UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2004/2005

October 2004

MAT 518-NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
[KAEDAH BERANGKA UNTUK PERSAMAAN PEMBEZAAN]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of **SIX** pages of printed material before begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **ENAM** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Answer **all FOUR** questions. All questions carry the same marks.

*Jawab **semua EMPAT** soalan. Semua soalan membawa jumlah markah yang sama.*

1. (a) Derive the Crank-Nicolson scheme for the heat equation $U_t = U_{xx}$. Investigate the consistency of the scheme.

   (b) Derive the CTCS scheme for the wave equation $U_{tt} = U_{xx}$. Investigate the consistency of the scheme.

[100 marks]

1. (a) Terbitkan skema Crank-Nicolson untuk persamaan haba $U_t = U_{xx}$. Semak *kekonsistenaan skema tesebut.*

   (b) Terbitkan skema CTCS untuk persamaan gelombang $U_{tt} = U_{xx}$. Semak *kekonsistenaan skema tersebut.*

[100 markah]

2. (a) Consider the one-dimensional linear convection equation $C_t + UC_x = 0$ where the symbols have their usual meanings. Write the FTCS scheme for this equation. Analyze its stability using the von Neumann method.

   (b) Consider the one-dimensional transport equation $C_t + UC_x - \alpha C_{xx} = 0$ where the symbols have their usual meanings. Write the FTCS scheme for this equation. Analyze its stability using the von Neumann method.

[100 marks]

2. (a) Pertimbangkan persamaan olahan (convection) linear satu dimensi $C_t + UC_x - \alpha C_{xx} = 0$ *di mana simbol-simbol mempunyai maksud yang biasa. Tulis skema FTCS untuk persamaan ini. Analisis stability skema ini menggunakan kaedah von Neumann.*

   (b) Pertimbangkan persamaan pengangkutan satu dimensi $C_t + UC_x - \alpha C_{xx} = 0$ *di mana simbol-simbol mempunyai maksud yang biasa. Tulis skema FTCS untuk persamaan ini. Analisis stability skema ini menggunakan kaedah von Neumann.*

[100 markah]

3. (a) Consider the Poisson equation

$$\nabla^2 u = xy(x-2)(y-2)$$

on the region $0 \le x \le 2,\ 0 \le y \le 2$ with u = 0 on all boundaries except for y=0, where u = 1.0.

Suppose the solution domain is discretised as shown in Figure 1. Assume that $\Delta x = \Delta y = h$.

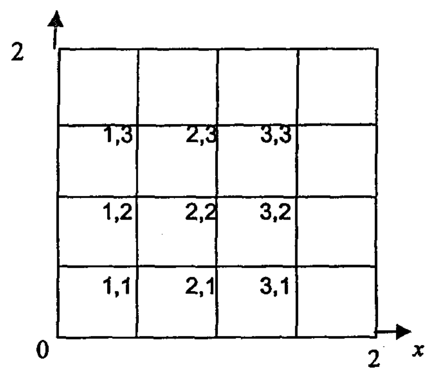


Figure 1

i. Write down the five-point difference equations approximating this problem at all the internal mesh points
ii. Write down the resulting system in the form $A\underline{u} = \underline{b}$.
iii. Calculate the spectral radius $\rho(B)$; where B is the Jacobi iteration matrix, when n = 25.
iv. Calculate the optimal relaxation parameter ω_b which maximises the rate of convergence of the S.O.R. method
v. Compute the spectral radius $\rho(L_{\omega_b})$, where L_{ω_b} is the S.O.R. iteration matrix
vi. What is the approximate number of iterations you would expect to get if the Jacobi, Gauss-Seidel and S.O.R. methods are used for mesh sizes n = 25 and tolerance $\varepsilon = 10^{-7}$.

(b) Prove that the eigenvalues λ of the S.O.R. iteration matrix in solving the system $A\underline{u} = \underline{b}$ are the roots of

$$\det\{(\lambda + \omega - 1)D - \lambda\omega L - \omega U\} = 0.$$

Here, $A = D\text{-}L\text{-}U$ with D, L, and U being diagonal, lower triangular and upper triangular matrices respectively.

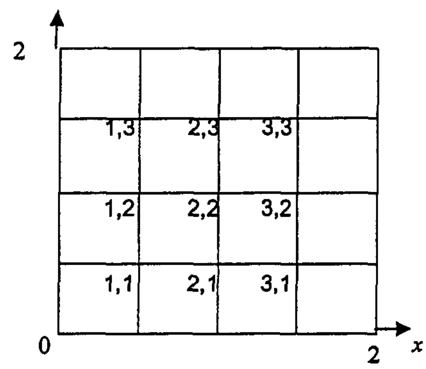
(100 marks)

3. (a) *Pertimbangkan persamaan Poisson*

$$\nabla^2 u = xy(x-2)(y-2)$$

pada rantau $0 \le x \le 2,\ 0 \le y \le 2$ *dengan* $u = 0$ *pada semua sempadan kecuali pada y=0, di mana u = 1.0.*

Katakan domain penyelesaian didiskretkan seperti ditunjukkan pada Rajah 1. Anggapkan $\Delta x = \Delta y = h$.



