UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2004/2005

October 2004

MAT 514 – MATHEMATICAL MODELLING
[PEMODELAN MATEMATIK]

Duration : 3 hours

Masa : 3 jam

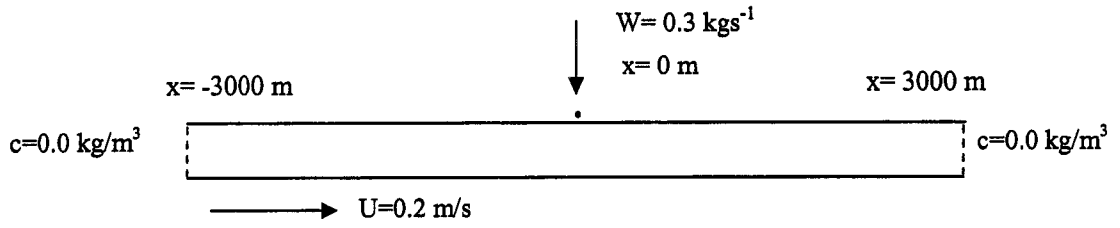
Please check that this examination paper consists of **SIX [6]** pages of printed material before you begin the examination.

Answer **all THREE** questions.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **ENAM [6]** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

Jawab **semua TIGA** soalan.

1. Given the following uniform river as shown below



with the properties $E = 0.5\ m^2/s$, $u = 0.2 m/s$, $Q = 2\ m^3/s$.
The length of the river is 6000 m.
A chemical is released into this river at the location $x = 0\ m$, at the rate $w = 0.3\ kg\ s^{-1}$. This chemical decays at the rate $\gamma = 10^{-4} s^{-1}$.

(a) Derive the following advection diffusion equation for the river above

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} + E\frac{\partial^2 c}{\partial x^2} - \gamma c + w \qquad (1)$$

and give the units for each of the terms.

(b) Obtain the exact solution for the above equation (1) subject to the conditions as stated above at steady state i.e. when $\frac{\partial c}{\partial t} = 0$.

(c) Sketch the solution for (b) above for $x \in (-\infty, \infty)$.

(d) Use the finite segment method to obtain the numerical solution for (b) above. Use three uniform segments. Form the linear system $A\underset{\sim}{x} = \underset{\sim}{b}$ and solve it.

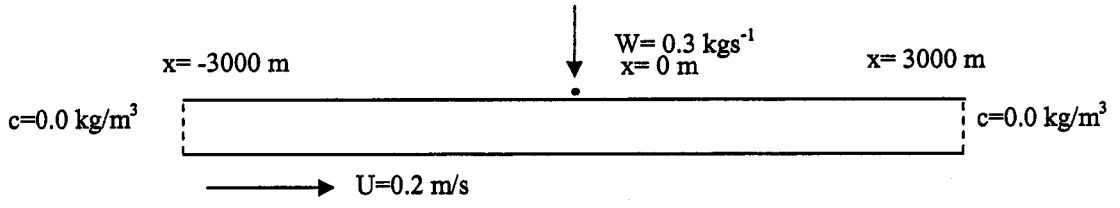
(e) Repeat (d) with the software I1D1SP and EIDISP. Compare the four solutions for (b), (d) and (e) in a single table or graph.

(f) Obtain more accurate numerical solutions using IIDISP and EIDISP with 5 segments and 10 segments respectively. Compare the accuracy.

(g) Discuss the rate of convergence of EIDISP.

[100 marks]

1. Diberi suatu sungai seragam yang tergambar berikut dengan



Ciri-ciri seperti $E = 0.5\ m^2/s$, $u = 0.2 m/s$, $Q = 2\ m^3/s$.
Panjang sungai ini ialah 6000 m.
Satu bahan kimia dilepaskan ke dalam sungai ini pada lokasi $x = 0\ m$, *pada kadar* $w = 0.3\ kg\ s^{-1}$. *Bahan Kimia ini merosot pada kadar* $\gamma = 10^{-4} s^{-1}$.

...3/-

(a) *Terbitkan persamaan aliran-sebaran bagi sungai di atas*

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} + E\frac{\partial^2 c}{\partial x^2} - \gamma c + w \qquad (1)$$

dan berikan unit untuk setiap sebutan.

