

UNIVERSITI SAINS MALAYSIA

First Semester Examination  
Academic Session 2004/2005

*October 2004*

**MAT 517 – COMPUTATIONAL LINEAR ALGEBRA AND FUNCTION  
APPROXIMATION  
[ALJABAR LINEAR PENGKOMPUTERAN DAN PENGHAMPIRAN FUNGSI]**

*Duration : 3 hours*

[Masa : 3 jam]

Please check that this examination paper consists of **EIGHT [8]** pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN [8]** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Answer all **FOUR [4]** questions.

Jawab semua **EMPAT [4]** soalan.

1. The Jacobi iterative method for finding an approximate solution to an  $n \times n$  linear system of the form  $\mathbf{Ax} = \mathbf{b}$  generates a sequence of approximation to  $\mathbf{x}$ , namely  $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ , by employing the following recursive equation to obtain the  $i$ th component of  $\mathbf{x}^{(k)}$ ,

$$x_i^{(k)} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^n (-a_{ij}x_j^{(k-1)}) + b_i}{a_{ii}}, \quad \text{for } i = 1, \dots, n. \quad (1.1)$$

where  $\mathbf{A} = [a_{ij}]$ ,  $\mathbf{x}^{(k)} = [x_i^{(k)}]$  and  $\mathbf{b} = [b_i]$ ,  $1 \leq i, j \leq n$ ,  $k \geq 1$ .

- (a) Show that the Jacobi iteration can be written in the form

$$\mathbf{x}^{(k)} = \mathbf{T}_J \mathbf{x}^{(k-1)} + \mathbf{c}_J,$$

where  $\mathbf{T}_J = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$  and  $\mathbf{c}_J = \mathbf{D}^{-1}\mathbf{b}$  such that  $\mathbf{D}$  is a diagonal matrix whose diagonal entries are those of  $\mathbf{A}$ ,  $-\mathbf{L}$  is the strictly lower triangular part of  $\mathbf{A}$  and  $-\mathbf{U}$  is the strictly upper triangular part of  $\mathbf{A}$ .

- (b) Modify equation (1.1) to give the Gauss-Seidel iterative technique and show that the Gauss-Seidel technique can be written in the form

$$\mathbf{x}^{(k)} = \mathbf{T}_G \mathbf{x}^{(k-1)} + \mathbf{c}_G$$

where  $\mathbf{T}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$  and  $\mathbf{c}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{b}$ , with matrices  $\mathbf{D}$ ,  $\mathbf{L}$  and  $\mathbf{U}$  are the same as given in part a).

- (c) Given that  $\mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1 \end{pmatrix}$ , find the spectral radius of  $\mathbf{T}_J$  and  $\mathbf{T}_G$  and, without solving the system  $\mathbf{Ax} = \mathbf{b}$ , confirm that both Jacobi and Gauss-Seidel iterative technique will converge to the unique solution of the system  $\mathbf{Ax} = \mathbf{b}$ , for any choice of  $\mathbf{b}$ .

[100 marks]

1. *Kaedah lelaran Jacobi untuk mencari penghampiran kepada penyelesaian sistem linear  $n \times n$  dalam bentuk  $\mathbf{Ax} = \mathbf{b}$ , membentuk satu jujukan penghampiran kepada  $\mathbf{x}$ , iaitu  $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ , melalui penggunaan persamaan rekursif berikut untuk mendapatkan komponen ke-  $i$  vektor  $\mathbf{x}^{(k)}$ ,*

$$x_i^{(k)} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^n (-a_{ij}x_j^{(k-1)}) + b_i}{a_{ii}}, \quad \text{untuk } i = 1, \dots, n.$$

(1.1)

di mana  $\mathbf{A} = [a_{ij}]$ ,  $\mathbf{x}^{(k)} = [x_i^{(k)}]$  dan  $\mathbf{b} = [b_i]$ ,  $1 \leq i, j \leq n$ ,  $k \geq 1$ .

- (a) Tunjukkan bahawa lelaran Jacobi boleh ditulis dalam bentuk  

$$\mathbf{x}^{(k)} = \mathbf{T}_J \mathbf{x}^{(k-1)} + \mathbf{c}_J,$$
 di mana  $\mathbf{T}_J = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$  dan  $\mathbf{c}_J = \mathbf{D}^{-1}\mathbf{b}$ ,  $\mathbf{D}$  ialah matriks pepenjuru yang mempunyai pemasangan pepenjuru yang sama dengan pepenjuru  $\mathbf{A}$ ,  $-\mathbf{L}$  ialah bahagian segitiga bawah tegas  $\mathbf{A}$  dan  $-\mathbf{U}$  ialah bahagian segitiga atas tegas  $\mathbf{A}$ .
- (b) Ubahsui persamaan (1.1) untuk memberikan teknik lelaran Gauss-Seidel dan tunjukkan bahawa teknik Gauss-Seidel boleh ditulis dalam bentuk  

