A NEW FINITE DIFFERENCE SCHEME BASED ON CENTROIDAL MEAN AVERAGING FOR THE GOURSAT PROBLEM

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Abstract: The Goursat partial differential equation arises in the simulation and modeling of several physical problems. Several finite difference schemes have been formulated for the Goursat equation. In this paper we develop a new finite difference scheme based on centroidal mean averaging of functional values and apply the scheme to solve two Goursat problems.

Keywords: Goursat problem, Finite difference schemes, Centroidal mean

1 Introduction

The Goursat problem is an initial value problem involving a hyperbolic partial differential equation which arises in several areas of physics and engineering. The standard numerical procedure is the use of a finite difference scheme with arithmetic mean averaging of functional values (Wazwaz, 1993). Wazwaz (1993), introduced a scheme based on harmonic mean averaging of functional values. Evans & Sanugi (1988), introduced a scheme based on geometric mean averaging of functional values. Comparative studies of these schemes have been carried out by Wazwaz (1993) as well as Ismail & Nasir (2004). In general the arithmetic mean scheme produces more accurate results and for linear problems it has an advantage in that the linearity of the scheme is preserved (i.e. no iterations are required).

In this paper we develop a scheme based on centroidal mean averaging of functional values (Centroidal Scheme), study its implementation and compare the results with that of a scheme based on arithmetic mean averaging of functional values (Arithmetic Scheme) on two Goursat problems.

2 The Centroidal Mean and Finite Difference Scheme

Let a and c be two natural numbers. Then the centroidal mean of a and c is (Faradj, 2004):

$$b = \frac{2(a^2 + ac + c^2)}{3(a + c)} \qquad \ldots(1)$$

The Goursat problem is of the form:

$$u_{xy} = f(x, y, u, u_x, u_y)$$
$$u(x,0) = \phi(x), u(0,y) = \psi(y), \phi(0) = \psi(0)$$
$$0 \le x \le a, 0 \le y \le b$$

...(2)

We consider the Goursat partial differential equations $u_{xy} = f$. The L.H.S is discretized as follows:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right)$$
$$= \frac{1}{\Delta x}\frac{\partial}{\partial y}\left(u(x+\Delta x, y) - u(x,y)\right)$$
$$= \frac{1}{\Delta x}\left[\frac{(u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y))}{\Delta y} - \frac{(u(x, y+\Delta y) - u(x,y))}{\Delta y}\right]$$
$$= \frac{1}{h^2}\left[u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y)) - u(x, y+\Delta y) + u(x,y)\right]$$
where $h = \Delta x = \Delta y$

...(3)

Hence equation (3) becomes:

$$u_{xy} = \frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2}$$

...(4)

The R.H.S is approximated at (i + 1/2, j + 1/2) as follows:

$$f_{i+\frac{1}{2}, j+\frac{1}{2}} = \text{Centroidal mean}\left(\text{Centroidal mean } f_{i+1,j} \text{ and } f_{i,j+1} ; \text{Centroidal mean } f_{i+1,j+1} \text{ and } f_{i,j}\right)$$

...(5)

From (5) we obtain:

$$f_{i+\frac{1}{2}, j+\frac{1}{2}} =$$

$$\frac{4}{9}\left[\frac{\left(\frac{f_{i+1,j+1}^2 + f_{i+1,j+1}f_{i,j} + f_{i,j}^2}{f_{i+1,j+1} + f_{i,j}}\right)^2 + \left(\frac{f_{i+1,j+1}^2 + f_{i+1,j+1}f_{i,j} + f_{i,j}^2}{f_{i+1,j+1} + f_{i,j}}\right)\left(\frac{f_{i+1,j}^2 + f_{i+1,j}f_{i,j+1} + f_{i,j+1}^2}{f_{i+1,j} + f_{i,j+1}}\right) + \left(\frac{f_{i+1,j}^2 + f_{i+1,j}f_{i,j+1} + f_{i,j+1}^2}{f_{i+1,j} + f_{i,j+1}}\right)^2}{\left(\frac{f_{i+1,j+1}^2 + f_{i+1,j+1}f_{i,j} + f_{i,j}^2}{f_{i+1,j+1} + f_{i,j}}\right) + \left(\frac{f_{i+1,j}^2 + f_{i+1,j}f_{i,j+1} + f_{i,j+1}^2}{f_{i+1,j} + f_{i,j+1}}\right)}\right]$$

...(6)

