UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MST 561 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TWELVE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi DUA BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all five** [5] questions.

*[**Arahan** :* *Jawab **semua lima** [5] soalan.]*

1. (a) Three employees, A, B and C in a firm have been identified. A manager and an assistant manager will be appointed among these three employees.
   (i) What is the set of all possible outcomes, Ω?
   (ii) What is the σ-field, S, of this experiment based on (i)?

[30 marks]

(b) A point is chosen randomly from the interval (0,1) and let the random variable X_1 represent the number which corresponds to that point. Then choose a point at random from the interval $(0, x_1)$, where x_1 is the experimental value of X_1; and let the random variable X_2 represent the number which corresponds to this point.
   (i) Find the marginal probability density function of X_1.
   (ii) Find the conditional probability density function of X_2, given $X_1 = x_1$.
   (iii) Compute $P(X_1 + X_2 \geq 1)$.

[40 marks]

(c) Let $p(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$, zero elsewhere, be the probability mass function of the random variable X.
   (i) Find the moment generating function of X.
   (ii) Find the mean and variance of X using (i).

[30 marks]

1. (a) *Tiga kakitangan A, B dan C dalam suatu firma telah dikenalpasti. Seorang pengurus dan seorang penolong pengurus akan dilantik di kalangan tiga kakitangan ini.*

   (i) *Apakah set semua kesudahan yang mungkin,* Ω?

   (ii) *Apakah medan-*σ, *S, untuk eksperimen ini berdasarkan (i)?*

[30 markah]

(b) *Satu titik dipilih secara rawak dari selang (0,1) dan biarkan pembolehubah rawak* X_1 *mewakili nombor yang sepadan dengan titik itu. Kemudian pilih satu titik secara rawak dari selang* $(0, x_1)$, *yang mana* x_1 *ialah nilai ujikaji bagi* X_1; *dan biarkan pembolehubah rawak* X_2 *mewakili nombor yang sepadan dengan titik ini.*

   (i) *Cari fungsi ketumpatan kebarangkalian sut bagi* X_1.

   (ii) *Cari fungsi ketumpatan kebarangkalian bersyarat* X_2, *diberi* $X_1 = x_1$.

   (iii) *Kira* $P(X_1 + X_2 \geq 1)$.

[40 markah]

(c) *Biarkan* $p(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$, *sifar di tempat lain, sebagai fungsi jisim kebarangkalian bagi pembolehubah rawak X.*

   (i) *Cari fungsi penjana momen bagi X.*

   (ii) *Cari min dan varians bagi X dengan menggunakan (i).*

[30 markah]

2. (a) Consider the random variable X with probability mass function

$$P(X=x)=p_x=\frac{\binom{n}{x}}{2^n},\ x=0,\ 1,2,\ \ldots,\ n.$$

Find the probability generating function of X.

[Hint : $(1+t)^n=\sum_{j=0}^{n}\binom{n}{j}t^j$]

[30 marks]

(b) Let $T=\frac{W}{\sqrt{V/r}}$, where W and V are respectively, the standard normal random variable and the chi-square random variable with r degrees of freedom. By using the necessary theorems, show that T^2 has an F distribution. What is the distribution of $\frac{1}{T^2}$?

[30 marks]

(c) Let X_1 and X_2 have the joint probability mass function $p(x_1,x_2)=\frac{x_1x_2}{36}$, $x_1=1$, 2, 3 and $x_2=1, 2, 3$, zero elsewhere.

(i) Find the joint probability mass function of $Y_1=X_1X_2$ and $Y_2=X_2$.

(ii) Find the marginal probability mass function of Y_1.

[40 marks]

2. (a) *Pertimbangkan pembolehubah rawak X dengan fungsi jisim kebarangkalian*

$$P(X = x) = p_x = \frac{\binom{n}{x}}{2^n}, x = 0, 1, 2, \ldots, n.$$

Cari fungsi penjana kebarangkalian bagi X.