$$\mathbf{x}^{(k)} = \mathbf{T}_G \mathbf{x}^{(k-1)} + \mathbf{c}_G$$
 di mana  $\mathbf{T}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U}$  dan  $\mathbf{c}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{b}$ . Matriks  $\mathbf{D}$ ,  $\mathbf{L}$  dan  $\mathbf{U}$  sama seperti yang diberikan dalam bahagian a).
- (c) Diberikan  $\mathbf{A} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1 \end{pmatrix}$ , cari jejari spektrum  $\mathbf{T}_J$  dan  $\mathbf{T}_G$ , dan, tanpa menyelesaikan sistem  $\mathbf{Ax} = \mathbf{b}$ , sahkan bahawa kedua-dua teknik lelaran Jacobi dan Gauss-Seidel akan menumpu kepada penyelesaian unik system tersebut untuk sebarang pilihan  $\mathbf{b}$ .

[100 markah]

2. Consider the regression model of the form  $\mathbf{b} = \mathbf{Ax}$  for some  $m \times n$  data matrix  $\mathbf{A}$ ,  $m \times 1$  observation vector  $\mathbf{b}$  and  $n \times 1$  vector of independent variables  $\mathbf{x}$ .
- (a) Show that  

$$\|\mathbf{Ax} - \mathbf{b}\|_2^2 = \mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{A}^T \mathbf{b} - \mathbf{b}^T \mathbf{Ax} + \mathbf{b}^T \mathbf{b}.$$
 (Note that for any vector  $\mathbf{y}$ ,  $\|\mathbf{y}\|_2^2 = \mathbf{y}^T \mathbf{y}$ ).
- (b) Show that both  $\mathbf{AA}^T$  and  $\mathbf{A}^T \mathbf{A}$  are square and symmetric.
- (c) Find the value  $\mathbf{x}_{LS}$  that minimizes squared error norm  $\|\mathbf{Ax} - \mathbf{b}\|_2^2$  by considering the partial derivative of  $\|\mathbf{Ax} - \mathbf{b}\|_2^2$  with respect to  $\mathbf{x}$  and show that

$$\mathbf{x}_{LS} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.$$

(HINT: Solve for  $\mathbf{x}$  such that  $\frac{\partial}{\partial \mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2 = \mathbf{0}$ )

- (d) Let  $A = U\Sigma V^T$  be the singular value decomposition of matrix  $A$  so that  $U$  and  $V$  are orthogonal matrices and  $\Sigma$  is a diagonal matrix whose diagonal elements are the singular values of  $A$ . By considering matrices  $AA^T$  and  $A^T A$  and their properties, describe how  $U, V$  and  $\Sigma$  can be obtained. Hence show that

$$\mathbf{x}_{LS} = V\Sigma^{-1}U^T \mathbf{b}$$

[100 marks]

2. Pertimbangkan model regresi dalam bentuk  $\mathbf{b} = A\mathbf{x}$  untuk suatu matriks data  $m \times n$   $A$ , vektor pemerhatian  $m \times 1$   $\mathbf{b}$  dan vektor pembolehubah bebas  $n \times 1$   $\mathbf{x}$ .

- (a) Tunjukkan bahawa

$$\|A\mathbf{x} - \mathbf{b}\|_2^2 = \mathbf{x}^T A^T A \mathbf{x} - \mathbf{x}^T A^T \mathbf{b} - \mathbf{b}^T A \mathbf{x} + \mathbf{b}^T \mathbf{b}.$$

(Perhatikan bahawa untuk sebarang vektor  $\mathbf{y}$ ,  $\|\mathbf{y}\|_2^2 = \mathbf{y}^T \mathbf{y}$ ).

- (b) Tunjukkan bahawa kedua-dua  $AA^T$  dan  $A^T A$  adalah bersegiempatsama dan bersimetri.

