# Constrained $C^1$ Interpolation on Rectangular Grids

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## Abstract

*This paper is concerned with the range restricted interpolation of data on rectangular grids. The interpolant is constrained to lie on the same side of the constraint surface as the data. Sufficient non-negativity conditions on the Bézier ordinates are derived to ensure the non-negativity of a bicubic Bézier patch. The method modifies Bézier ordinates locally to fulfill the sufficient non-negativity conditions. The $C^1$ interpolating surface is constructed piecewise as a convex combination of two bicubic Bézier patches with the same set of boundary Bézier ordinates. The set of admissible constraint surfaces include polynoimal surfaces of the form z = C(x, y) where $C(x, y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{i,j}x^i y^j$ and the $a_{i,j}$ are real numbers, as well as $C^1$ spline surfaces consisting of polynomial pieces of the form z = C(x, y) on the rectangular grid. Some graphical examples are presented.*

## 1. Introduction

The problem of positivity preserving interpolation is of practical interest. Physical quantities like concentration when represented visually should not admit negative values since negative values are physically meaningless. A number of works have been done for the univariate case but not much has been done for the more general problem, range restricted interpolation for both the univariate and bivariate cases. As we are concerned with the constrained bivariate interpolation to data on a rectangular grid, we mention some of the work done in this respect. Mulansky and Schmidt [5] present a non-negative interpolation to gridded data by $C^1$ biquadratic splines on a refined rectangular grid. They derive sufficient non-negativity conditions on the first partial derivatives and mixed second partial derivatives by using the corresponding results for the univariate quadratic splines with additional knots [4] and the tensor product structure for the spline space. There exist an infinite number of interpolants meeting the constraints. The selection of the interpolant is based on a fit-and-modify approach or the minimization of a suitable objective functional.

Brodlie, Butt & Mashwama [1] construct $C^1$ bicubic splines on rectangular grids addressing the problem of generating interpolants subject to linear constraints as lower and upper bounds. The results of Schmidt and Heß [6] for the univariate case are used to derive sufficient non-negativity conditions on the first partial derivatives and second mixed partial derivatives. These derivatives are estimated and the estimated values are projected onto the valid intervals defined by the sufficient non-negativity conditions. This interpolation method is local.

In [7] a local $C^1$ scheme for interpolating data on a rectangular grid subject to lower and upper constant bounds has been constructed. The interpolant is piecewise an average of two blending surfaces, each being obtained by blending between two boundary curves of the patch by using univariate rational cubics. The weights of the rational cubics are chosen to ensure that the blending surfaces lie within the given bounds.

In this paper the construction of range restricted bivariate $C^1$ interpolants to data on a rectangular grid is considered. We derive in Section 2 sufficient conditions on the Bézier ordinates to ensure non-negativity for a bicubic Bézier patch by using the univariate result on non-negativity in [3] and some simple observations. A local scheme applying these sufficient non-negativity conditions for $C^1$ non-negativity preserving interpolation is constructed in Section 3. The interpolating surface is obtained piecewise as the convex combination of two bicubic Bézier patches, each with the same set of boundary Bézier ordinates. In Section 4, we extend the results to range restricted interpolation which considers as lower and upper constraints polynomial surfaces of the form $C(x, y) = \sum_{i=0}^{3}\sum_{j=0} a_{i,j}x^i y^j$, where the $a_{i,j}$ are real numbers, as well as $C^1$ spline surfaces consisiting of polynomial pieces of the form z = C(x, y) on the rectangular grid. Two numerical examples are presented graphically in the last section.

## 2. Sufficient non-negativity conditions for a bicubic Bézier patch

A bicubic Bézier patch is defined as

$$P(u, v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,j}B_i^3(u)B_j^3(v)\,, \quad u, v \in [0, 1], \quad (2.1)$$

where $B_k^3$ are cubic Bernstein polynomials and $b_{i,j}$ are the Bézier ordinates of $P$. Consider a unit square with vertices $U_1$, $U_2$, $U_3$ and $U_4$. Given positive Bézier ordinates at the vertices, *i.e.* we may assume

$$\{b_{0,0}, b_{3,0}, b_{0,3}, b_{3,3}\} = \{ \alpha \ell, \beta \ell, \gamma \ell, \ell \}, \qquad (2.2)$$

with $\ell > 0$ and $\alpha \geq \beta \geq \gamma \geq 1$, our aim is to derive sufficient conditions on the remaining Bézier ordinates for the Bézier patch to be non-negative.