[Petua : $(1+t)^n = \sum_{j=0}^{n} \binom{n}{j} t^j$ *]*

[30 markah]

(b) *Biarkan* $T = \frac{W}{\sqrt{V/r}}$, *yang mana W dan V adalah masing-masing pembolehubah rawak normal piawai dan pembolehubah rawak khi-kuasadua dengan r darjah kebebasan. Dengan menggunakan teorem tertentu, tunjukkan bahawa* T^2 *mempunyai taburan F. Apakah taburan untuk* $\frac{1}{T^2}$*?*

[30 markah]

(c) *Biarkan* X_1 *dan* X_2 *mempunyai fungsi jisim kebarangkalian tercantum* $p(x_1, x_2) = \frac{x_1 x_2}{36}$, $x_1 = 1, 2, 3$ *dan* $x_2 = 1, 2, 3$, *sifar di tempat lain.*

(i) *Cari fungsi jisim kebarangkalian tercantum bagi* $Y_1 = X_1 X_2$ *dan* $Y_2 = X_2$.

(ii) *Cari fungsi jisim kebarangkalian sut bagi* Y_1.

[40 markah]

3. (a) Let Y_1 denote the minimum statistic of a random sample of size n from a distribution that has probability density function $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Find the limiting distribution of Z_n.

[30 marks]

(b) Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli, $b(1,\theta)$ distribution, where $0 < \theta < 1$. Then $Y = \sum_{i=1}^{n} X_i$ follows a binomial, Bin(n,θ) distribution with probability mass function

$$f(y \mid \theta) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, & y = 0,1,2,...,n \\ 0 & , \text{elsewhere} \end{cases}.$$

Assume that the prior probability density function of the random variable Θ is given by

$$h(\theta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, \text{ elsewhere} \end{cases},$$

where α and β are known positive constants.

(i) Find the posterior probability density function for Θ.

(ii) Find the Bayes' solution, $w(y)$ for θ with respect to the prior probability density function, $h(\theta)$ using the loss function, $L[\theta, w(y)] = [\theta - w(y)]^2$.

[40 marks]

(c) Let $X_1, X_2, ..., X_n$ be a random sample from a beta distribution with the following probability density function:

$$f(x_i;\theta) = \frac{\Gamma(2\theta)}{[\Gamma(\theta)]^2}[x_i(1-x_i)]^{\theta-1}, \quad 0 < x_i < 1, \ \theta > 0$$

Find a sufficient statistic for θ.

[30 marks]

3. (a) *Biarkan* Y_1 *menandai statistik minimum untuk suatu sampel rawak dengan saiz n daripada taburan yang mempunyai fungsi ketumpatan kebarangkalian* $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, *sifar di tempat lain. Biarkan* $Z_n = n(Y_1 - \theta)$. *Cari taburan penghad bagi* Z_n.

[30 markah]

(b) *Biarkan* $X_1, X_2, \ldots, X_n$ *mewakili sampel rawak daripada taburan Bernoulli,* $b(1, \theta)$, *yang mana* $0 < \theta < 1$. *Maka,* $Y = \sum_{i=1}^{n} X_i$ *mengikuti taburan binomial,* $Bin(n, \theta)$ *dengan fungsi jisim kebarangkalian*

$$f(y \mid \theta) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, & y = 0, 1, 2, \ldots, n \\ 0, & \text{di tempat lain} \end{cases}.$$

Andaikan bahawa fungsi ketumpatan kebarangkalian priori bagi pembolehubah rawak Θ *diberi oleh*

$$h(\theta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, & 0 < \theta < 1 \\ 0, & \text{di tempat lain} \end{cases},$$

yang mana α *dan* β *merupakan pemalar positif dengan nilai yang diketahui.*

(i) *Cari fungsi ketumpatan kebarangkalian posterior bagi* Θ.