- (c) Cari nilai  $\mathbf{x}_{LS}$  yang meminimumkan norma ralat kuasa dua  $\|A\mathbf{x} - \mathbf{b}\|_2^2$  dengan mempertimbangkan beza separa  $\|A\mathbf{x} - \mathbf{b}\|_2^2$  terhadap  $\mathbf{x}$  dan tunjukkan bahawa

$$\mathbf{x}_{LS} = (A^T A)^{-1} A^T \mathbf{b}.$$

(PETUNJUK: Selesaikan persamaan  $\frac{\partial}{\partial \mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2 = \mathbf{0}$  untuk  $\mathbf{x}$ )

- (d) Biar  $A = U\Sigma V^T$  menjadi penghuraian nilai singular matriks  $A$  yang mana matriks  $U$  dan  $V$  adalah berortogon dan  $\Sigma$  ialah matriks pepenjuru dengan memasukkan pepenjurnya adalah nilai-nilai singular matriks  $A$ . Dengan mempertimbangkan matriks  $AA^T$ ,  $A^T A$  dan ciri-ciri matriks tersebut, huraikan bagaimana  $U, V$  dan  $\Sigma$  boleh diperolehi. Seterusnya tunjukkan bahawa

$$\mathbf{x}_{LS} = V\Sigma^{-1}U^T \mathbf{b}$$

[100 markah]

3. (a) The  $n$ th degree Lagrange interpolating polynomial for some function  $f$  is defined as

$$P(x) = f(x_0)L_{n,0}(x) + \dots + f(x_n)L_{n,n}(x)$$

where, for each  $k = 0, 1, \dots, n$ ,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

and  $x_0, x_1, \dots, x_n$  are  $k+1$  distinct numbers such that

$$f(x_k) = P(x_k) \text{ for each } k = 0, 1, \dots, n.$$

- (i) Let  $P_3(x)$  be the Lagrange interpolating polynomial for the data  $(0,0)$ ,  $(0.5, y)$ ,  $(1,3)$  and  $(2,2)$ . Find  $y$  if the coefficient of  $x^3$  in  $P_3(x)$  is 5.
- (ii) For the function  $f(x) = \sqrt{1+x}$  let  $x_0 = 0$ ,  $x_1 = 0.6$  and  $x_2 = 0.9$ . Construct the Lagrange interpolating polynomial of degree 2 to approximate  $f(0.45)$  and find the actual error.
- (b) The Hermite polynomial of degree at most  $2n+1$  for some function  $f \in C^1[a, b]$  is given by

$$H_{2n+1}(x) = \sum_{j=0}^n f(x_j)H_{n,j}(x) + \sum_{j=0}^n f'(x_j)\hat{H}_{n,j}(x),$$

where

$$H_{n,j}(x) = [1 - 2(x-x_j)L'_{n,j}(x_j)]L^2_{n,j}(x)$$

and

$$\hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x),$$

with distinct numbers  $x_0, x_1, \dots, x_n \in [a, b]$ .  $L_{n,j}(x)$  denotes the  $j$ th Lagrange coefficient of degree  $n$  as defined in part a).

- (i) Show that the Hermite polynomial with definition given above agrees with  $f$  at  $x_0, x_1, \dots, x_n$  (i.e. show that  $f(x_k) = H_{2n+1}(x_k)$  for each  $k = 0, 1, \dots, n$ ).
- (ii) Show that the Hermite polynomial with definition given above agrees with  $f'$  at  $x_0, x_1, \dots, x_n$  (i.e. show that  $f'(x_k) = H'_{2n+1}(x_k)$  for each  $k = 0, 1, \dots, n$ ).
- (iii) Show that  $H_{2n+1}(x)$  is the unique polynomial of least degree agreeing with  $f$  and  $f'$  at  $x_0, x_1, \dots, x_n$ . (HINT: Assume that  $P(x)$  is another such polynomial then consider  $D = H_{2n+1} - P$  and  $D'$  at  $x_0, x_1, \dots, x_n$ ).

[100 marks]

3. (a) *Polinomial interpolasi Lagrange berdarjah  $n$  untuk sesuatu fungsi ditakrifkan sebagai*

$$P(x) = f(x_0)L_{n,0}(x) + \dots + f(x_n)L_{n,n}(x)$$

di mana, untuk setiap  $k = 0, 1, \dots, n$ ,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

dan  $x_0, x_1, \dots, x_n$  adalah  $k+1$  nombor yang berbeza supaya

$$f(x_k) = P(x_k) \text{ untuk setiap } k = 0, 1, \dots, n.$$

- (i) *Biar  $P_3(x)$  mewakili polinomial interpolasi Lagrange untuk data  $(0,0)$ ,  $(0.5, y)$ ,  $(1,3)$  dan  $(2,2)$ . Cari  $y$  jika pekali  $x^3$  dalam  $P_3(x)$  ialah 5.*
- (ii) *Untuk fungsi  $f(x) = \sqrt{1+x}$  biar  $x_0 = 0$ ,  $x_1 = 0.6$  dan  $x_2 = 0.9$ . Bina polinomial interpolasi Lagrange berdarjah 2 untuk mendapatkan penghampiran kepada  $f(0.45)$  dan cari ralat sebenar penghampiran tersebut.*
- (b) *Polinomial Hermite berdarjah paling tinggi  $2n+1$  untuk sesuatu fungsi  $f \in C^1[a, b]$  diberikan oleh*

$$H_{2n+1}(x) = \sum_{j=0}^n f(x_j)H_{n,j}(x) + \sum_{j=0}^n f'(x_j)\hat{H}_{n,j}(x),$$

di mana

$$H_{n,j}(x) = [1 - 2(x-x_j)L'_{n,j}(x_j)]L^2_{n,j}(x)$$

dan

$$\hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x),$$

dengan nombor-nombor berbeza  $x_0, x_1, \dots, x_n \in [a, b]$ .  $L_{n,j}(x)$  melambangkan pekali Lagrange ke  $j$  berdarjah  $n$  dan ditakrifkan seperti dalam bahagian a).