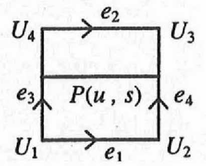


**Figure 1. Notation on a rectangle.**

Denote the edges along $U_1U_2$, $U_4U_3$, $U_1U_4$ and $U_2U_3$ by $e_1$, $e_2$, $e_3$ and $e_4$ respectively (see Figure 1). A curve on $P$ along a line segment parallel to $e_1$ is given by

$$P(u, s) = \Sigma_{i=0}^3 \Sigma_{j=0}^3 b_{i,j} B_i^3(u) B_j^3(s), \quad u \in [0, 1], \qquad (2.3)$$

where $s$ is a constant between 0 and 1. Observe that for each fixed $s$, $P(u, s)$, $u \in [0, 1]$, is a cubic Bézier curve with $\Sigma_{j=0}^3 b_{i,j} B_j^3(s)$, $i = 1, 2, 3$ as the Bézier ordinates. Clearly, if all these curves are non-negative, then the patch $P(u, v)$ is non-negative. Similarly, a curve on $P$ along a line segment parallel to $e_3$ is given by

$$P(t, v) = \Sigma_{i=0}^3 \Sigma_{j=0}^3 b_{i,j} B_i^3(t) B_j^3(v), \quad v \in [0, 1], \qquad (2.4)$$

where $t$ is a constant between 0 and 1.

Based upon Theorem 1 quoted from [3] and some simple observations below we derive our sufficient non-negativity conditions in Proposition 1.

**Theorem 1** Let $r(x) = A(1-x)^3 + 3B(1-x)^2x + 3C(1-x)x^2 + Dx^3$, $x \in (0, 1)$, where $A, D > 0$, and $B < 0$ and /or $C < 0$. Then $r(x) < 0$ for some $x \in (0, 1)$ [resp. $r(x) = 0$ for only one point in (0, 1)] if and only if

$$3B^2C^2 + 6ABCD - 4(AC^3 + B^3D) - A^2D^2 > 0 \text{ [resp. } = 0]. \qquad (2.5)$$

Denote $\Delta = 3B^2C^2 + 6ABCD - 4(AC^3 + B^3D) - A^2D^2$. Observe that if $A, D > 0$, $B = -A/3$ and $C = -D/3$, then $\Delta = 4AD(A-D)^2$. Thus in this case if $A \neq D$, then $r(x)$ as defined above will be negative for some $x \in (0, 1)$.

However if $\min\{A, D\} = \ell > 0$, $B = C = -\ell/3a$, where $a > 1$, then $r(x) \geq \ell(a-1)/4a$, $\forall x \in [0, 1]$. (2.6)

**Proposition 1** Let $P(u, v) = \Sigma_{i=0}^3 \Sigma_{j=0}^3 b_{i,j} B_i^3(u) B_j^3(v)$, $u$, $v \in [0, 1]$, where $\{b_{0,0}, b_{3,0}, b_{0,3}, b_{3,3}\}$ is as given in (2.2). Let $\lambda = \gamma$ if $\ell$ and $\gamma\ell$ are values at the diagonal vertices, otherwise $\lambda = \beta$.

If $b_{1,0}, b_{2,0}, b_{1,3}, b_{2,3}, b_{0,1}, b_{0,2}, b_{3,1}, b_{3,2}, b_{1,1}, b_{2,1}, b_{1,2}, b_{2,2} \geq -\ell/3a$, where $a$ is the smallest solution in (1, 5] of

$$-27\lambda^2 a^4 + 108\lambda^2 a^3 + (288\lambda - 162\lambda^2) a^2 + (108\lambda^2 - 320\lambda + 256) a - 27\lambda^2 + 32\lambda = 0, \qquad (2.7)$$

then $P(u, v) \geq 0$, $\forall u, v \in [0, 1]$.

**Proof** In view of (2.6), in order that the four boundary curves $P(u, 0)$, $P(u, 1)$, $P(0, v)$, $P(1, v)$ are positive, it suffices to have $b_{1,0}, b_{2,0}, b_{1,3}, b_{2,3}, b_{0,1}, b_{0,2}, b_{3,1}, b_{3,2} \geq -\ell/3a$, $a > 1$. Without loss of generality, we may assume $b_{0,0} = \ell$. We shall distinguish two cases.

(i) when $b_{3,3} = \gamma\ell$ (i.e. $\ell$ and $\gamma\ell$ are at diagonal vertices)
(ii) when $b_{3,3} \neq \gamma\ell$ (i.e. $\ell$ and $\gamma\ell$ are at adjacent vertices).

