UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MSG 388 – Mathematical Algorithms for Computer Graphics
[Algoritma Matematik untuk Grafik Komputer]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all three** [3] questions.

*[**Arahan:** Jawab **semua tiga** [3] soalan.]*

1. (a) Given a polynomial

$$P(x) = 2x^2 + x, \qquad x \in [-1, 1].$$

Find the Bézier representation of function P in terms of parameter $t \in [0, 1]$.

(b) Given four Bézier points $(1, 0)$, $(1, 1)$, $(2, 1)$ and $(2, 0)$ of a cubic Bézier curve $\boldsymbol{P}(t)$, use the de Casteljau algorithm to evaluate the tangent vector to the curve $\boldsymbol{P}(t)$ at $t = 0.25$.

(c) Given a quadratic rational Bézier curve of the form

$$\boldsymbol{R}(t) = \frac{\boldsymbol{C}_0 B_0^2(t) + w\boldsymbol{C}_1 B_1^2(t) + \boldsymbol{C}_2 B_2^2(t)}{B_0^2(t) + wB_1^2(t) + B_2^2(t)}, \qquad t \in [0, 1],$$

where

$$B_i^2(t) = \frac{2!}{i!\,(2-i)!} t^i (1-t)^{2-i}, \qquad i = 0, 1, 2,$$

are the Bernstein polynomials, $\boldsymbol{C}_i \in \mathbb{R}^2$ are the Bézier points and the weight $w \geq 0$. Let $\boldsymbol{C}_0 = (1, 1)$, $\boldsymbol{C}_1 = (2, -2)$ and $\boldsymbol{C}_2 = (3, 1)$.

(i) Suppose $w = 0$, evaluate the point $\boldsymbol{R}(t)$ at $t = 0.5$.

(ii) If w tends to infinity, determine the location of point $\boldsymbol{R}(0.5)$.

(iii) When $w = 1$, find the intersection points of the curve $\boldsymbol{R}(t)$ with x-axis.

(iv) Evaluate the weight w such that the curve $\boldsymbol{R}(t)$ is a circular arc.

[100 marks]

1. (a) *Diberi polinomial*

$$P(x) = 2x^2 + x, \qquad x \in [-1, 1].$$

Cari perwakilan Bézier bagi fungsi P dengan parameter $t \in [0, 1]$.

(b) *Diberi empat titik Bézier* $(1, 0)$, $(1, 1)$, $(2, 1)$ *dan* $(2, 0)$ *bagi satu lengkung Bézier kubik* $\boldsymbol{P}(t)$, *gunakan algoritma de Casteljau untuk menilai vektor tangen kepada lengkung* $\boldsymbol{P}(t)$ *pada* $t = 0.25$.

(c) *Diberi suatu lengkung Bézier nisbah kuadratik dalam bentuk*

$$\boldsymbol{R}(t) = \frac{\boldsymbol{C}_0 B_0^2(t) + w\boldsymbol{C}_1 B_1^2(t) + \boldsymbol{C}_2 B_2^2(t)}{B_0^2(t) + w B_1^2(t) + B_2^2(t)}, \qquad t \in [0, 1],$$

di mana

$$B_i^2(t) = \frac{2!}{i!\,(2-i)!} t^i (1-t)^{2-i}, \qquad i = 0, 1, 2,$$

ialah polinomial Bernstein, $\boldsymbol{C}_i \in \mathbb{R}^2$ *adalah titik Bézier dan pemberat* $w \geq 0$. *Katakan* $\boldsymbol{C}_0 = (1, 1)$, $\boldsymbol{C}_1 = (2, -2)$ *dan* $\boldsymbol{C}_2 = (3, 1)$.

(i) *Andaikan* $w = 0$, *nilaikan titik* $\boldsymbol{R}(t)$ *pada* $t = 0.5$.

(ii) *Jika* w *menumpu ke infiniti, tentukan kedudukan titik* $\boldsymbol{R}(0.5)$.

(iii) *Apabila* $w = 1$, *cari titik persilangan bagi lengkung* $\boldsymbol{R}(t)$ *dengan paksi-*x.

(iv) *Nilaikan pemberat* w *supaya lengkung* $\boldsymbol{R}(t)$ *adalah suatu lengkok bulatan.*

[100 markah]

2. (a) Let $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n+k})$ be a knot vector. The normalized B-splines of order k are defined recursively by

$$N_i^1(u) = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad k > 1.$$

Suppose $\boldsymbol{u} = (-2, -1, 0, 1, 2, 3)$. Then a uniform B-spline curve of order 3 is defined by

$$\boldsymbol{P}(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} N_0^3(u) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} N_1^3(u) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} N_2^3(u), \qquad u \in [0, 1]$$

where $N_i^3(u)$ are the normalized B-splines of order 3.

(i) Show that $\sum_{i=0}^{2} N_i^3(u) = 1$, for $u \in [0, 1]$.

(ii) Find the control points of a cubic Bézier curve such that the curve coincides with $\boldsymbol{P}(u)$.