Thus the Centroidal Scheme for the Goursat problem is given by:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=$$

$$\frac{4}{9}\left[\frac{\left(\frac{f_{i+1,j+1}^2+f_{i+1,j+1}f_{i,j}+f_{i,j}^2}{f_{i+1,j+1}+f_{i,j}}\right)^2+\left(\frac{f_{i+1,j+1}^2+f_{i+1,j+1}f_{i,j}+f_{i,j}^2}{f_{i+1,j+1}+f_{i,j}}\right)\left(\frac{f_{i+1,j}^2+f_{i+1,j}f_{i,j+1}+f_{i,j+1}^2}{f_{i+1,j}+f_{i,j+1}}\right)+\left(\frac{f_{i+1,j}^2+f_{i+1,j}f_{i,j+1}+f_{i,j+1}^2}{f_{i+1,j}+f_{i,j+1}}\right)^2}{\left(\frac{f_{i+1,j+1}^2+f_{i+1,j+1}f_{i,j}+f_{i,j}^2}{f_{i+1,j+1}+f_{i,j}}\right)+\left(\frac{f_{i+1,j}^2+f_{i+1,j}f_{i,j+1}+f_{i,j+1}^2}{f_{i+1,j}+f_{i,j+1}}\right)}\right]$$

...(7)

The Arithmetic Scheme for the Goursat problem is given by Wazwaz (1993):

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=\frac{1}{4}\left(f_{i+1,j+1}+f_{i,j}+f_{i+1,j}+f_{i,j+1}\right)$$

...(8)

Here the L.H.S of $u_{xy}=f$ is discretized as before and the discretization of the R.H.S is obtained as follows:

$$f_{i+\frac{1}{2},j+\frac{1}{2}}=\text{Arithmetic mean}\left(\text{Arithmetic mean } f_{i+1,j} \text{ and } f_{i,j+1}\text{; Arithmetic mean } f_{i+1,j+1} \text{ and } f_{i,j}\right)$$

3 Numerical Experiments

We consider the non linear Goursat problem:

$$u_{xy}=e^{2u}$$

$$u(x,0)=\frac{x}{2}-\ln(1+e^x)$$

$$u(0,y)=\frac{y}{2}-\ln(1+e^y)$$

$$0\le x\le 4, 0\le y\le 4$$

...(9)

The analytical solution for (9) can be found in Wazwaz (1993).

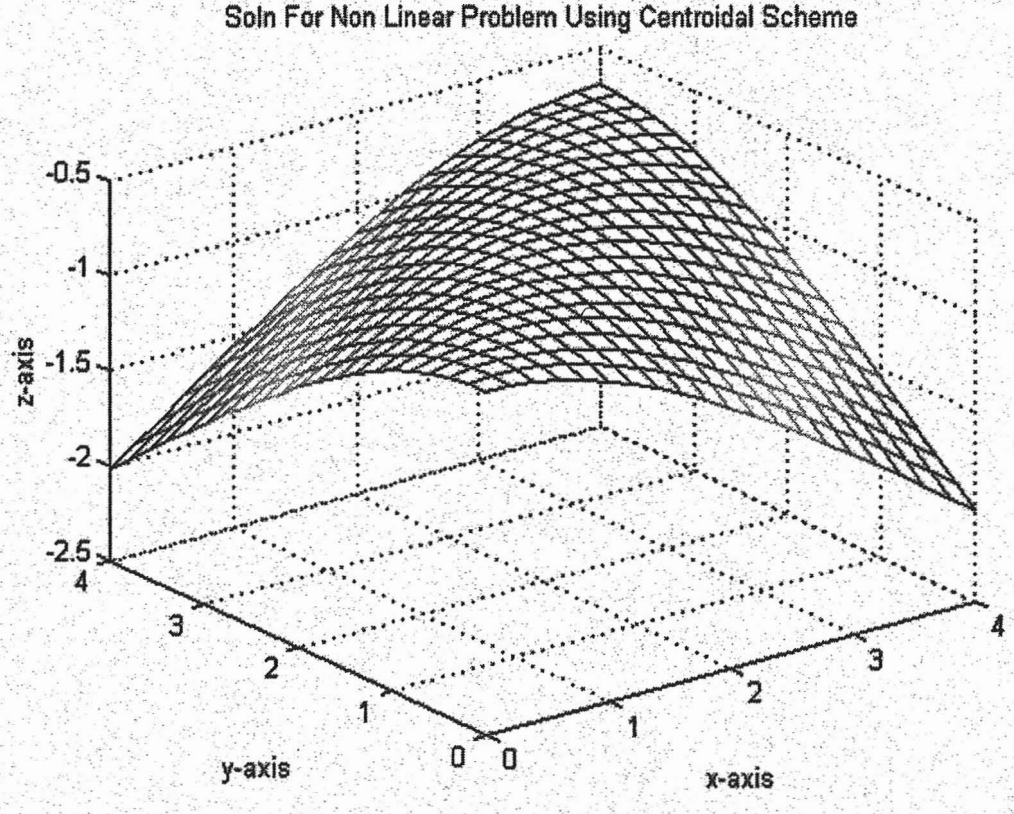
A computer program to implement the Centroidal Scheme was developed. For the non linear Goursat problem (9) with
h = 0.05, we obtained