For case (i), we consider a curve along a line segment parallel to $e_1$ which is given by (2.3). We will derive conditions for the curve in (2.3) to be non-negative. Let

$$A(s) = \Sigma_{j=0}^3 b_{0,j} B_j^3(s), \quad B(s) = \Sigma_{j=0}^3 b_{1,j} B_j^3(s),$$
$$C(s) = \Sigma_{j=0}^3 b_{2,j} B_j^3(s) \text{ and } D(s) = \Sigma_{j=0}^3 b_{3,j} B_j^3(s), \qquad (2.8)$$

then (2.3) becomes

$$P(u, s) = (1-u)^3 A(s) + 3(1-u)^2 u B(s) + 3(1-u)u^2 C(s) + u^3 D(s). \qquad (2.9)$$

Observe that $A(s) \geq \ell(a-1)/4a$, $D(s) \geq \gamma\ell(a-1)/4a$, $B(s), C(s) \geq -\ell/3a$, $0 \leq s \leq 1$. By Theorem 1, in order for $P(u, s) \geq 0$, it suffices to have

$$3B^2C^2 + 6ABCD - 4(AC^3 + B^3D) - A^2D^2 = 0,$$

where $A = \ell(a-1)/4a$, $D = \gamma\ell(a-1)/4a$ and $B = C = -\ell/3a$. That is, it suffices to have $f(a) = 0$ where

$$f(a) = -27\gamma^2 a^4 + 108\gamma^2 a^3 + (288\gamma - 162\gamma^2) a^2 + (108\gamma^2 - 320\gamma + 256) a - 27\gamma^2 + 32\gamma.$$

Observe that $f(1) = 256 > 0$ and $f(5) \leq 0$, thus there exists $a_0 \in (1, 5]$ with $f(a_0) = 0$. If there exists more than one $a_0$, then the smallest one in (1, 5] is chosen.

For case (ii) where $b_{3,3} \neq \gamma\ell$, there are two possibilities; either $b_{3,0} = \gamma\ell$ or $b_{0,3} = \gamma\ell$. When $b_{3,0} = \gamma\ell$, we consider again a curve along a line segment parallel to $e_1$ which is given by (2.9). Then we know $A(s) \geq \ell(a-1)/4a$, $D(s) \geq \beta\ell(a-1)/4a$, $B(s), C(s) \geq -\ell/3a$, $s \in [0, 1]$. For $P(u, s) \geq 0$, it suffices to have $a$ satisfying

$$-27\beta^2 a^4 + 108\beta^2 a^3 + (288\beta - 162\beta^2) a^2 + (108\beta^2 - 320\beta + 256) a - 27\beta^2 + 32\beta = 0. \qquad (2.10)$$

When case (ii) occurs with $b_{3,0} = \gamma\ell$, the same argument above is repeated by considering a curve along a line segment parallel to $e_3$ and we will also obtain (2.10).

Suppose now the boundary Bézier ordinates have already been determined by using the derivatives defined at the vertices and their values have been ensured to be not less than the lower bound $-\ell/3a$ as stated in Proposition 1. Then we shall derive two sets of less stringent lower bounds for inner Bézier ordinates, *i.e.* a lower bound which is less than or equal to $-\ell/3a$ suggested in Proposition 1, by considering curves in (2.3) and (2.4) respectively. We shall need the following lemma which can be obtained using simple calculus.

**Lemma 1** Let $r(x)$ be as in Theorem 1.

(i) Suppose $A = 0$ and $D > 0$.
For $r(x) \geq 0$, it is necessary that $B \geq 0$.
If $C < 0$, then $r(x) \geq 0$ if and only if $B \geq 0$ and $3C^2 \leq 4BD$.

(ii) Suppose $A > 0$ and $D = 0$.
For $r(x) \geq 0$, it is necessary that $C \geq 0$.
If $B < 0$, then $r(x) \geq 0$ if and only if $C \geq 0$ and $3B^2 \leq 4AC$.

(iii) If $A = D = 0$, then $r(x) \geq 0$ if and only if
$B \geq 0$ and $C \geq 0$.

Let us first consider $P$ as a surface consisting of curves along line segments parallel to $e_1$, *i.e.* curves $P(u, s)$ as given by (2.3). These are cubic Bézier curves with Bézier ordinates given by $A(s)$, $B(s)$, $C(s)$ and $D(s)$ as in (2.8). As all the Bézier ordinates for $A(s)$ and $D(s)$, $s \in [0, 1]$ are already fixed, the minimum values $A$ and $D$ respectively for $A(s)$ and $D(s)$, $s \in [0, 1]$, can be easily obtained. Suppose that $B(s) \geq B$, $C(s) \geq C$ for $s \in [0, 1]$ and $B = C$, we would like to find an optimum negative value for $B$ and $C$ by using Theorem 1 so that $P(u, v) \geq 0$ for $u, v \in [0, 1]$ and then by using $B$ and $C$ to determine a less stringent lower bound for the inner Bézier ordinates.

