UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MSS 302 – Real Analysis
[Analisis Nyata]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all eleven** [11] questions.

*[**Arahan:** Jawab **semua sebelas** [11] soalan.]*

PART I

1. Explain the meaning of the following statements:
   (a) A Lebesgue measure space over a set $A \subset \mathbb{R}$.
   (b) The Lebesgue integral of a simple function f on A.
   (c) The Lebesgue integral of a non-negative measurable function f on A.
   (d) A Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$.

[8 marks]

2. (a) Let f and g be measurable functions on A. What does it mean by $f < g$ almost everywhere on A?
   (b) Let (f_n) be a sequence of measurable functions on A. What does it mean by (f_n) converges almost everywhere to f on A?

[4 marks]

3. Let $A \subset \mathbb{R}$ and let $(A, \mathfrak{M}_A, m)$ be a measure space with $m(A) < \infty$. Suppose (f_n) is sequence of measurable functions on $A \subset \mathbb{R}$ that converges pointwisely to a function f.
   (a) What is the condition for f to be Lebesgue integrable?
   (b) What is the condition for the equation $\lim_n \int_A f_n \, dm = \int_A \lim_n f_n$ to be true?
   (c) What is the condition for the equation $\sum_n^\infty \int_A f_n \, dm = \int_A \sum_n^\infty f_n$ to be true?

[6 marks]

4. (a) When does a Riemann integrable function $f : [a,b] \to \mathbb{R}$ become Lebesgue integrable on $[a,b]$?
   (b) When does a Riemann integrable function $f : [a,\infty) \to \mathbb{R}$ become Lebesgue integrable on $[a,\infty)$?

[4 marks]

5. (a) What is an L^p space for $1 \le p < \infty$, and what is the L^p norm?
   (b) What is an L^∞ space, what is the L^∞ norm?
   (c) Why the L^2 space is more special than other L^p spaces?
   (d) When will you get $L^{p_1} \subseteq L^{p_2}$ for $p_1, p_2 \in [1,\infty)$?

[8 marks]

BAHAGIAN I

1. *Jelaskan maksud untuk pernyataan berikut:*
   (a) *Ruang sukatan Lebesgue terhadap sebuah set* $A \subset \mathbb{R}$.
   (b) *Kamiran Lebesgue untuk function ringkas f pada A.*
   (c) *Kamiran Lebesgue untuk fungsi tersukatkan tak negatif f pada A.*
   (d) *Fungsi* $f : \mathbb{R} \to \mathbb{R}$ *terkamirkan secara Lebesgue.*

[8 markah]

2. (a) *Andaikan f dan g fungsi tersukatkan pada A. Apakah maksud* $f < g$ *secara hampir di mana-mana pada A?*
   (b) *Andaikan* (f_n) *suatu jujukan fungsi tersukatkan pada A. Apakah maksud* (f_n) *menumpu secara hampir di mana-mana kepada f pada A?*

[4 markah]

3. *Diberi* $A \subset \mathbb{R}$ *dan* $(A, \mathfrak{M}_A, m)$ *suatu ruang tersukat dengan* $m(A) < \infty$. *Andaikan* (f_n) *ialah suatu jujukan fungsi tersukatkan pada* $A \subset \mathbb{R}$ *yang menumpu secara titik demi titik kepada fungsi f.*
   (a) *Apakah syarat untuk f supaya f terkamirkan secara Lebesgue?*
   (b) *Apakah syarat untuk persamaan* $\lim_n \int_A f_n \, dm = \int_A \lim_n f_n$ *adalah benar?*
   (c) *Apakah syarat untuk persamaan* $\sum_n^{\infty} \int_A f_n \, dm = \int_A \sum_n^{\infty} f_n$ *adalah benar?*

[6 markah]

4. (a) *Bilakah suatu fungsi* $f : [a,b] \to \mathbb{R}$ *yang terkamirkan secara Riemann menjadi terkamirkan secara Lebesgue pada* $[a,b]$?
   (b) *Bilakah suatu fungsi* $f : [a,\infty) \to \mathbb{R}$ *yang terkamirkan secara Riemann menjadi terkamirkan secara Lebesgue pada* $[a,\infty)$?