(iii) If knot $u = 0.75$ is inserted twice into $\boldsymbol{u}$, find the de Boor points of a B-spline curve of order 3 such that the curve coincides with $\boldsymbol{P}(u)$.

(b) Given knots $(u_i, v_j) = (i-2, j-2)$, $i = 0, 1, \ldots, 5$, $j = 0, 1, \ldots, 5$. A uniform tensor-product B-spline surface of order $(3, 3)$ is defined by

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \boldsymbol{D}_{i,j} N_j^3(v) N_i^3(u), \qquad 0 \le u, v \le 1$$

where N_s^3, $s = 0, 1, 2$, are the normalized B-splines of order 3 and $\boldsymbol{D}_{i,j}$ are the de Boor points.

(i) State the surface point $\boldsymbol{S}(0, 0)$ in terms of the given de Boor points.

(ii) Find the cross boundary derivative of $\boldsymbol{S}(u, v)$ at boundary $u = 0$.

[100 marks]

2. (a) *Katakan* $\boldsymbol{u} = (u_0\,,\ u_1\,,\ \ldots,\ u_{n+k})$ *ialah suatu vektor knot. Splin-B ternormal berperingkat* k *ditakrif secara rekursi sebagai*

$$N_i^1(u) = \begin{cases} 1, & u_i \le u < u_{i+1} \\ 0, & \text{di tempat lain} \end{cases}$$

dan

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u), \qquad k > 1.$$

Andaikan $\boldsymbol{u} = (-2\,,\ -1\,,\ 0\,,\ 1\,,\ 2\,,\ 3)$. *Kemudian lengkung splin-B seragam berperingkat* 3 *ditakrif sebagai*

$$\boldsymbol{P}(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} N_0^3(u) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} N_1^3(u) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} N_2^3(u), \qquad u \in [0,\ 1]$$

di mana $N_i^3(u)$ *ialah splin-B ternormal berperingkat* 3.

(i) *Tunjukkan bahawa* $\sum_{i=0}^{2} N_i^3(u) = 1$, *bagi* $u \in [0,\ 1]$.

(ii) *Cari titik-titik kawalan bagi suatu lengkung Bézier kubik supaya lengkung ini sama dengan lengkung* $\boldsymbol{P}(u)$.

(iii) *Jika knot* $u = 0.75$ *dimasukkan dua kali ke dalam* $\boldsymbol{u}$, *cari titik-titik de Boor bagi suatu lengkung splin-B berperingkat* 3 *supaya lengkung ini sama dengan lengkung* $\boldsymbol{P}(u)$.

(b) *Diberi knot* $(u_i,\ v_j) = (i-2,\ j-2)$, $i = 0,\ 1,\ \ldots,\ 5$, $j = 0,\ 1,\ \ldots,\ 5$. *Permukaan hasil darab tensor splin-B seragam berperingkat* $(3,\ 3)$ *ditakrifkan sebagai*

$$\boldsymbol{S}(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \boldsymbol{D}_{i,j}\, N_j^3(v)\, N_i^3(u), \qquad 0 \le u, v \le 1$$

di mana N_s^3, $s = 0\,,\ 1\,,\ 2$, *ialah splin-B ternormal berperingkat* 3 *dan* $\boldsymbol{D}_{i,j}$ *merupakan titik de Boor.*

(i) *Nyatakan titik permukaan* $\boldsymbol{S}(0, 0)$ *dalam sebutan titik-titik de Boor.*

(ii) *Cari terbitan silang sempadan bagi* $\boldsymbol{S}(u, v)$ *pada sempadan* $u = 0$.

[100 markah]

3. (a) Suppose a tensor-product Bézier surface of degree $(2, 2)$ is defined by

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{C}_{i,j}\,B_j^2(v)\,B_i^2(u), \qquad 0\le u, v\le 1$$

where B_s^2, $s=0, 1, 2$, are the Bernstein polynomials of degree 2 and $\boldsymbol{C}_{i,j}$ are the Bézier points.

(i) Surface $\boldsymbol{S}(u,v)$ can be represented in matrix form as

$$\boldsymbol{S}(u,v)=\begin{bmatrix} u^2 & u & 1\end{bmatrix} M \begin{bmatrix} \boldsymbol{C}_{0,0} & \boldsymbol{C}_{0,1} & \boldsymbol{C}_{0,2} \\ \boldsymbol{C}_{1,0} & \boldsymbol{C}_{1,1} & \boldsymbol{C}_{1,2} \\ \boldsymbol{C}_{2,0} & \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2}\end{bmatrix} M^{\mathrm{T}} \begin{bmatrix} v^2 \\ v \\ 1\end{bmatrix}, \qquad 0\le u, v\le 1.$$

Find the 3×3 matrix M.

(ii) Prove that the surface $\boldsymbol{S}(u,v)$, $0\le u, v\le 1$ lies within the convex hull of its Bézier net.