By Theorem 1, for $P(u, s) \geq 0$ it suffices to have

$$3B^2C^2 + 6ABCD - 4(AC^3 + B^3D) - A^2D^2 = 0.$$

This equation is solved for $B$ and $C$ with $B = C$. Observe that $B, C \leq -\ell/3a$.

The lower bound for $b_{1,1}$ and $b_{1,2}$ is obtained from the relation $B(s) \geq B$, *i.e.* $(1-s)^3(b_{1,0} - B) + 3(1-s)^2s(b_{1,1} - B) + 3(1-s)s^2(b_{1,2} - B) + s^3(b_{1,3} - B) \geq 0$. Let $b_{1,1} - B = b_{1,2} - B = m_1$. As $b_{1,0}, b_{1,3} \geq -\ell/3a$, then $(b_{1,0} - B) \geq 0$ and $(b_{1,3} - B) \geq 0$. For the case $(b_{1,0} - B) > 0$ and $(b_{1,3} - B) > 0$, by Theorem 1, in order that $B(s) - B \geq 0$ it suffices to have

$$3m_1^4 + 6m_1^2(b_{1,0} - B)(b_{1,3} - B) - 4m_1^3[(b_{1,0} - B) + (b_{1,3} - B)] - (b_{1,0} - B)^2(b_{1,3} - B)^2 = 0.$$

This equation is solved for the value of $m_1$ which is negative. For the other three cases, *i.e.* $(b_{1,0} - B) = 0$ and $(b_{1,3} - B) > 0$, $(b_{1,0} - B) > 0$ and $(b_{1,3} - B) = 0$, $(b_{1,0} - B) = 0$ and $(b_{1,3} - B) = 0$, by Lemma 1, in order that $B(s) - B \geq 0$ it suffices to have $m_1 = 0$. Thus $B(s) - B \geq 0$ if $b_{1,1} \geq k_B$ and $b_{1,2} \geq k_B$, where $k_B = m_1 + B$ and $m_1 \leq 0$.

Similarly the lower bound $k_C$ for $b_{2,1}$ and $b_{2,2}$ is obtained from the relation $C(s) \geq C$, i.e. $(1-s)^3(b_{2,0} - C) + 3(1-s)^2s(b_{2,1} - C) + 3(1-s)s^2(b_{2,2} - C) + s^3(b_{2,3} - C) \geq 0$. We conclude that by ensuring $b_{1,1} \geq k_B$, $b_{1,2} \geq k_B$, $b_{2,1} \geq k_C$ and $b_{2,2} \geq k_C$, then $P(u, s) \geq 0$, $\forall\, u, s \in [0, 1]$.

In order to obtain the second set of lower bounds, we consider $P$ as a set of curves along line segments parallel to $e_3$, *i.e.* curves of the form given by (2.4). By repeating similar arguments as above, we obtained lower bounds $\ell_B$ and $\ell_C$ such that by ensuring $b_{1,1} \geq \ell_B$, $b_{2,1} \geq \ell_B$, $b_{1,2} \geq \ell_C$ and $b_{2,2} \geq \ell_C$, then $P(s, v) \geq 0$, $\forall\, s, v \in [0,1]$.

Hence, $P(u, v) \geq 0$, $\forall\, u, v \in [0, 1]$ by ensuring

$$b_{1,1} \geq h_1, \quad b_{1,2} \geq h_2, \quad b_{2,1} \geq h_3, \quad b_{2,2} \geq h_4 \qquad (2.11)$$

where $h_1$, $h_2$, $h_3$, $h_4$ are given by one of the following two sets, $Q = \{ h_1 = k_B, h_2 = k_B, h_3 = k_C, h_4 = k_C \}$ or $T = \{ h_1 = \ell_B, h_2 = \ell_C, h_3 = \ell_B, h_4 = \ell_C \}$. Either of these sets of lower bounds could be used.

Observe that indeed $h_1, h_2, h_3, h_4 \leq -\ell/3a$ and so they are less stringent lower bounds for the inner Bézier ordinates than the one stated in Proposition 1.

## 3. Generation of non-negativity preserving interpolating surface

Given functional values $\{ f_{p,q} \}$ at the grid nodes $V_{p,q}(x_p, y_q)$ of a rectangular grid with $f_{p,q} > 0$, $0 \leq p \leq m$, $0 \leq q \leq n$, where $p$, $q$ are positive integers and $x_0 < x_1 < \ldots < x_m$, $y_0 < y_1 < \ldots < y_n$. We would like to construct an interpolating $C^1$ non-negativity preserving functional surface $S(x, y)$ through all the $f_{p,q}$.

To construct the interpolating surface $S$, first we define $S(x_p, y_q) = f_{p,q}$, $p = 1, \cdots, m$, $q = 1, \cdots, n$. The partial derivatives $S_x$ and $S_y$ at each grid node are estimated by using the three points difference approximation. The twist at each grid node is estimated by using Adini's twist [2]. Though the resulting surface is dependent upon the derivative estimation method, we shall focus on the generation of the non-negative surface patches.

