Differential Subordination and Superordination on Schwarzian Derivatives

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ABSTRACT. Let q_1 and q_2 be convex univalent in the unit disk. In this paper, we give an application of differential subordination and superordination to obtain sufficient conditions on the Schwarzian derivatives for normalized analytic functions f to satisfy

$$q_1(z) \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2(z).$$

1. Introduction

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}[a, n]$ be the class of analytic functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let C_β denote the usual class of strongly convex functions of order β $(0 < \beta \leq 1)$ in Δ, $C := C_0$ and $\{f; z\}$ denote the Schwarzian derivative of f defined by

$$\{f; z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Owa and Obradović [1] proved that if $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f; z\}\right] > 0,$$

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then $f \in C$. For a given $0 < \beta < 1$, Darus [2] have determined α such that

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + 2z^2\{f;z\} \prec \left(\frac{1+z}{1-z}\right)^\alpha$$

implies $f \in C_\beta$. Here $\prec$ denotes subordination.

If f is subordinate to F, then F is superordinate to f [3]. Recently Miller and Mocanu[4] considered certain first and second order differential superordinations. Using the results of Miller and Mocanu[4], Bulboaca have considered certain classes of first order differential superordinations [5] as well as superordination-preserving integral operators [6]. We in [7] have used the results of Bulboaca [5] to obtain some sufficient conditions for normalized analytic functions f to satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$$

where q_1, q_2 are given univalent functions in Δ. In the present paper, we give an application of the theory of differential subordination and superordination to Schwarzian derivatives to obtain sufficient conditions for the functions f to satisfy

$$q_1(z) \prec 1+\frac{zf''(z)}{f'(z)} \prec q_2(z).$$

In our present investigation, we need the following:

Definition 1. [4, Definition 2, p. 817] *Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where*

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1.1. [7] *Let $q(z)$ be convex univalent in Δ and $\alpha, \beta, \gamma \in \mathbb{C}$. Further assume that*

$$\Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma}q(z)\right] \geq 0.$$

If $\psi(z) \in \mathcal{H}[q(0),1] \cap Q$, $\alpha\psi(z) + \beta\psi^2(z) + \gamma z\psi'(z)$ is univalent in Δ, then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \alpha\psi(z) + \beta\psi^2(z) + \gamma z\psi'(z)$$

implies $q(z) \prec \psi(z)$ and $q(z)$ is the best subordinant.

Lemma 1.2. [8] *Let $p(z)$ and $q(z)$ be analytic in Δ, $q(z)$ be convex univalent, α, β and γ be complex and $\gamma \neq 0$. Further assume that*

$$\Re\left\{\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma}q(z) + \left(1+\frac{zq''(z)}{q'(z)}\right)\right\} > 0.$$

If $p(z) = 1 + c_1 z + \cdots$ is analytic in Δ and satisfies

$$\alpha p(z) + \beta p^2(z) + \gamma z p'(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2. The main results

By making use of Lemma 1.1 and 1.2, we first prove the following:

THEOREM 2.1. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ. Further, assume that $\Re q_1(z) \leq 0$ and*

$$\Re\left(1+\frac{zq_2''(z)}{q_2'(z)}-q_2(z)\right)>0.$$

If $f\in\mathcal{A}$, $1+zf''(z)/f'(z)\in Q$, $z^2\{f;z\}$ is univalent in Δ and

$$1+2zq_1'(z)-q_1^2(z)\prec 2z^2\{f;z\}\prec 1+2zq_2'(z)-q_2^2(z), \tag{2.1}$$

then $q_1(z)\prec 1+zf''(z)/f'(z)\prec q_2(z)$. The functions $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

PROOF. Let the function $p(z)$ be defined by

$$p(z):=1+\frac{zf''(z)}{f'(z)}. \tag{2.2}$$

Then we have

$$2z^2\{f;z\}=1+2zp'(z)-p^2(z) \tag{2.3}$$

and (2.1) becomes

$$2zq_1'(z)-q_1^2(z)\prec 2zp'(z)-p^2(z)\prec 2zq_2'(z)-q_2^2(z).$$

The result now follows from Lemmas 1.1 and 1.2. We omit the details. □

Also we have the following:

THEOREM 2.2. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ. If $f\in\mathcal{A}$, $1+zf''(z)/f'(z)\in Q$, $\left(1+\frac{zf''(z)}{f'(z)}\right)^2+2z^2\{f;z\}$ is univalent in Δ and*

$$1+2zq_1'(z)\prec\left(1+\frac{zf''(z)}{f'(z)}\right)^2+2z^2\{f;z\}\prec 1+2zq_2'(z), \tag{2.4}$$

then $q_1(z)\prec 1+zf''(z)/f'(z)\prec q_2(z)$ and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

PROOF. Let the function $p(z)$ be defined by (2.2). In view of (2.2) and (2.3), the relation (2.4) becomes

$$zq_1'(z)\prec zp'(z)\prec zq_2'(z).$$

The result now follows from Lemmas 1.1 and 1.2. □

COROLLARY 2.3. *If $f\in\mathcal{A}$ satisfies*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^2+2z^2\{f;z\}\in\mathbb{C}\setminus[-\infty,0],$$

then $f\in C$.

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