(b) Let $\Delta\boldsymbol{ABC}$ denote a triangle with the vertices $\boldsymbol{A}=(x_1, y_1)$, $\boldsymbol{B}=(x_2, y_2)$ and $\boldsymbol{C}=(x_3, y_3)$. Any point $\boldsymbol{P}$ inside the triangle $\Delta\boldsymbol{ABC}$ can be represented as $\boldsymbol{P}=u\boldsymbol{A}+v\boldsymbol{B}+w\boldsymbol{C}$ where $u, v, w\ge 0$ and $u+v+w=1$. Show that

$$u = \frac{\text{area of } \Delta\boldsymbol{PBC}}{\text{area of } \Delta\boldsymbol{ABC}}.$$

(c) The generalized Bernstein polynomial of degree n is defined by

$$B^n_{i,j,k}(u, v, w)=\frac{n!}{i!\,j!\,k!}u^i v^j w^k, \qquad u, v, w\ge 0, \qquad u+v+w=1$$

for the integers $i, j, k\ge 0$ and $i+j+k=n$.

(i) Show that

$$B^n_{i,j,k}(u,v,w) = uB^{n-1}_{i-1,j,k}(u,v,w) + vB^{n-1}_{i,j-1,k}(u,v,w) + wB^{n-1}_{i,j,k-1}(u,v,w).$$

(ii) Evaluate the directional derivative of $B^3_{0,1,2}(u, v, w)$ with the vector $\boldsymbol{d}=(-3, 2, 1)$ at $(u, v, w)=(0, \frac{1}{3}, \frac{2}{3})$.

[100 marks]

3. (a) *Andaikan suatu permukaan hasil darab tensor Bézier berdarjah* (2, 2) *ditakrif sebagai*

$$\boldsymbol{S}(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}\boldsymbol{C}_{i,j}\,B_j^2(v)\,B_i^2(u), \qquad 0\le u, v\le 1$$

di mana B_s^2, $s=0$, 1, 2, *ialah polinomial Bernstein berdarjah* 2 *dan* $\boldsymbol{C}_{i,j}$ *merupakan titik Bézier.*

(i) *Permukaan* $\boldsymbol{S}(u,v)$ *boleh diwakili dalam bentuk matriks sebagai*

$$\boldsymbol{S}(u,v)=\begin{bmatrix}u^2 & u & 1\end{bmatrix} M \begin{bmatrix}\boldsymbol{C}_{0,0} & \boldsymbol{C}_{0,1} & \boldsymbol{C}_{0,2}\\ \boldsymbol{C}_{1,0} & \boldsymbol{C}_{1,1} & \boldsymbol{C}_{1,2}\\ \boldsymbol{C}_{2,0} & \boldsymbol{C}_{2,1} & \boldsymbol{C}_{2,2}\end{bmatrix} M^{\mathrm{T}} \begin{bmatrix}v^2\\ v\\ 1\end{bmatrix}, \qquad 0\le u, v\le 1.$$

Cari matriks M *yang berperingkat* 3×3.

(ii) *Buktikan bahawa permukaan* $\boldsymbol{S}(u,v)$, $0\le u$, $v\le 1$ *terletak di dalam hul cembung bagi jairng Bézier.*

(b) *Katakan* $\Delta\boldsymbol{ABC}$ *menandai suatu segitiga dengan bucu* $\boldsymbol{A}=(x_1, y_1)$, $\boldsymbol{B}=(x_2, y_2)$ *dan* $\boldsymbol{C}=(x_3, y_3)$. *Sebarang titik* $\boldsymbol{P}$ *di dalam segitiga* $\Delta\boldsymbol{ABC}$ *boleh diwakili sebagai* $\boldsymbol{P}=u\boldsymbol{A}+v\boldsymbol{B}+w\boldsymbol{C}$ *di mana* u, v, $w\ge 0$ *dan* $u+v+w=1$. *Tunjukkan bahawa*

$$u = \frac{\text{luas segitiga } \Delta \boldsymbol{PBC}}{\text{luas segitiga } \Delta \boldsymbol{ABC}}.$$

(c) *Polinomial Bernstein teritlak berdarjah* n *ditakrifkan sebagai*

$$B_{i,j,k}^n(u, v, w)=\frac{n!}{i!\ j!\ k!}u^i v^j w^k, \qquad u, v, w\ge 0, \qquad u+v+w=1$$

bagi integer i, j, $k\ge 0$ *dan* $i+j+k=n$.

(i) *Tunjukkan bahawa*

$$B_{i,j,k}^n(u,v,w) = uB_{i-1,j,k}^{n-1}(u,v,w) + vB_{i,j-1,k}^{n-1}(u,v,w) + wB_{i,j,k-1}^{n-1}(u,v,w).$$

(ii) *Nilaikan terbitan berarah bagi* $B_{0,1,2}^3(u, v, w)$ *dengan vektor* $\boldsymbol{d}=(-3, 2, 1)$ *pada* $(u, v, w)=(0, \frac{1}{3}, \frac{2}{3})$.

[100 markah]

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