- (i) *Tunjukkan bahawa polinomial Hermite dengan takrifan di atas menyamai nilai  $f$  pada  $x_0, x_1, \dots, x_n$  (dalam perkataan lain tunjukkan bahawa  $f(x_k) = H_{2n+1}(x_k)$  untuk setiap  $k = 0, 1, \dots, n$ ).*
- (ii) *Tunjukkan bahawa polinomial Hermite dengan takrifan di atas menyamai nilai  $f'$  pada  $x_0, x_1, \dots, x_n$  (dalam perkataan lain tunjukkan bahawa  $f'(x_k) = H'_{2n+1}(x_k)$  untuk setiap  $k = 0, 1, \dots, n$ ).*
- (iii) *Tunjukkan bahawa  $H_{2n+1}(x)$  ialah polinomial unik berdarjah paling rendah yang menyamai nilai  $f$  dan  $f'$  pada  $x_0, x_1, \dots, x_n$ . (PETUNJUK: Andaikan  $P(x)$  ialah polynomial lain yang menepati ciri-ciri tersebut, kemudian pertimbangkan  $D = H_{2n+1} - P$  dan  $D'$  pada  $x_0, x_1, \dots, x_n$ ).*

[100 markah]

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4. The Chebyshev polynomials  $\{T_n(x)\}$  are defined as follows. For  $x \in [-1, 1]$ ,

$$T_n(x) = \cos(n \cos^{-1} x), \quad \text{for each } n \geq 0.$$

- (a) Show that  
 (i)  $T_0(x) = 1$ ;  
 (ii)  $T_1(x) = x$ ;
- (b) For  $n \geq 1$ , show that

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

(HINT: Use the substitution  $\theta = \cos^{-1} x$  and trigonometric identities

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

on  $T_{n+1}(\theta)$  and  $T_{n-1}(\theta)$ ).

- (c) Show that the Chebyshev polynomials are orthogonal on  $(-1, 1)$  with respect to the weight function  $w(x) = (1-x^2)^{-1/2}$ .
- (d) Confirm that  $\tilde{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ , for each  $k = 1, 2, \dots, n$  are the zeros of the Chebyshev polynomial  $T_n(x)$  which is of degree  $n \geq 1$ .
- (e) Use the zeros of the Chebyshev polynomial  $T_3(x)$  to construct the Lagrange interpolating polynomial of degree 2 for the function  $f(x) = \ln(x+2)$  on the interval  $[-1, 1]$ .

[100 marks]

4. *Polinomial Chebyshev  $\{T_n(x)\}$  ditakrifkan seperti berikut. Untuk  $x \in [-1, 1]$ ,*

$$T_n(x) = \cos(n \cos^{-1} x), \quad \text{untuk setiap } n \geq 0.$$

- (b) *Tunjukkan bahawa*  
 (i)  $T_0(x) = 1$ ;  
 (ii)  $T_1(x) = x$ ;
- (b) *Untuk  $n \geq 1$ , tunjukkan bahawa*

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

(PETUNJUK: Gunakan penggantian  $\theta = \cos^{-1} x$  dan identiti trigonometri  
 $\cos(a+b) = \cos a \cos b - \sin a \sin b$   
 $\cos(a-b) = \cos a \cos b + \sin a \sin b$   
ke atas  $T_{n+1}(\theta)$  dan  $T_{n-1}(\theta)$ ).

- (c) Tunjukkan bahawa polinomial Chebyshev berortogon pada  $(-1,1)$  tertakluk kepada fungsi pemberat  $w(x) = (1-x^2)^{-1/2}$ .
- (d) Sahkan bahawa  $\tilde{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ , untuk setiap  $k=1,2,\dots,n$  adalah pensifar polinomial Chebyshev  $T_n(x)$  berdarjah  $n \geq 1$ .
- (e) Gunakan pensifar polinomial Chebyshev  $T_3(x)$  untuk membina polinomial interpolasi Lagrange berdarjah 2 untuk fungsi  $f(x) = \ln(x+2)$  pada selang  $[-1,1]$ .

[100 markah]