On each rectangle in the domain, $S$ will be constructed as a convex combination of two bicubic Bézier patches. These two bicubic Bézier patches, denoted as $P_1$ and $P_2$, will have the same set of boundary Bézier ordinates but they may have different inner Bézier ordinates. Denote the edges of the rectangle under consideration as $e_1$, $e_2$, $e_3$ and $e_4$ (see Figure 1). These patches are constructed such that $P_1$ is $C^1$ with its adjacent patches along the common boundaries $e_3$ and $e_4$ while $P_2$ is $C^1$ with its adjacent patches along the common boundaries $e_1$ and $e_2$. Let the bicubic Bézier rectangular patches $P_k$, $k = 1, 2$ be given as $P_k(u, v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,j}^k B_i^3(u) B_j^3(v)$, $u, v \in [0, 1]$.

It suffices just to consider the patch $P_1$ since $P_1$ and $P_2$ are constructed in the same manner. The Bézier ordinates $b_{0,0}^1$, $b_{1,0}^1$ and $b_{0,1}^1$ are given by

$$b_{0,0}^1 = P_1(0, 0) = S(V_{p,q}) = f_{p,q},$$

$$b_{1,0}^1 = P_1(0,0) + \frac{1}{3}\frac{\partial P_1}{\partial u}(0,0)$$
$$= S(V_{p,q}) + \frac{1}{3}(x_{p+1,q} - x_{p,q})\frac{\partial S}{\partial x}(V_{p,q}),$$
$$b_{0,1}^1 = P_1(0,0) + \frac{1}{3}\frac{\partial P_1}{\partial v}(0,0)$$
$$= S(V_{p,q}) + \frac{1}{3}(y_{p,q+1} - y_{p,q})\frac{\partial S}{\partial y}(V_{p,q})$$

where $V_{p,q}$, $V_{p+1,q}$, $V_{p+1,q+1}$, $V_{p,q+1}$ are the vertices of the domain rectangle under consideration. All the $b_{i,j}^1$ (except $b_{1,1}^1$, $b_{1,2}^1$, $b_{2,1}^1$, $b_{2,2}^1$) are similarly determined.

With these Bézier ordinates, the resulting Bézier patch may not ensure non-negativity. In view of Proposition 1, we shall impose upon these boundary Bézier ordinates the condition $b_{i,j} \geq -\ell/3a$, where $\ell = \min\{S(V_{p,q}), S(V_{p+1,q}), S(V_{p+1,q+1}), S(V_{p,q+1})\}$ and $a$ as described by (2.7). This is achieved by modifying the partial derivatives at the vertices if necessary. The derivatives $S_x$ and $S_y$ at a vertex are modified by scaling each of them with a positive factor $\alpha < 1$. The scaling factor $\alpha$ is obtained by taking into account all the rectangular patches sharing that vertex.

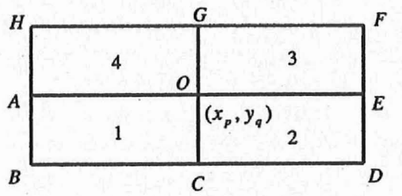


**Figure 2. Vertex *O* with its associated rectangles.**

Consider the four rectangles which share vertex $V_{p,q}(x_p, y_q)$. For simplicity, we denote the vertex $V_{p,q}$ as $O$ and its adjacent vertices as $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$ respectively (see Figure 2).

For rectangle 1, $\alpha_{OA}^1$ is defined as follows. If

$$S(O) - \frac{1}{3}\frac{\partial S}{\partial x}(O)(x_p - x_{p-1}) \geq -\frac{\ell_1}{3a_1},$$

where $\ell_1 = \min\{S(O), S(A), S(B), S(C)\}$ and $a_1$ is as described by (2.7) for rectangle 1, then $\alpha_{OA}^1 = 1$, otherwise $\alpha_{OA}^1$ is defined by the equation

$$S(O) - \frac{1}{3}\alpha_{OA}^1\frac{\partial S}{\partial x}(O)(x_p - x_{p-1}) = -\frac{\ell_1}{3a_1}.$$

Similarly for the scalar $\alpha_{OC}^1$, if

$$S(O) - \frac{1}{3}\frac{\partial S}{\partial y}(O)(y_q - y_{q-1}) \geq -\frac{\ell_1}{3a_1},$$

then $\alpha_{OC}^1 = 1$, otherwise $\alpha_{OC}^1$ is given by the equation

$$S(O) - \frac{1}{3}\alpha_{OC}^1\frac{\partial S}{\partial y}(O)(y_q - y_{q-1}) = -\frac{\ell_1}{3a_1}.$$

Then we define $\alpha_1 = \min\{\alpha_{OA}^1, \alpha_{OC}^1\}$. $\alpha_2$, $\alpha_3$ and $\alpha_4$ are similarly defined for rectangles 2, 3 and 4 respectively. Lastly, in order to fulfill the non-negativity preserving conditions stated in Proposition 1 for all the boundary Bézier ordinates adjacent to $O$, we choose $\alpha_O = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

For the boundary node $U$ which belongs to one or two rectangles of the rectangular grid, $\alpha_U$ is defined in a way similar to that for a node shared by four rectangles. The only difference is that we consider only one or two rectangles instead of four.

