DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract

Let \( q_1, q_2 \) be univalent in \( \Delta := \{ z : |z| < 1 \} \). We give some applications of first order differential superordinations to obtain sufficient conditions for normalized analytic functions \( f(z) \) to satisfy

\[ q_1(z) < z f'(z)/f(z) < q_2(z). \]

1. Introduction

Let \( \mathcal{H} \) be the class of analytic functions in \( \Delta := \{ z : |z| < 1 \} \) and \( \mathcal{H}(a, n) \) be the subclass of \( \mathcal{H} \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \). Let \( \mathcal{A} \) be the class of all analytic functions \( f(z) = z + a_2 z^2 + \ldots (z \in \Delta) \). Let \( p, h \in \mathcal{H} \) and let \( \psi(r, s, t; z) : \mathbb{C}^3 \to \mathbb{C} \). If \( p \) and \( \psi(p(z), z p'(z), z^2 p''(z); z) \) are univalent and if \( p \) satisfies the second order superordination

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then \( p \) is a solution of the differential superordination (1.1).

subordinate to \( F \), then \( F \) is superordinate to \( f \). An analytic function called a subordinant if \( q < p \) for all \( p \) satisfying (1.1). A univalent subordinant \( \tilde{q} \) that satisfies \( q < \tilde{q} \) for all subordinants \( q \) of (1.1) is called a best subordinant. Recently Miller and Mocanu [3] obtained conditions on \( h \), \( q \) and \( \phi \) for which the following implication holds:

\[
h(z) < \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) < p(z).
\]

Using the results of Miller and Mocanu [3], Bulboaca [2] have considered classes of first order differential superordinations as superordination-preserving integral operators [1]. In the present paper we give some applications of first order differential superordinations in \( A \).

In our present investigation, we shall need the following:

\textbf{Definition 1.1} [3, Definition 2, p. 817]. Denote by \( Q \), the set of functions \( f(z) \) that are analytic and injective on \( \Delta - \text{E}(f) \), where

\[
\text{E}(f) = \{ \zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty \},
\]

and are such that \( f'(\zeta) \neq 0 \) for all \( \zeta \in \partial \Delta - \text{E}(f) \).

\textbf{Lemma 1.2} [2]. Let \( q(z) \) be univalent in the unit disk \( \Delta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(\Delta) \). Suppose that

(1) \( \Re \{q'(\phi(q(z)))/q'(q(z))\} \geq 0 \) for \( z \in \Delta \),

(2) \( zq'(z)q'(q(z)) \) is starlike univalent in \( \Delta \).

If \( p(z) \in \mathcal{H}(q(0), 1) \cap Q \), with \( p(\Delta) \subseteq D \), and \( \phi(p(z)) + zp'(z)q(q(z)) \) univalent in \( \Delta \), then

\[
\mathcal{S}(q(z)) + zq'(z)q(q(z)) < \mathcal{S}(p(z)) + zp'(z)q(p(z))
\]

implies \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.
By making use of Lemma 1.2, we obtain the following results.

Lemma 2.1. Let $q(z)$ be convex univalent in $\Delta$ and $\alpha, \beta, \gamma \in \mathbb{C}$. Further assume that

$$\Re \left[ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) \right] \geq 0.$$ 

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $\alpha p(z) + \beta p^2(z) + \gamma z p'(z)$ is univalent in $\Delta$, then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) < \alpha p(z) + \beta p^2(z) + \gamma z p'(z)$$

implies $q(z) < p(z)$ and $q(z)$ is the best subordinant.

Proof. Define the functions $\Theta$ and $\Phi$ by

$$\Theta(w) := \alpha w + \beta w^2 \quad \text{and} \quad \Phi(w) := \gamma.$$ 

Clearly, $\Theta(w)$ and $\Phi(w)$ are analytic in $\mathbb{C}$. Also

$$\Re \left[ \frac{\Theta(q(z))}{\Phi(q(z))} \right] = \Re \left[ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) \right] \geq 0$$

and the function $\gamma z q'(z)$ is starlike univalent in $\Delta$. Lemma 2.1 now follows by an application of Lemma 1.2.

