UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MGM 503 – Combinatorics
[Kombinatorik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) There are k different types of objects. Derive a formula for the number of n-permutations of n_r elements of the r^{th} type, for $r = 1, 2, ..., k$, where $n = n_1 + ... + n_k$. What is the number of permutations of the letters in the word MISSISSIPPI? In how many ways can we permute the word MISSISSIPPI so that no two letters 'I' appear together?

(b) Give a combinatorial proof of the relation

$$C_n^k C_{n-k}^{m-k} = C_m^k C_n^m .$$

Deduce that

$$C_n^0 C_n^{n-m} + C_n^1 C_{n-1}^{n-m} + ... + C_n^m C_{n-m}^{n-m} = 2^m C_n^m .$$

[60 marks]

2. (a) If $k \leq n+1$, show that the number of sequences of k ones and n zeros with no two consecutive ones is C_{n+1}^k. Using this find the number of ways in which 5 out of 12 books on a book-shelf be selected if a selection must not include two neighbouring books.

(b) Give a combinatorial proof of the relation

$$C_n^n + C_{n+1}^n + C_{n+2}^n + ... + C_{n+m-1}^n = C_{n+m}^{n+1}$$

where C_n^k is the number of k-combinations of n elements. Deduce that

$$1 + 2 + ... + m = \frac{1}{2} m(m+1) .$$

[60 marks]

1. (a) *Terdapat k objek berlainan jenis. Terbitkan satu formula untuk bilangan n pilihatur bagi n_r unsur dari r jenis, $r = 1, 2, ..., k$, dan $n = n_1 + ... + n_k$. Berapakah pilihatur huruf dalam perkataan MISSISSIPPI? Berapakah cara kita dapat memilihatur perkataan MISSISSIPPI supaya tiada dua abjad 'I' muncul bersama?*

   (b) *Beri pembuktian kombinatorik untuk hubungan*

$$C_n^k C_{n-k}^{m-k} = C_m^k C_n^m .$$

   Deduksikan bahawa

$$C_n^0 C_n^{n-m} + C_n^1 C_{n-1}^{n-m} + ... + C_n^m C_{n-m}^{n-m} = 2^m C_n^m .$$

[60 markah]

2. (a) *Jika $k \leq n+1$, tunjukkan bahawa bilangan jujukan untuk k nombor satu dan n nombor sifar tanpa dua nombor satu berturutan ialah C_{n+1}^k. Gunakan ini untuk mencari bilangan cara memilih 5 daripada 12 buah buku di atas sebuah rak buku jika pilihan dibuat tanpa memilih dua buku bersebelahan.*

   (b) *Beri pembuktian kombinatorik untuk hubungan*

$$C_n^n + C_{n+1}^n + C_{n+2}^n + ... + C_{n+m-1}^n = C_{n+m}^{n+1},$$

   dengan C_n^k ialah bilangan k gabungan dari n unsur. Deduksikan bahawa

$$1 + 2 + ... + m = \frac{1}{2} m(m+1) .$$

[60 markah]

3. (a) If there are n_1 objects of the first kind, n_2 objects of the second kind, …, n_k objects of the kth kind, show that these objects can be divided between two people in $(n_1+1)(n_2+1)...(n_k+1)$ ways. Using this formula find the number of divisors of a natural number $N = p_1^{n_1}p_2^{n_2}...p_k^{n_k}$ where $p_1, p_2, ..., p_k$ are distinct primes. If we impose the restriction that each of the two people is to get at least s_1 objects of the first kind, s_2 of the second kind, …, s_k objects of the kth kind,

show that the number of ways in which the division can be carried out between the two people is $(n_1-2s_1+1)(n_2-2s_2+1)...(n_k-2s_k+1)$.

(b) State the Principle of Inclusion-Exclusion. Using this principle, find how many non-negative integer solutions does the equation $x_1+x_2+x_3=11$ have where x_1, x_2, x_3 are non-negative integers such that $x_1 \le 3, x_2 \le 4, x_3 \le 6$.

[50 marks]

4. (a) Solve the linear recurrence relation

$$a_{n+2} = 6a_{n+1} - 9a_n, a_0 = 0, a_1 = 1.$$

(b) Solve the linear recurrence relation using generating functions:

$$a_n - a_{n-1} - 6a_{n-2} = -2^n, a_0 = 1, a_1 = -3.$$

[30 marks]

3. (a) *Jika terdapat n_1 objek jenis pertama, n_2 objek jenis kedua, ..., n_k objek jenis ke-k, tunjukkan bahawa objek-objek ini dapat dibahagi di antara dua orang dengan* $(n_1+1)(n_2+1)...(n_k+1)$ *cara. Guna formula ini untuk mencari bilangan pembahagi suatu nombor asli* $N = p_1^{n_1}p_2^{n_2}...p_k^{n_k}$, *dengan* $p_1, p_2, ..., p_k$ *ialah nombor perdana berbeza. Jika kita mengamalkan kekangan bahawa setiap orang mendapat sekurang-kurangnya s_1 objek jenis pertama, s_2 objek jenis kedua, ..., s_k objek jenis ke-k, tunjukkan bahawa bilangan cara pembahagian dapat dibuat di antara dua orang tersebut ialah* $(n_1-2s_1+1)(n_2-2s_2+1)...(n_k-2s_k+1)$.

(b) *Nyatakan Prinsip Inklusi-Eksklusi. Gunakan prinsip ini untuk mencari bilangan penyelesaian integer tak negatif dalam persamaan* $x_1+x_2+x_3=11$, *dengan* x_1, x_2, x_3 *ialah integer tak negatif sedemikian* $x_1 \le 3, x_2 \le 4, x_3 \le 6$.

[50 markah]

4. (a) *Selesaikan hubungan jadi semula linear*

$$a_{n+2} = 6a_{n+1} - 9a_n, a_0 = 0, a_1 = 1.$$

(b) *Selesaikan hubungan jadi semula linear dengan menggunakan fungsi penjana:*

$$a_n - a_{n-1} - 6a_{n-2} = -2^n, a_0 = 1, a_1 = -3.$$

[30 markah]

- ooo O ooo -