RELATIVE ERRORS FOR CENTROIDAL SCHEME

Y \ X	1.0	2.0	3.0	4.0
1.0	6.7089767e-005	6.9078097e-005	1.9699678e-005	4.7306911e-006
2.0	6.9078097e-005	2.1450144e-004	1.8999699e-004	9.1226313e-005
3.0	1.9699678e-005	1.8999699e-004	4.3252802e-004	4.1681908e-004
4.0	4.7306911e-006	9.1226313e-005	4.1681908e-004	8.9231928e-004

Average error of the Centroidal Scheme = 1.3000668e-004

Fig.1: Solution in graphic form for problem (9)



RELATIVE ERRORS FOR ARITHMETIC SCHEME

X

Y	1.0	2.0	3.0	4.0
1.0	7.2890713e-005	9.7076691e-005	6.4976523e-005	4.0385671e-005
2.0	9.7076691e-005	2.9467327e-004	3.4128209e-004	2.5890833e-004
3.0	6.4976523e-005	3.4128209e-004	8.0424181e-004	9.5132232e-004
4.0	4.0385671e-005	2.5890833e-004	9.5132232e-004	2.1149666e-003

Average error of arithmetic scheme = 2.5268253e-004

For grid sizes h = 0.025, 0.100, 0.200 and 0.500 we obtained the following results

	h = 0.025	h = 0.100	h = 0.200	h = 0.500
Average error of Centroidal scheme	3.2117909e-005	5.3298255e-004	2.2472913e-003	1.6844006e-002
Average error of Arithmetic scheme	6.2298718e-005	1.0387387e-003	4.3798764e-003	3.1510452e-002

We consider the derivative linear Goursat problem below:

$$u_{xy} = \frac{u_x + u_y + u}{3}$$

$$u(x,0) = e^x$$

$$u(0,y) = e^y$$

$$0 \le x \le 4,\ 0 \le y \le 4$$

...(10)

The analytical solution for (10) can be found in Day (1966).

A computer program to implement the Centroidal Scheme was developed. For the derivative linear Goursat problem (10) with h = 0.0125, we obtained

RELATIVE ERRORS FOR CENTROIDAL SCHEME

X

Y	0.5	1.0	1.5	2.0
0.5	7.6361544e-004	1.3389312e-003	1.7720348e-003	2.0978319e-003
1.0	1.3395230e-003	2.3944425e-003	3.2231452e-003	3.8726349e-003
1.5	1.7735183e-003	3.2244443e-003	4.4061765e-003	5.3647931e-003
2.0	2.1003223e-003	3.8756082e-003	.3667861e-003	6.6123506e-003

Average error of the Centroidal Scheme = 2.1333016e-003

RELATIVE ERRORS FOR ARITHMETIC SCHEME

X

Y	0.5	1.0	1.5	2.0
0.5	7.6521334e-004	1.3420271e-003	1.7764802e-003	2.1034593e-003
1.0	1.3420271e-003	2.3994449e-003	3.2305250e-003	3.8822038e-003
1.5	1.7764802e-003	3.2305250e-003	4.4153686e-003	5.3769759e-003
2.0	2.1034593e-003	3.8822038e-003	5.3769759e-003	6.6261263e-003

Average error of Arithmetic Scheme = 2.1377548e-003

For grid sizes h = 0.005, 0.01, 0.02 and 0.025 we obtained the following results

	h = 0.005	h = 0.01	h = 0.02	h = 0.025
Average error of Centroidal Scheme	8.5423982e-004	1.7072556e-003	3.4095780e-003	4.2588655e-003
Average error of Arithmetic Scheme	8.5495225e-004	1.7101055e-003	3.4209800e-003	4.2766835e-003

4 Conclusions

In this paper we have developed a new scheme for the Goursat problem. We have implemented the scheme on a non linear Goursat problem and a derivative linear Goursat problem and found the scheme to be more accurate than the Arithmetic Scheme (for these two problem). However, we note that this new scheme is non linear and thus, even for, linear Goursat partial differential equation, would require iteration.

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