(ii) *Cari penyelesaian Bayes,* $w(y)$ *bagi* θ *terhadap fungsi ketumpatan kebarangkalian priori,* $h(\theta)$ *dengan menggunakan fungsi kerugian* $L[\theta, w(y)] = [\theta - w(y)]^2$.

[40 markah]

(c) *Biarkan* $X_1, X_2, \ldots, X_n$ *sebagai suatu sampel rawak daripada taburan beta dengan fungsi ketumpatan kebarangkalian berikut:*

$$f(x_i; \theta) = \frac{\Gamma(2\theta)}{[\Gamma(\theta)]^2} [x_i(1-x_i)]^{\theta-1}, \quad 0 < x_i < 1, \ \theta > 0$$

Cari suatu statistik cukup bagi θ.

[30 markah]

4. (a) Let $X_1, X_2, ..., X_n$ denote a random sample of size $n > 2$ from the $N(0,\theta)$ distribution.

(i) Find the maximum likelihood estimator of θ.

(ii) Find the Cramer–Rao lower bound for the variance of unbiased estimators of θ.

(iii) Is the maximum likelihood estimator of θ in (i) an efficient estimator of θ?

[40 marks]

(b) Assume that $X_1, X_2, ..., X_n$ is a random sample having an exponential probability density function, $f(x;\lambda) = \lambda e^{-\lambda x}$, $x > 0$. Show that $\hat{\lambda} = Y_1$ is not consistent for λ.

[20 marks]

(c) Assume that X is a single observation from a distribution with density function

$$f(x;\alpha) = \alpha e^{-\alpha x}, \ 0 < x < \infty, \ \alpha > 0.$$

(i) Is $2\alpha X$ a pivotal quantity? Explain.

(ii) Find the confidence coefficient for the random interval $(X, 3X)$ if this interval is a confidence interval for α.

(iii) What is the mathematical expectation of the length of the random interval in (ii)?

[40 marks]

4. (a) *Biarkan* $X_1, X_2, ..., X_n$ *menandai suatu sampel rawak dengan saiz* $n > 2$ *daripada taburan* $N(0, \theta)$.

(i) *Cari penganggar kebolehjadian maksimum bagi* θ.

(ii) *Cari batas bawah Cramer-Rao untuk varians penganggar-penganggar saksama* θ.

(iii) *Adakah penganggar kebolehjadian maksimum bagi* θ *dalam (i) suatu penganggar cekap bagi* θ?

[40 markah]

(b) *Andaikan bahawa* $X_1, X_2, ..., X_n$ *suatu sampel rawak dengan fungsi ketumpatan kebarangkalian eksponen,* $f(x;\lambda) = \lambda e^{-\lambda x}$, $x > 0$. *Tunjukkan bahawa* $\hat{\lambda} = Y_1$ *tidak konsisten untuk* λ.

[20 markah]

(c) *Andaikan bahawa X ialah suatu cerapan tunggal daripada taburan dengan fungsi ketumpatan*

$$f(x;\alpha) = \alpha e^{-\alpha x}, \ 0 < x < \infty, \ \alpha > 0.$$

(i) *Adakah* $2\alpha X$ *suatu kuantiti pangsian? Jelaskan.*

(ii) *Cari pekali keyakinan bagi selang rawak (X, 3X) jika selang ini ialah selang keyakinan bagi* α.

(iii) *Apakah jangkaan matematik bagi panjang selang rawak dalam (ii)?*

[40 markah]

5. (a) Let $X_1, X_2, \ldots, X_{10}$ denote a random sample of size 10 from a Poisson distribution with mean θ. Show that the critical region C defined by $\sum_{i=1}^{10} x_i \geq c'$ is a best critical region for testing $H_0 : \theta = 0.1$ versus $H_1 : \theta = 0.5$. If $c' = 3$, determine for this test, the size of the Type-I error, α and the power at $\theta = 0.5$.

