UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2004/2005

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MST 561 – STATISTICAL INFERENCE
[PENTAABIRAN STATISTIK]

Duration : 3 hours

[Masa : 3 jam]

Please check that this examination paper consists of **NINE [9]** pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi ***SEMBILAN [9]*** *muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Answer **all FIVE** questions.

Jawab **semua LIMA** soalan.

1. (a) Let $X_1, X_2, ..., X_n$ be independent, identically distributed random variables with common probability density function.

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Let $Y_1 = X_{(1)}, Y_2 = X_{(2)}, ..., Y_n = X_{(n)}$ be the order statistics for $X_1, X_2, ..., X_n$.

Let $Z_1 = nY_1, Z_2 = (n-1)(Y_2 - Y_1), ..., Z_n = (Y_n - Y_{n-1})$.

Show that $Z_1, Z_2, ..., Z_n$ are independent and identically distributed random variables.

[50 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a Poisson distribution with parameter $\lambda = 1$, i.e.

$$P(X_i = x) = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, ..., \quad i = 1, 2, ..., n$$

(i) Derive the m.g.f. for X_i.

(ii) Let $\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}$. Consider $Y_n = \sqrt{n}(\overline{X}_n - 1)$. Show that the m.g.f for Y_n is given by exp $\left[-t\sqrt{n} + n\,(e^{t/\sqrt{n}} - 1)\right]$

(iii) Hence, obtain the limiting distribution of Y_n when $n \to \infty$.

[50 marks]

1. (a) Katakan $X_1, X_2, ..., X_n$ adalah pembolehubah-pembolehubah rawak takbersandar dan tertabur secaman dengan fungsi ketumpatan kebarangkalian sepunya

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Katakan $Y_1 = X_{(1)}, Y_2 = X_{(2)}, ..., Y_n = X_{(n)}$ adalah statistik tertib bagi $X_1, X_2, ..., X_n$.

Katakan $Z_1 = nY_1, Z_2 = (n-1)(Y_2 - Y_1), ..., Z_n = (Y_n - Y_{n-1})$.

Tunjukkan bahawa $Z_1, Z_2, ..., Z_n$ adalah pembolehubah-pembolehubah rawak takbersandar dan tertabur secaman.

[50 markah]

(b) Katakan $X_1, X_2, ..., X_n$ satu sampel rawak saiz n daripada suatu taburan Poisson dengan parameter $\lambda = 1$, iaitu,

$$P(X_i = x) = \frac{e^{-1}}{x!}, \quad x = 0, 1, 2, ..., \quad i = 1, 2, ..., n$$

(i) *Terbitkan f.p.m bagi* X_i.

(ii) *Biarkan* $\overline{X}_n = \dfrac{X_1 + X_2 + ... + X_n}{n}$

Pertimbangkan $Y_n = \sqrt{n}(\overline{X}_n - 1)$. *Tunjukkan f.p.m bagi* Y_n *diberi oleh exp* $\left[-t\sqrt{n} + n(e^{t/\sqrt{n}} - 1)\right]$.

(iii) *Demikian, dapatkan taburan penghad bagi* Y_n *apabila* $n \to \infty$.

[50 markah]

2. (a) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics from a random sample of size 4 taken from a distribution with p.d.f.

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

(i) Obtain the joint p.d.f. of Y_3 and Y_4.

(ii) Find the conditional p.d.f. of Y_3 given $Y_4 = y_4$

(iii) Evaluate $E(Y_3 \mid y_4)$.

[30 marks]

(b) State and prove Chebychev's inequality.

[30 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}\exp(-e^{-(x-\theta)}), x \geq \theta, -\infty < \theta < \infty \\ 0 \qquad\qquad\qquad , x < \theta \end{cases}$$

(i) Find the MLE for θ.

(ii) Find the MLE for $P_\theta\{X_1 - \theta \geq 1\}$.

[40 marks]

2. (a) *Katakan* $Y_1 < Y_2 < Y_3 < Y_4$ *statistik tertib daripada suatu sampel rawak saiz* $n = 4$ *yang diambil daripada suatu taburan dengan f.k.k.*

$$f(x,\theta) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{di tempat lain} \end{cases}.$$

(i) *Dapatkan f.k.k. tercantum bagi* Y_3 *dan* Y_4.

(ii) *Cari f.k.k. bersyarat bagi* Y_3 *diberi* $Y_4 = y_4$.

(iii) *Nilaikan* $E(Y_3 \mid y_4)$.

[30 markah]

(b) *Nyatakan dan buktikan ketaksamaan Chebychev.*

[30 markah]

(c) *Katakan* $X_1, X_2, ..., X_n$ *adalah suatu sampel rawak dari fungsi ketumpatan kebarangkalian*

$$f(x,\theta) = \begin{cases} e^{-(x-\theta)}\exp(-e^{-(x-\theta)}), x \geq \theta, -\infty < \theta < \infty \\ 0 \qquad , x < \theta \end{cases}$$

(i) *Carikan MLE* θ.