If at any node $O$, $\alpha_O < 1$, then the $x$ and $y$ partial derivatives at $O$ are redefined as $\alpha_O$ times the corresponding initial values and the Bézier ordinates adjacent to $O$ in each rectangle are redetermined by using the modified derivatives $S_x(O)$ and $S_y(O)$. The boundary Bézier ordinates of all the Bézier patches are determined by repeating the above process at all the nodes $V_{i,j}$.

Now to define the inner Bézier ordinate $b_W^k$, for $P_k$, $k = 1, 2$ where $W \in \{(1,1), (1,2), (2,1), (2,2)\}$, initial values for $b_W^k$ are determined by the mixed second order partial derivatives for $P_k$. For example, $b_{1,1}^k$ is given by

$$b_{1,1}^k = \frac{1}{9}\frac{\partial^2 P_k}{\partial u \partial v}(0,0) + b_{1,0}^k + b_{0,1}^k - b_{0,0}^k$$
$$= \frac{1}{9}(x_{p+1,q} - x_{p,q})(y_{p,q+1} - y_{p,q})\frac{\partial^2 S}{\partial x \partial y}(V_{p,q})$$
$$+ b_{1,0}^k + b_{0,1}^k - S(V_{p,q})$$

and the expressions for $b_{2,1}^k$, $b_{1,2}^k$ and $b_{2,2}^k$ are similar.

These initial values for the inner Bézier ordinates are the same for both patches $P_1$ and $P_2$. However the resulting Bézier patch with these inner Bézier ordinates may not ensure non-negativity. We modify the inner Bézier ordinates if necessary in order to ensure the surface patch $P_k$ is non-negative and is $C^1$ across two of the boundaries of the rectangle. It suffices to describe the determination of the inner Bézier ordinates for $P_1$.

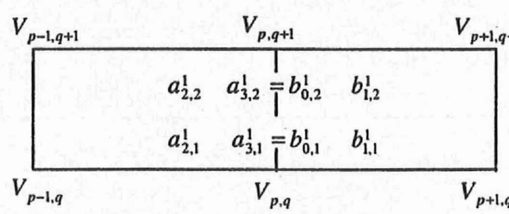


**Figure 3. Two adjacent rectangular patches.**

Consider two adjacent bicubic patches $Q_1$ and $P_1$ with Bézier ordinates $a_{i,j}^1$ and $b_{i,j}^1$ respectively (see Figure 3). With $a_{3,j}^1 = b_{0,j}^1$, $j = 0, 1, 2, 3$, the necessary and sufficient conditions for $C^1$ continuity along the boundaries $V_{p,q}V_{p,q+1}$ where $p \neq 0$ and $p \neq m$ are

$$b_{1,j}^1 = b_{0,j}^1 + h_p \, (a_{3,j}^1 - a_{2,j}^1) / h_{p-1}, \qquad (3.1)$$

where $h_k = x_{k+1} - x_k$, $k = p-1, p$, $j = 0, 1, 2, 3$. Observe that the cases $j = 0$ and $j = 3$ in (3.1) will be automatically fulfilled since the Bézier ordinates adjacent to $V_{p,q}$ and $V_{p,q+1}$ are determined by using the $x$ and $y$ partial derivatives at the corresponding vertices.

If $a_{2,1}^1 < h_3^Q$ while $b_{1,1}^1 \geq h_1^P$, where $h_3^Q$ and $h_1^P$ are the lower bounds obtained in (2.11) for $a_{2,1}^1$ and $b_{1,1}^1$ respectively. We reset $a_{2,1}^1$ to be equal to $h_3^Q$ and make an adjustment to $b_{1,1}^1$ according to (3.1). For the other case when $b_{1,1}^1 < h_1^P$ while $a_{2,1}^1 \geq h_3^Q$, we reset $b_{1,1}^1$ equal to $h_1^P$ and adjust $a_{2,1}^1$ according to (3.1) in order that the two adjacent patches are $C^1$ along common boundary. Observe that the case where $a_{2,1}^1 < h_3^Q$ and $b_{1,1}^1 < h_1^P$ will never occur. Indeed if $a_{2,1}^1 < h_3^Q$, then $a_{2,1}^1 < b_{0,1}^1$, and in order that the $C^1$ condition holds, we have $b_{1,1}^1 \geq b_{0,1}^1$; hence $b_{1,1}^1 \geq h_1^P$. Using the same argument, we examine $a_{2,2}^1$ and $b_{1,2}^1$, and modify if necessary.

