UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2008/2009

November 2008

MAT 518 – Numerical Methods for Differential Equations
[Kaedah Berangka untuk Persamaan Pembezaan]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of FIVE pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi LIMA muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **all four** [4] questions.

*[**Arahan:** Jawab **semua empat** [4] soalan.]*

1. (a) Consider the diffusion equation $u_t = \alpha^2 u_{xx}$. Write down the FTCS scheme for this equation. What is the condition for stability?

(b) Consider the 2 dimensional diffusion equation $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$.

(i) Write down the FTCS scheme. What is the condition for stability? Comment on the difference between the stability condition for the one dimensional case with the two dimensional case.

(ii) Write down an implicit scheme for $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$. Comment on the stability. What is the main problem of the implicit scheme?

(iii) Describe the ADI scheme for $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$.

[100 marks]

2. Consider the equation $u_t + u_x = 0$. Write down the Forward Time Backward Space (upwind) scheme. Using the Fourier method, compute the amplification factor.

[100 marks]

3. (a) Suppose the matrix coefficient of the system $A\underline{u} = \underline{b}$ is

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}, \qquad \text{with} \qquad \underline{b} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

(i) Using the initial vector $\underline{u}^{(0)} = (0,0,1)^{\mathrm{T}}$ as the starting value, generate two iterations ($\underline{u}^{(1)}, \underline{u}^{(2)}$) of the second order Richardson's method for this 3x3 system.

(ii) What is the rate of convergence, R_∞, of this iterative method in solving this system?

(b) Given the block system $\sum_{j=1}^{4} A_{i,j} \underline{U}_j = \underline{C}_i$, $i=1,2,3,4$.

Write the block Gauss-Seidel iterative method in solving this system. Write the corresponding block SOR iterative method in solving the same system.

[100 marks]

1. (a) *Pertimbangkan persamaan* $u_t = \alpha^2 u_{xx}$. *Tuliskan skema FTCS untuk persamaan ini. Apakah syarat kestabilan?*

(b) *Pertimbangkan persamaan resapan 2 dimensi* $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$.

(i) *Tulis skema FTCS. Apakah syarat kestabilan? Komen perbezaan antara syarat kestabilan untuk kes satu dimensi dengan kes dua dimensi.*

(ii) *Tulis suatu skema tersirat untuk* $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$. *Komen berkaitan kestabilan. Apakah masalah utama skema ini?*

(iii) *Huraikan skema ADI untuk* $u_t = \alpha^2 \left(u_{xx} + u_{yy}\right)$.

[100 markah]

2. *Pertimbangkan persamaan* $u_t + u_x = 0$. *Tuliskan skema beza ke depan dalam masa, beza ke belakang dalam ruang (upwind). Dengan menggunakan kaedah Fourier, kira faktor amplikasikan.*

[100 markah]

3. (a) *Katakan matriks koefisien bagi sistem* $A\underline{u} = \underline{b}$ *ialah*

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}, \qquad \textit{dengan} \qquad \underline{b} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

(i) *Dengan menggunakan vektor awalan* $\underline{u}^{(0)} = (0,0,1)^T$ *sebagai nilai permulaan, janakan dua lelaran* ($\underline{u}^{(1)}, \underline{u}^{(2)}$) *kaedah Richardson peringkat dua bagi sistem 3x3 ini.*

(ii) *Apakah kadar penumpuan,* R_∞, *bagi kaedah lelaran ini dalam menyelesaikan sistem ini?*

(b) *Diberikan sistem blok* $\sum_{j=1}^{4} A_{i,j} \underline{U}_j = \underline{C}_i$, $i=1,2,3,4$.

Tuliskan kaedah lelaran blok Gauss-Seidel dalam menyelesaikan sistem ini. Tuliskan kaedah lelaran blok SOR yang bersepadan dalam menyelesaikan sistem yang sama.

[100 markah]

4. (a) Suppose that the matrix A resulted from a finite difference discretization of an elliptic partial differential equation is as follows:

$$A = \begin{bmatrix} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{bmatrix}.$$

(i) Using appropriate theorems discussed in class, find all the eigenvalues of A.

(ii) Use the information obtained from part 4(a)(i) to find all the eigenvalues of the point Jacobi iteration matrix T_j.

(iii) Find the spectral radius $\rho(T_j)$ of the point Jacobi method.

(iv) Find the spectral radius $\rho(T_G)$ of the point Gauss-Seidel method.

(v) Find the theoretical optimal relaxation parameter of the point S.O.R. method.

(vi) Estimate the number of iterations you would expect to get if the Jacobi, Gauss-Seidel and S.O.R. methods are used for this mesh size and tolerance $\varepsilon = 10^{-5}$.

(b) For the five-point finite difference Dirichlet problem of Laplace equation in $0 < y < \pi,\ 0 < x < \pi$, show that the optimal relaxation parameter of the point S.O.R. iteration is $\lambda = \dfrac{2}{1+\sin h}$ (here h is the mesh size).

[100 marks]

4. (a) *Katakan matriks A yang terhasil daripada pendiskretan beza terhingga satu persamaan pembezaan separa eliptik adalah seperti berikut:*

$$A = \begin{bmatrix} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{bmatrix}.$$

(i) *Dengan menggunakan teorem yang bersesuaian yang dibincangkan dalam kuliah, cari semua nilai eigen bagi A.*

(ii) *Gunakan maklumat yang diperoleh dari bahagian 4(a)(i) untuk mencari semua nilai eigen bagi matriks lelaran Jacobi titik* T_j .

(iii) *Cari jejari spekrum* $\rho(T_j)$ *bagi kaedah Jacobi titik.*

(iv) *Cari jejari spektrum* $\rho(T_G)$ *bagi kaedah Gauss-Seidel titik.*

(v) *Cari parameter pengenduran optimum secara teori bagi kaedah S.O.R. titik.*

(vi) *Anggarkan bilangan lelaran yang anda jangka didapati jika kaedah-kaedah Jacobi, Gauss-Seidel dan S.O.R. digunakan untuk saiz mesy ini dan tolerans* $\varepsilon = 10^{-5}$.

(b) *Bagi masalah Dirichlet lima-titik beza terhingga untuk persamaan Laplace dalam* $0 < y < \pi,\ 0 < x < \pi$, *tunjukkan bahawa parameter pengenduran optimum bagi lelaran S.O.R. titik ialah* $\lambda = \dfrac{2}{1+\sin h}$ *(di sini* h *ialah saiz mesy).*

[100 markah]

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