Remark 1. When $\alpha = 1$ and $\beta = 0$, Lemma 2.1 reduces to [3, Theorem 8, p. 822]. When $\alpha = \beta = 0$ and $\gamma = 1$ Lemma 2.1 reduces to [3, Theorem 9, p. 823].

By making use of Lemma 2.1, we now prove the following:

Theorem 2.2. Let $\alpha \in \mathbb{C}$. Let $q(z)$ be convex univalent in $\Delta$ and $\Re q(z)$

$$\geq \Re \frac{\alpha - 1}{2\alpha}.$$ 

If $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$ is univalent in $\Delta$, then

$$(1 - \alpha) q(z) + \alpha q^2(z) + \alpha z q'(z) - \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$$
implies
\[ q(z) < \frac{zf'(z)}{f(z)} \]
and \( q(z) \) is the best subordinate.

**Proof.** Define the function \( p(z) \) by
\[ p(z) := \frac{zf'(z)}{f(z)}. \]
Then a computation shows that
\[ \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} = (1 - \alpha)p(z) + \alpha p^2(z) + \alpha zp'(z). \]
By using Lemma 2.1, we have the result.

Together with the corresponding result for differential subordinants (see Ravichandran [4]), we obtain the following "sandwich result".

**Corollary 2.3.** Let \( q_1(z) \) and \( q_2(z) \) be convex univalent in \( \alpha \in \mathbb{C} \). Assume that \( \Re q_i(z) \geq \Re \frac{\alpha - 1}{2\alpha} \) for \( i = 1, 2 \). If \( f \in A \), \( zf' \in \mathcal{H}(1, 1) \cap Q \), \( \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} \) is univalent in \( \Delta \), then
\[ (1 - \alpha)q_1(z) + \alpha q_1^2(z) + \alpha zq_1'(z) < \frac{zf'(z)}{f(z)} + \alpha \frac{z^2f''(z)}{f(z)} \]
implies
\[ q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z) \]
and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinate and dominants.

**Lemma 2.4.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( \alpha, \beta \). Assume that \( \Re(\alpha \beta q(z)) \geq 0 \) and \( zq'(z)/q(z) \) is starlike univalent in \( \mathcal{H}(q(0), 1) \cap Q \). \( p(z) = 0, \) \( \alpha p(z) + \beta \frac{zp'(z)}{p(z)} \) is univalent in \( \Delta \).
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\[ \alpha q(z) + \beta \frac{zq'(z)}{q(z)} < \alpha p(z) + \beta \frac{zp'(z)}{p(z)} \]

implies \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

**Proof.** The Lemma 2.4 follows from Lemma 1.2 when the functions \( \vartheta \) and \( \varphi \) are given by \( \vartheta(w) := \alpha w \) and \( \varphi(w) := \beta/w \).

By making use of Lemma 2.4, we now prove the following:

**Theorem 2.5.** Let \( \alpha \in \mathbb{C} \). Let \( q(z) \neq 0 \) be univalent in \( \Delta \). Further assume that \( \Re[\alpha q(z)] \geq 0 \) and \( zq''(z)/q(z) \) is starlike univalent in \( \Delta \). If \( f \in A, 0 \neq zf'(z)/f(z) \in \mathcal{H}(1,1) \cap Q, (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) is univalent in \( \Delta \), then

\[ q(z) + \alpha \frac{zq'(z)}{q(z)} < (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \]

implies

\[ q(z) < \frac{zf'(z)}{f(z)} \]

and \( q(z) \) is the best subordinant.

**Proof.** Theorem 2.5 follows from Lemma 2.4 by taking \( p(z) \) to be the function given by \( p(z) := zf'(z)/f(z) \).

Together with the corresponding result for differential subordination (see Ravichandran and Darus [6]), we obtain the following:

**Corollary 2.6.** Let \( \alpha \in \mathbb{C} \). Let \( q_i(z) \neq 0 \) (\( i = 1, 2 \)) be univalent in \( \Delta \). Further assume that \( \Re[\alpha q_i(z)] \geq 0 \) for \( i = 1, 2 \) and \( zq_i''(z)/q_i(z) \) \( (i = 1, 2) \) is starlike univalent in \( \Delta \). If \( f \in A, 0 \neq zf'(z)/f(z) \in \mathcal{H}(1,1) \cap Q, (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) is univalent in \( \Delta \), then

\[ q_1(z) + \alpha \frac{zq'_1(z)}{q_1(z)} < (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < q_2(z) + \alpha \frac{zq'_2(z)}{q_2(z)} \]
implies
\[ q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z) \]
and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and dominant.