Along the boundary $V_{0,q}V_{0,q+1}$, inner Bézier ordinates adjacent to it are determined by ensuring that they satisfy (2.11). We determine inner Bézier ordinates adjacent to boundary $V_{m,q}V_{m,q+1}$ in same manner.

The inner Bézier ordinates for the patch $P_2$ on each rectangle are determined in a similar way, so that $P_2$ is non-negative and $C^1$ with its adjacent patch across the common boundaries $e_1$ and $e_2$. The interpolating surface patch $P$ on the rectangle is defined as a convex combination, that is $P = c_1P_1 + c_2P_2$ where

$$c_1 = v^2(1-v)^2 / (u^2(1-u)^2 + v^2(1-v)^2),$$
$$c_2 = u^2(1-u)^2 / (u^2(1-u)^2 + v^2(1-v)^2).$$

$c_1$ and $c_2$ ensure that $P = P_1$ and $\partial P / \partial u = \partial P_1 / \partial u$ on $e_3$ and $e_4$, $P = P_2$ and $\partial P / \partial v = \partial P_2 / \partial v$ on $e_1$ and $e_2$. Hence $P$ interpolates all the given data at the vertices of the rectangle and is $C^1$ across all its boundaries. The interpolating surface $S$ is the composite surface defined as $S\,|_R = P_R$, where $R$ is a rectangle in the domain and $P_R$ is the patch constructed as described above on the rectangle $R$. Thus $S$ is a non-negativity preserving $C^1$ surface interpolating the given data.

## 4. Generation of range restricted interpolating surface

We would like to extend our scheme for a larger set of constraints besides the plane $z = 0$. We consider the constraint surfaces of the form $z = C(x, y)$ where

$$C(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{i,j} x^i y^j, \; a_{i,j} \text{ are real numbers} \qquad (4.1)$$

because $C(x, y)$ can be expressed as a bicubic Bézier patch on each rectangle in the domain.

Given the functional data points $(x_i, y_j, f_{i,j})$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, lying on one side of the constraint surface $z = C(x, y)$. We would like to generate a $C^1$ interpolating surface $z = S(x, y)$ which lies on the same side of the given constraint surface as the data.

Suppose that the data points lie above the constraint surface. The partial derivatives $S_x$, $S_y$ and $S_{xy}$ at $(x_i, y_j)$ are estimated as in Section 3. Let $G(x, y) = S(x, y) - C(x, y)$. A new set of data points $(x_i, y_j, f_{i,j}^*)$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, is derived from the original data set and the constraint function $C(x, y)$ by letting $f_{i,j}^* = f_{i,j} - C(x_i, y_j)$. Then the construction of the $C^1$ interpolating function $S(x, y)$ subject to the constraint surface $z = C(x, y)$ is reduced to the construction of the function $G(x, y)$ so that it is non-negative and $C^1$, with $G(x_i, y_j) = f_{i,j}^*$. With the initial partial derivatives of $G$ as

$$G_x(x_i, y_j) = S_x(x_i, y_j) - C_x(x_i, y_j),$$
$$G_y(x_i, y_j) = S_y(x_i, y_j) - C_y(x_i, y_j),$$
$$G_{xy}(x_i, y_j) = S_{xy}(x_i, y_j) - C_{xy}(x_i, y_j),$$

they are modified if necessary so that the sufficient non-negativity conditions are fulfilled and the construction of $G$ proceeds as in Section 3. As the non-negativity preserving interpolating surface $G$ is piecewise a convex combination of bicubic Bézier patches, thus so is $S$.

The arguments above have also been extended to admit constraint which is a $C^1$ piecewise polynomial surface consisting of polynomial pieces of the form in (4.1) on the rectangular grid.

## 5. Numerical examples

We shall illustrate the range restricted interpolation scheme with two numerical examples. The linear interpolant to the test data of the first example is shown in Figure 4. The given data lie between the constraint planes z = 6.51 and z = –0.01. The unconstrained interpolating surface which is piecewise a convex combination of two bicubic Bézier rectangular patches is shown in Figure 5. It oscillates and crosses the upper and lower bounding planes. The range restricted interpolating surface in Figure 6 does not oscillate unnecessarily and it lies within the upper and lower bounding planes.

