $\rho \neq 1$

$$L = \frac{\rho\left[1-(k+1)\rho^k + k\rho^{k+1}\right]}{\left(1-\rho^{k+1}\right)(1-\rho)}$$

$$L_q = L - (1-P_0) = L - \frac{\rho\left(1-\rho^k\right)}{1-\rho^{k+1}}$$

$$W = L/\lambda' \quad , \quad \lambda' = \mu(L - L_q)$$

$$W_q = W - 1/\mu = L_q / \lambda'$$

Untuk $\rho = 1$

$$L = \frac{k}{2}$$

7. Model M/M/s/k :

$$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & (0 \leq n < s) \\ \dfrac{1}{s^{n-2}\, s!}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & (s \leq n \leq k) \end{cases}$$

$$P_0 = \begin{cases} \left[\displaystyle\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(\lambda/\mu)^s}{s!} \; \frac{1-\left(\frac{\lambda}{s\mu}\right)^{k-s+1}}{1-\frac{\lambda}{s\mu}}\right]^{-1} & \left(\dfrac{\lambda}{s\mu} \neq 1\right) \\ \left[\displaystyle\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(\lambda/\mu)^s}{s!} (k - s + 1)\right]^{-1} & \left(\dfrac{\lambda}{s\mu} = 1\right) \end{cases}$$

$$L_q = \frac{P_0 (s\rho)^s \rho}{s!(1-\rho)^2}\left[1 - \rho^{k-s+1} - (1-\rho)(k-s+1)\rho^{k-s}\right]$$

$$L = L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)(\rho s)^n}{n!}$$

$$W = \frac{L}{\lambda'} \quad , \quad \lambda' = \lambda(1 - P_k)$$

$$W_q = W - \frac{1}{\mu}$$

$$W_q = \frac{L_q}{\lambda'}$$

8. Model M/M/s/s :

$$P_n = \frac{(\lambda/\mu)^n / n!}{\sum_{i=0}^{s} \left(\frac{\lambda}{\mu}\right)^i / i!} \qquad (0 \le n \le s)$$

$$P_s = \frac{(sp)^s / s!}{\sum_{i=0}^{s} (s\rho)^i / i!} \qquad \left(\rho = \frac{\lambda}{s\mu}\right)$$

$$L = \frac{\lambda}{\mu}(1 - P_s) \quad , \quad W = \frac{L}{\lambda'} \text{ dengan } \lambda' = \lambda(1 - P_s)$$

9. Model M/M/∞ :

$$P_n = \frac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n!} \qquad (n \ge 0)$$

$$L = \lambda/\mu \quad W = \frac{1}{\mu}$$

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10. **Layanan Berkeadaan**

$$\mu_n = \begin{cases} \mu_1 & (1 \le n \le k) \\ \mu & (n \ge k) \end{cases}$$

$$P_0 = \left[\frac{1-\rho_1{}^k}{1-\rho_1} + \frac{\rho\rho_1{}^{k-1}}{1-\rho} \right]^{-1} \qquad (\rho_1 = \lambda/\mu_1,\ \rho = \lambda/\mu < 1)$$

$$L = P_0 \left[\frac{\rho_1\left[1+(k-1)\rho_1{}^k - k\rho_1{}^{k-1}\right]}{(1-\rho_1)^2} + \frac{\rho\rho_1{}^{k-1}[k-(k-1)\rho]}{(1-\rho)^2} \right]$$

$$L_q = L - (1 - P_0)$$

$$W = \frac{L}{\lambda} \qquad W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1-P_0}{\lambda}$$

$$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu_1}\right)^n P_0 & (0 \le n < k) \\ \dfrac{\lambda^n}{\mu_1{}^{k-1}\,\mu^{n-k+1}} P_0 & (n \ge k) \end{cases}$$

11. M/M/1 dengan saiz sumber input terhad sebanyak M.

$$P_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$P_n = \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu}\right)^n P_0 \quad \text{bagi } n = 1, 2, \ldots, M$$

$$L = M - \frac{\mu}{\lambda}[1-P_0]$$

$$L_q = M - \frac{\lambda + \mu}{\lambda}(1 - P_0)$$

$$W = \frac{L}{\lambda'}, \quad W_q = \frac{L_q}{\lambda'} \quad \text{dengan } \lambda' = \lambda(M-L)$$

TABLE 1.8 TWO-DIGIT RANDOM-NUMBER TABLE

03	26	48	92	38	96	41	04	35	84
71	44	81	46	44	47	07	20	58	04
33	75	06	41	87	72	63	88	59	54
53	71	27	13	37	45	89	61	30	26
41	15	43	91	46	81	57	39	34	86
16	18	75	11	26	80	93	97	29	33
88	50	00	56	70	19	90	00	93	95
13	10	08	15	29	33	75	70	43	05
15	72	73	69	27	75	72	95	99	56
64	10	99	02	18	26	78	69	19	12
98	66	53	86	34	71	09	88	56	08
43	05	06	19	91	78	03	65	08	16
69	82	02	61	98	50	74	84	60	41
06	40	10	24	68	42	39	97	25	55
34	86	83	41	33	83	85	92	32	29
46	05	92	36	82	04	67	05	18	69
28	73	59	56	43	88	61	17	07	48
35	53	49	39	98	14	16	76	69	10
90	90	18	27	75	08	75	17	55	68
62	32	97	16	33	66	02	34	62	26