(ii) *Carikan MLE bagi* $P_\theta\{X_1 - \theta \geq 1\}$.

[40 markah]

3. (a) What are the roles of the following theorems in estimation theory:

(i) Neyman – Fisher Factorization Theorem

(ii) Rao – Blackwell Theorem.

(iii) Lehmann – Scheffe Theorem.

(iv) Cramer – Rao Theorem?

Give examples to illustrate your answer.

[40 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0 \\ 0 , & \text{elsewhere} \end{cases}.$$

(i) Show that $T = \sum_{i=1}^{n} X_i$ is a statistic which is sufficient and complete for λ.

(ii) Let $U(X_1) = \begin{cases} 0 \text{ if } X_1 < K, \\ 1 \text{ if } X_1 \geq K, \end{cases}$

where K is a known constant. Show than $U(X_1)$ is unbiased for $e^{-K\lambda}$.

(iii) Hence, show that the uniformly minimum variance unbiased estimator for $e^{-K\lambda}$ is given by

$$g(T) = \begin{cases} \left(\dfrac{T-K}{T}\right)^{n-1} & \text{when } T \geq K \\ 0 & \text{when } T < K \end{cases}$$

[60 marks]

3. (a) *Apakah peranan teorem-teorem berikut di dalam teori penganggaran:*

(i) *Teorem Pemfaktoran Neyman-Fisher,*

(ii) *Teorem Rao-Blackwell,*

(iii) *Teorem Lehmann-Scheffe,*

(iv) *Teorem Cramer-Rao?*

Beri contoh-contoh untuk mengilustrasi jawapan anda.

[40 markah]

(b) *Katakan* $X_1, X_2, ..., X_n$ *suatu sampel rawak daripada suatu taburan dengan f.k.k.*

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \lambda > 0 \\ 0, & \text{di tempat lain} \end{cases}$$

(i) *Tunjukkan* $T = \sum_{i=1}^{n} X_i$ *ialah suatu statistik yang cukup dan lengkap bagi* λ.

(ii) *Katakan* $U(X_1) = \begin{cases} 0 \text{ jika } X_1 < K, \\ 1 \text{ jika } X_1 \geq K, \end{cases}$

di mana K ialah suatu pemalar yang diketahui. Tunjukkan $U(X_1)$ *saksama bagi* $e^{-K\lambda}$.

(iii) *Demikian, tunjukkan bahawa penganggar saksama bervarians minimum secara seragam bagi* $e^{-K\lambda}$ *diberi oleh*

$$g(T) = \begin{cases} \left(\dfrac{T-K}{T}\right)^{n-1} & \text{apabila } T \geq K \\ 0 & \text{apabila } T < K \end{cases}$$

[60 markah]

4. (a) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with p.d.f.

$$f(x|\ \theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \theta > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Suppose than the prior distribution for Θ is given by the p.d.f.

$$\pi(\theta) = \begin{cases} \dfrac{\lambda^r \theta^{r-1} e^{-\theta\lambda}}{\Gamma(r)}, & \theta > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where r and λ are known positive constarts.

(i) Find the posterior p.d.f. for Θ

(ii) Find the Bayes estimator for θ with respect to the prior $\pi(\theta)$ above using the squared error loss function

[40 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample of size from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \theta(1+x)^{-(1+\theta)}, & 0 < x < \infty, \theta > 0 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find the m.l.e. for (a) θ and (b) $\dfrac{1}{\theta}$.

(ii) Does there exist a complete and sufficient statistic for θ?

(iii) Find a uniformly minimum variance unbiased estimator for (a) $\frac{1}{\theta}$ dan (b) θ.

(iv) Find the Cramer-Rao lower bound for the variances of unbiased estimators of $\frac{1}{\theta}$.

[60 marks]

4. (a) *Katakan $X_1, X_2, \ldots, X_n$ suatu sampel rawak daripada suatu taburan dengan f.k.k.*

$$f(x|\ \theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \theta > 0 \\ 0 & , \text{ di tempat lain} \end{cases}$$

Andaikan bahawa taburan priori bagi Θ diberi oleh f.k.k.

$$\pi(\theta) = \begin{cases} \dfrac{\lambda^r \theta^{r-1} e^{-\theta\lambda}}{\Gamma(r)}, & \theta > 0 \\ 0, & \text{di tempat lain} \end{cases}$$

di mana r and λ adalah pemalar positif yang diketahui.

(i) *Cari f.k.k. posterior bagi Θ*

(ii) *Cari penganggar Bayes bagi θ terhadap prior $\pi(\theta)$ di atas dengan menggunakan fungsi kerugian ralat kuasa dua.*

[40 markah]

(b) *Katakan $X_1, X_2, \ldots, X_n$ suatu sampel rawak saiz n daripada suatu taburan dengan f.k.k.*

$$f(x;\theta) = \begin{cases} \theta(1+x)^{-(1+\theta)}, 0 < x < \infty, \theta > 0 \\ 0 \quad , \text{ di tempat lain} \end{cases}$$

(i) *Cari p.k.m bagi (a) θ dan (b) $\frac{1}{\theta}$.*

(ii) *Adakah wujudnya suatu statistik yang cukup dan lengkap bagi θ?*

(iii) *Cari suatu penganggar saksama bervarians minimum secara seragam bagi (a) $\frac{1}{\theta}$ dan (b) θ.*

(iv) *Cari batas bawah Cramer-Rao bagi varians penganggar saksama untuk $\frac{1}{\theta}$.*

[60 markah]

5. (a) State the Neyman-Pearson fundamental lemma.