Rajah 1

i. *Tuliskan persamaan beza lima-titik yang menganggar masalah ini pada semua titik mesh dalaman.*
ii. *Tuliskan sistem yang terhasil dalam bentuk* $A\underline{u} = \underline{b}$.
iii. *Kirakan jejari spektrum* $\rho(B)$; *di mana B ialah matriks lelaran Jacobi, apabila* $n = 25$.
iv. *Kirakan parameter pengenduran optimum* ω_b *yang memaksimumkan kadar penumpuan kaedah S.O.R.*
v. *Kirakan jejari spektrum* $\rho(L_{\omega_b})$, *di mana* L_{ω_b} *ialah matriks lelaran S.O.R.*
vi. *Apakah anggaran bilangan lelaran yang anda dapat jangkakan jika kaedah Jacobi, Gauss-Seidel dan S.O.R. digunakan untuk saiz mesh* $n = 25$ *dan toleran* $\varepsilon = 10^{-7}$.

(b) *Buktikan bahawa nilai eigen* λ *bagi matriks lelaran S.O.R. dalam menyelesaikan sistem* $A\underline{u} = \underline{b}$ *ialah punca persamaan*

$$pen\{(\lambda + \omega - 1)D - \lambda\omega L - \omega U\} = 0$$

Di sini A = D-L-U dengan D, L, dan U adalah matriks pepenjuru, segitiga bawah dan segitiga atas masing-masing.

(100 markah)

4. (a) Suppose the matrix coefficient of the system $A\underline{u} = \underline{b}$ is

$$A = \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}, \qquad \text{with} \qquad \underline{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

i. Write down the formula of the Simultaneous Displacement method (with extrapolated constant α) to solve this system
ii. What is the rate of convergence, R_∞, of this iterative method in solving this system?

iii. Using the initial vector $\underline{u}^{(0)} = (1,1,1)^T$ as the starting value, generate two iterations ($\underline{u}^{(1)}$ and $\underline{u}^{(2)}$) of the method for this 3x3 system.

iv. Write down the formula for the second order Richardson's method to solve this system

v. What is the rate of convergence, R_∞, of this iterative method?

vi. Using the same initial vector as in (iii), generate two iterations ($\underline{u}^{(1)}$ and $\underline{u}^{(2)}$) of the Richardson's method for this 3x3 system.

(b) The linear system of equations

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \underline{x} = \underline{b}$$

where a is real, can under certain conditions be solved by the iterative method

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix} \underline{x}^{(k+1)} = \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix} \underline{x}^{(k)} + \omega \underline{b}$$

i. For which values of a is the method convergent when $\omega = 1$?

ii. For $a = 0.5$, find the value of ω which minimizes the spectral radius of the matrix

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}$$

(100 marks)

4. (a) Katakan matriks koefisien bagi sistem $A\underline{u} = \underline{b}$ adalah

$$A = \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}, \qquad \textit{dengan} \qquad \underline{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

i. Tuliskan rumus kaedah 'Simultaneous Displacement' (dengan pemalar ekstrapolasi α) untuk menyelesaikan sistem ini.

ii. Apakah kadar penumpuan, R_∞, bagi kaedah ini dalam menyelesaikan sistem tersebut?

iii. Dengan menggunakan vektor nilai awal $\underline{u}^{(0)} = (1,1,1)^T$ sebagai nilai permulaan, janakan dua lelaran ($\underline{u}^{(1)}$ dan $\underline{u}^{(2)}$) kaedah ini bagi sistem 3x3 ini.

iv. Tuliskan rumus kaedah Richardson peringkat dua untuk menyelesaikan system ini.

v. Apakah kadar penumpuan, R_∞, bagi kaedah ini?

vi. *Dengan menggunakan vektor nilai awal yang sama seperti (iii), janakan dua lelaran ($\underline{u}^{(1)}$ dan $\underline{u}^{(2)}$) kaedah Richardson ini bagi sistem 3x3 ini.*

(b) *Sistem persamaan linear*

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \underline{x} = \underline{b}$$

di mana a adalah nyata, boleh di bawah syarat tertentu diselesaikan dengan kaedah lelaran

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix} \underline{x}^{(k+1)} = \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix} \underline{x}^{(k)} + \omega \underline{b}$$

i. *Bagi nilai a yang manakah kaedah ini akan menumpu untuk $\omega = 1$?*

ii. *Bagi $a = 0.5$, dapatkan nilai ω yang meminimumkan jejari spektrum matriks*

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{pmatrix}$$

(100 markah)

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