[40 marks]

(b) Let $X_1, X_2, \ldots, X_n$ denote a random sample having the probability density function, $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, $\Theta = \{\theta : \theta > 0\}$. Find the generalized likelihood ratio test of size-α to test $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$.

[30 marks]

(c) Assume that X is a single observation from a distribution with probability density function

$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1; \quad \theta > 0.$$

To test $H_0 : \theta \leq 1$ vs. $H_1 : \theta > 1$, the test that is used is reject H_0 if and only if $X \geq \frac{1}{2}$. Find the power function and the size of this test.

[30 marks]

5. (a) *Biarkan* $X_1, X_2, ..., X_{10}$ *menandai suatu sampel rawak saiz 10 daripada taburan Poisson dengan min* θ. *Tunjukkan bahawa rantau genting C yang ditakrifkan oleh* $\sum_{i=1}^{10} x_i \geq c'$ *adalah suatu rantau genting terbaik untuk menguji* $H_0 : \theta = 0.1$ *lawan* $H_1 : \theta = 0.5$. *Jika* $c' = 3$, *cari untuk ujian ini, saiz ralat Jenis-I,* α *dan kuasanya pada* $\theta = 0.5$.

[40 markah]

(b) *Biarkan* $X_1, X_2, ..., X_n$ *menandai suatu sampel rawak yang mempunyai fungsi ketumpatan kebarangkalian,* $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, $\Theta = \{\theta : \theta > 0\}$. *Cari ujian nisbah kebolehjadian teritlak saiz-*α *untuk menguji* $H_0 : \theta \leq \theta_0$ *lawan* $H_1 : \theta > \theta_0$.

[30 markah]

(c) *Andaikan bahawa X adalah suatu cerapan tunggal daripada taburan dengan fungsi ketumpatan kebarangkalian*

$$f(x;\theta) = \theta x^{\theta - 1}, \quad 0 < x < 1; \quad \theta > 0.$$

Untuk menguji $H_0 : \theta \leq 1$ *lawan* $H_1 : \theta > 1$, *ujian yang digunakan adalah tolak* H_0 *jika dan hanya jika* $X \geq \frac{1}{2}$. *Cari fungsi kuasa dan saiz ujian ini.*

[30 markah]

APPENDIX / *LAMPIRAN*

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x)=\frac{1}{N}\, I_{\{1,2,\cdots,N\}}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^{N}\frac{1}{N}\, e^{jt}$
Bernoulli	$f(x)=p^x q^{1-x}\, I_{\{0,1\}}(x)$	p	pq	$q+pe^t$
Binomial	$f(x)=\binom{n}{x} p^x q^{n-x}\, I_{\{0,1,\cdots,n\}}(x)$	np	npq	$\left(q+pe^t\right)^n$
Geometri	$f(x)=pq^x\, I_{\{0,1,\cdots\}}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t},\ qe^t<1$
Poisson	$f(x)=e^{-\lambda}\frac{\lambda^x}{x!}\, I_{\{0,1,\cdots\}}(x)$	λ	λ	$exp(\lambda(e^t-1))$
Seragam	$f(x)=\frac{1}{b-a}\, I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t},\ t\neq 0$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, exp\{-(x-\mu)^2/2\sigma^2\}\, I_{(-\infty,\infty)}(x)$	μ	σ^2	$exp\{\mu t+(\sigma t)^2/2\}$
Eksponen	$f(x)=\lambda e^{-\lambda x}\, I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t},\ t<\lambda$
Gama	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\, e^{-\lambda x} x^{\alpha-1}\, I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,\ t<\lambda$
Khi Kuasa Dua	$f(x)=\left(\frac{1}{2}\right)^{r/2}\frac{1}{\Gamma(r/2)}\, e^{-x/2} x^{(r/2)-1}\, I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2},\ t<\frac{1}{2}$
Beta	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	

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