The data of the second example sampled from the surface $z = \cos xy$ lie above the $C^1$ constraint surface

$$z = \begin{cases} -0.28x^2 - 0.3y^2 + 0.9, & (x, y) \in [-2.5, 0] \times [-2.5, 2.5] \\ -0.3x^3 - 0.3y^2 + 0.166x^2 + 0.9, & (x, y) \in [0, 2.5] \times [-2.5, 2.5] \end{cases}.$$

Figure 7 shows the linear interpolant. The unconstrained interpolating surface has crossed the constraint surface as shown in Figure 8 while the constrained interpolating surface in Figure 9 stays above the constraint surface as required.

## Acknowledgements

We gratefully acknowledge the financial support of the Fundamental Research Grant.

## References

[1] Brodlie K. W., Butt S. and Mashwama P. Visualization of Surface Data to Preserve Positivity and Other Simple Constraints. *Computers and Graphics* 19, 585-594, 1995.

[2] Farin G. *Curves and Surfaces for Computer Aided Geometric Design*, 4th ed, Academic Press Inc., 1996.

[3] Goodman T.N.T., Ong B.H. and Unsworth K. Constrained Interpolation Using Rational Cubic Splines. In *Nurbs for Curve and Surface Design*, Farin G. (ed.), SIAM, 59 – 74, 1991.

[4] Mulansky, B. and J. W. Schmidt. Powell-Sabin Splines in Range Restricted Interpolation of Scattered Data. *Computing* 53, 137 – 154, 1994.

[5] Mulansky, B. and Schmidt J.W. Nonnegative Interpolation by Biquadratic Splines on Refined Rectangular Grids. In *Wavelets, Images and Surface Fitting*, P.-J. Laurent, A. Le Mehaute, and L. L. Schumaker (eds.), A K Peters, Wellesley, 379 – 386, 1994.

[6] Schmidt Jochen W. and Heß Walter. Positivity of Cubic Polynomials on Intervals and Positive Spline Interpolation, *Bit* 28, 340 – 362, 1988.

[7] Wong Heng Cheang, Positivity Preserving Surface Interpolation, M. Sc Thesis, Universiti Sains Malaysia, 1994.

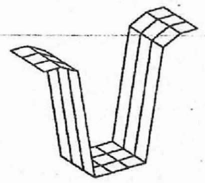

**Figure 4. The linear interpolant.**

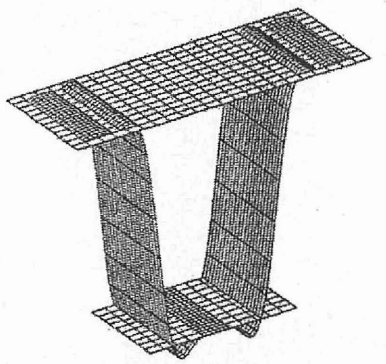

**Figure 5. The unconstrained interpolating surface and the two constraint planes.**

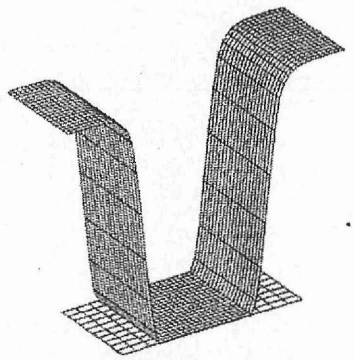

**Figure 6. The constrained interpolating surface (without displaying the upper constraint plane).**

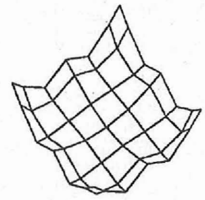

**Figure 7. The linear interpolant.**

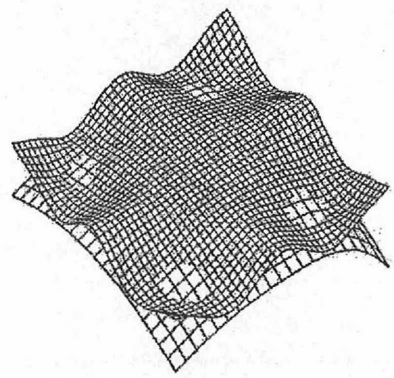

**Figure 8. The unconstrained interpolating surface and the constraint surface.**

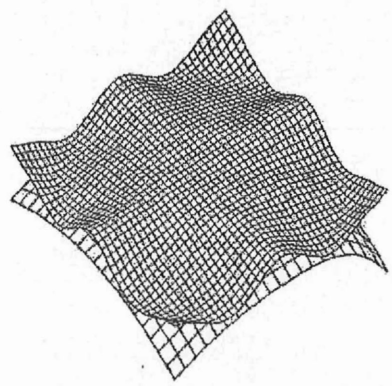

**Figure 9. The constrained interpolating surface and the constraint surface.**