[10 marks]

(b) Describe the concepts which are relevant in the theory of hypothesis testing.

[20 marks]

(c) Let X_1, X_2 be a random sample of size 2 from a distribution with p.d.f.

$$f(x;\theta)=\begin{cases}(1+\theta)x^{\theta} & ,0\le x\le 1,\theta>0\\ 0 & , \text{ elsewhere}\end{cases}$$

Show that the uniformly most powerful test for testing $H_0:\theta=2$ versus $H_i:\theta=\theta_1<2$ is given by

$$\phi(x_1,x_2)=\begin{cases}1 & \text{if } x_1x_2\le\frac{3}{4}\\ 0 & \text{if } x_1x_2>\frac{3}{4}\end{cases}.$$

What is the size of the above test?

[30 marks]

(d) Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with p.d.f. $N(\mu,\sigma^2)$, $-\infty<\mu<\infty$, $0<\sigma^2<\infty$.

Let $Y_1, Y_2, ..., Y_m$ be a random sample from a distribution with p.d.f. $N(\eta,\tau^2)$, $-\infty<\eta<\infty$, $0<\tau^2<\infty$.

Suppose that the two samples are independent. Show that the likelihood ratio test for testing $H_0:\sigma^2=\tau^2, \mu$ and η unknown, versus $H_0:\sigma^2\ne\tau^2, \mu$ and η unknown, can be based on the statistic

$$F=\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2/(n-1)}{\sum_{j=1}^{m}(Y_j-\bar{Y})^2/(m-1)},$$

where $\bar{X}=\sum_{i=1}^{n}X_i\Big/n$ and $\bar{Y}=\sum_{j=1}^{m}Y_j\Big/m$.

[40 marks]

5. (a) *Nyatakan lema asasi Neyman-Pearson.*

[10 markah]

(b) *Huraikan konsep-konsep yang berkaitan di dalam teori pengujian hipotesis.*

[20 markah]

(c) *Katakan X_1, X_2 sampel rawak saiz 2 daripada suatu taburan dengan f.k.k.*

$$f(x;\theta)=\begin{cases}(1+\theta)x^{\theta} & ,0\le x\le 1,\theta>0\\ 0 & , \text{ di tempat lain}\end{cases}$$

Tunjukkan bahawa ujian yang paling berkuasa secara seragam untuk menguji $H_0 : \theta = 2$ *lawan* $H_i : \theta = \theta_1 < 2$ *diberi oleh*

$$\phi(x_1, x_2) = \begin{cases} 1 & \text{jika} \quad x_1 x_2 \le \dfrac{3}{4} \\ 0 & \text{jika} \quad x_1 x_2 > \dfrac{3}{4} \end{cases}.$$

Apakah saiz ujian di atas?

[30 markah]

(d) *Katakan* $X_1, X_2, \ldots, X_n$ *sampel rawak daripada taburan dengan f.k.k..* $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$.
Katakan $Y_1, Y_2, \ldots, Y_m$ *sampel rawak daripada taburan dengan f.k.k.* $N(\eta, \tau^2)$, $-\infty < \eta < \infty$, $0 < \tau^2 < \infty$.
Andaikan dua sampel tersebut tak bersandar. Tunjukkan bahawa ujian nisbah kebolehjadian untuk menguji $H_0 : \sigma^2 = \tau^2, \mu$ dan η *anu, lawan* $H_0 : \sigma^2 \ne \tau^2, \mu$ dan η *anu, boleh didasarkan pada statistik.*

$$F = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2 / (n-1)}{\sum_{j=1}^{m} (Y_j - \overline{Y})^2 / (m-1)},$$

di mana $\overline{X} = \sum_{i=1}^{n} X_i \Big/ n$ dan $\overline{Y} = \sum_{j=1}^{m} Y_j \Big/ m$.

[40 markah]

Taburan	*Fungsi Ketumpatan* f	*Min* $\mu = E[X]$	*Varians* $\sigma^2 = E[(X-\mu)^2]$	*Fungsi penjana Momen*
Seragam Diskrit	$f(x) = \frac{1}{N} I_{\{1,2,\ldots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\ldots,n\}}(x)$	np	npq	$(q + pe^t)^n$
Geometri	$f(x) = p q^x I_{\{0,1,\ldots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$	λ	λ	$\exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-(x-\mu)^2/2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$, $t < \lambda$
Gamma	$f(x) = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda x} x^{n-1} I_{(0,\infty)}(x)$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$, $t < \lambda$
Khi kuasa dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}$, $t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	—

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