[4 markah]

5. (a) *Apakah ruang* L^p *untuk* $1 \le p < \infty$, *dan apakah norma untuk* L^p?
   (b) *Apakah ruang* L^∞, *apakah norma untuk* L^∞?
   (c) *Kenapa ruang* L^2 *lebih istimewa dari ruang* L^p *lain?*
   (d) *Bilakah anda mendapat* $L^{p_1} \subseteq L^{p_2}$ *untuk* $p_1, p_2 \in [1,\infty)$?

[8 markah]

PART II

1. Show (by using the definition) that every countable set has outer measure zero, and then show it is measurable.

[8 marks]

2. Give an example of a collection $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ such that $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n)$. Prove your answer.

[7 marks]

3. (a) Prove that if $f: A \to \mathbb{R}$ is a measurable function, and $f = g$ almost everywhere on A, then g is measurable.

   (b) Show that if f is a non-negative measurable function on $\mathbb{R}$, then

   $$f = 0 \text{ a.e.} \qquad \text{if and only if } \int_{\mathbb{R}} f\, dm = 0.$$

[7+10=17 marks]

4. Let $f \in L(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Suppose $g(x) = \sin(\alpha x)$ for every $x \in \mathbb{R}$. Show that $g \in L(\mathbb{R})$.

[7 marks]

5. Show that if f is a Lebesgue integrable function on $(\mathbb{R}, \mathfrak{M}, m)$, then

   $$\left|\int f\, dm\right| \leq \int |f|\, dm.$$

   Prove that the equality occurs if $f \leq 0$ almost everywhere on $\mathbb{R}$.

[7+8=15 marks]

6. (a) Suppose f is a bounded and measurable function on a bounded measurable set A. Prove that $f \in L^2(A, \mathfrak{M}_A, m)$.

   (b) Let $f:[0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Show that $f \in L([0,1])$ but $f \notin L^2([0,1])$.

[4+12=16 marks]

BAHAGIAN II

1. *Tunjuk (dengan menggunakan takrif) bahawa setiap set terbilangkan mempunyai sukat terkeluar sifar, dan kemudian tunjukkan bahawa ia adalah tersukatkan.*
[8 markah]

2. *Beri satu contoh pungutan* $\{A_n\}_{n=1}^{\infty}$ *dari subset untuk* $\mathbb{R}$ *sedemikian* $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n)$. *Buktikan jawapan anda.*
[7 markah]

3. (a) *Buktikan bahawa jika* $f : A \to \mathbb{R}$ *fungsi tersukatkan, dan* $f = g$ *secara hampir di mana-mana pada A, maka g tersukatkan.*
   (b) *Tunjukkan bahawa jika f fungsi tersukatkan tak negative pada* $\mathbb{R}$, *maka* $f = 0$ *a.e. jika dan hanya jika* $\int_{\mathbb{R}} f\, dm = 0.$
[7+10=17 markah]

4. *Andaikan* $f \in L(\mathbb{R})$ *dan* $\alpha \in \mathbb{R}$, *dan* $g(x) = \sin(\alpha x)$ *untuk semua* $x \in \mathbb{R}$. *Tunjukkan bahawa* $g \in L(\mathbb{R})$
[7 markah]

5. *Tunjukkan bahawa jika f fungsi terkamirkan secara Lebesgue pada* $(\mathbb{R}, \mathfrak{M}, m)$, *maka*
$$\left|\int f\, dm\right| \leq \int |f|\, dm.$$
Buktikan bahawa kesamaan berlaku jika $f \leq 0$ *hampir di mana-mana pada* $\mathbb{R}$.
[7+8=15 markah]

6. (a) *Andaikan f fungsi yang terbatas dan tersukatkan pada suatu set tersukatkan A yang terbatas. Buktikan bahawa* $f \in L^2(A, \mathfrak{M}_A, m)$.
   (b) *Andaikan* $f : [0,1] \to \mathbb{R}$ *tertakrif sebagai* $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & jika\ x \neq 0 \\ 0 & jika\ x = 0. \end{cases}$
   Tunjukkan bahawa $f \in L([0,1])$ *tetapi* $f \notin L^2([0,1])$.
[4+12=16 markah]

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