By making use of Lemma 2.4, we obtain the following:

**Theorem 2.7.** Let \( q(z) \neq 0 \) be univalent in \( \Delta \) and \( zq'(z)/q(z) \) be univalent in \( \Delta \). If \( f \in A, \ 0 \neq z^2 f'(z)/f^2(z) \in H(1,1) \cap Q, \), then
\[ \frac{zq(z)}{q(z)} < \frac{(zf')''(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} \]
implies
\[ q(z) < \frac{z^2f'(z)}{f^2(z)} \]
and \( q(z) \) is the best subordinant.

Together with the corresponding result for differential subordinant (see Ravichandran [4]), we obtain the following:

**Corollary 2.8.** Let \( q_i(z) \neq 0 \) be univalent in \( \Delta \) and \( zq_i'(z)/q_i(z) \) be starlike univalent in \( \Delta \) for \( i = 1, 2 \). If \( f \in A, \ 0 \neq z^2 f'(z)/f^2(z) \in H(1,1) \cap Q, \), then
\[ \frac{zq_i(z)}{q_i(z)} < \frac{(zf')''(z)}{f'(z)} - 2 \frac{zf'(z)}{f(z)} < \frac{zq_2(z)}{q_2(z)} \]
implies
\[ q_1(z) < \frac{z^2f'(z)}{f^2(z)} < q_2(z) \]
and \(q_1(z)\) and \(q_2(z)\) are respectively the best subordinant and b.

dominant.

**Lemma 2.9.** Let \(q(z) \neq 0\) be univalent in \(\Delta\) and \(zq'(z)/q^2(z)\) be starli
univalent in \(\Delta\). If \(p(z) \in \mathcal{T}(q(0), 1) \cap Q, p(z) \neq 0, zp'(z)/p^2(z)\) is univa
in \(\Delta\), then

\[
\frac{zq'(z)}{q^2(z)} \prec \frac{zp'(z)}{p^2(z)}
\]

implies \(q(z) \prec p(z)\) and \(q(z)\) is the best subordinant.

**Proof.** Lemma 2.9 follows from Lemma 1.2 when \(\theta(w) := 0\) a
\[q(w) := 1/w^2.\]

**Theorem 2.10.** Let \(q(z) \neq 0\) be univalent in \(\Delta\) and \(zq'(z)/q^2(z)\)
starlike univalent in \(\Delta\). If \(f \in \mathcal{A}, 0 \neq zf'(z)/f(z) \in \mathcal{T}(1, 1) \cap Q, 1 + zf''(z)/f'(z)
is univalent in \(\Delta\), then

\[
1 + \frac{zq'(z)}{q^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}
\]

implies \(q(z) \prec zf'(z)/f(z)\) and \(q(z)\) is the best subordinant.

**Proof.** The result follows from Lemma 2.9 by taking \(p(z) = zf'(z)/f(z)\).

Together with the corresponding result for differential subordinati
(see Ravichandran and Darus [5]), we obtain the following:

**Theorem 2.11.** Let \(q_i(z) \neq 0\) be univalent in \(\Delta\) and \(zq_i'(z)/q_i^2(z)\)
starlike univalent in \(\Delta\) for \(i = 1, 2\). If \(f \in \mathcal{A}, 0 \neq zf'(z)/f(z) \in \mathcal{T}(1, 1) \cap Q, 1 + zf''(z)/f'(z)\) is univalent in \(\Delta\), then

\[
1 + \frac{zq_1'(z)}{q_1^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq_2'(z)}{q_2^2(z)}
\]

implies \(q_1(z) \prec zf'(z)/f(z) \prec q_2(z)\) and \(q_1(z)\) and \(q_2(z)\) are respec
the best subordinant and the best dominant.
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References


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