(b) *Dapatkan penyelesaian tepat bagi persamaan (1) tertakluk kepada syarat-syarat untuk sungai di atas apabila keadaan mantap tercapai iaitu* $\frac{\partial c}{\partial t} = 0$.

(c) *Lakarkan penyelesaian di (b) untuk* $x \in (-\infty, \infty)$.

(d) *Gunakan kaedah segmen terhingga untuk mendapatkan penyelesaian berangka untuk (b) di atas. Gunakan tiga segmen terhingga yang seragam. Bentukkan sistem linear* $A\underset{\sim}{x} = \underset{\sim}{b}$ *dan selesaikan.*

(e) *Ulangi (d) dengan menggunakan perisian I1D1SP dan EIDISP. Bandingkan empat jawapan di bahagian (b), (d) dan (e) dalam satu jadual atau rajah.*

(f) *Dapatkan penyelesaian berangka yang lebih jitu melalui IIDISP dan EIDISP dengan 5 segmen dan 10 segmen masing-masing. Bincangkan kejituan masing-masing.*

(g) *Bincangkan kadar penumpuan kaedah EIDISP.*

[100 markah]

2. Consider the following heat equation

$$\frac{d}{dx}\left[KA\frac{dT}{dx}\right] - hp(T - T_\infty) = 0, \quad x \in [0, 10cm] \qquad (2)$$

subject to the following conditions.

$K = 70 \quad w/cm/^0C$, $A = \pi\ cm^2$, $h = 15w/cm^2/^0C$,

$P = 2\pi\ cm$, $T(0) = 80^0C$, $T_\infty = 10^0C$ and $\frac{dT}{dx} = 0$ *at* $x = 10cm$

(a) Use the finite element software HEAT04 with 10 uniform elements to solve (2). Compare the solution with the analytical solution, with the help of a single table or graph.

(b) Repeat the above question now with $T_\infty = 0^0C$.

(c) Solve the heat equation (2) using the software EIDISP and IIDISP with 10 uniform segments. Compare the three solutions as obtained above.

(d) Discuss the accuracy of the three solutions above.

(e) Obtain more accurate numerical solution by means of HEAT04, IIDISP and EIDISP for (2) above. Tabulate the solutions in a single table.

[100 marks]

2. *Pertimbangkan persamaan haba*

$$\frac{d}{dx}\left[KA\frac{dT}{dx}\right]-hp(T-T_\infty)=0,\quad x\in[0,10cm] \qquad (2)$$

tertakluk kepada syarat-syarat berikut.

$K=70\ \ w/cm/^0C,\ A=\pi\ cm^2,\ h=15w/cm^2/^0C,$

$P=2\pi\ cm,\ \ T(0)=80^0C,\ \ T_\infty=10^0C$ dan $\frac{dT}{dx}=0$ *pada* $x=10cm$

(a) Gunakan kaedah unsur terhingga melalui perisian HEAT04 dengan 10 unsur seragam untuk menyelesaikan (2). Bandingkan jawapan ini dengan jawapan analitik, melalui satu jadual dan satu rajah.

(b) Ulangi soalan di atas dengan $T_\infty=0^0C$.

(c) Selesaikan persamaan haba (2) dengan kaedah EIDISP dan IIDISP dengan 10 segmen seragam. Bandingkan ketiga-tiga jawapan.

(d) Bincangkan kejituan ketiga-tiga kaedah di atas.

(e) Dapatkan penyelesaian berangka melalui HEAT04, IIDISP dan EIDISP yang lebih jitu untuk (2). Tunjukkan jawapan melalui satu jadual.

[100 markah]

3. A diffusion equation at steady state is given by

$$a\ddot{y}+by+c=0,\quad x\in J=[x_o,x_m]. \qquad (3)$$

Equation (3) can be solved by the finite element method as follows.

$$\int_{e_i}\underset{\sim}{N}(a\ddot{y}+by+c)\,dx=\underset{\sim}{0},\ \text{for}\ i=1,2,3,..,M. \qquad (4)$$

In the above, J has been divided into M elements e_i so that $J=\bigcup_{i=1}^{M}e_i$. Further, $\underset{\sim}{N}$ is the shape function defined by the following.

$$\underset{\sim}{N}=(N_1,N_2)^T,\quad N_1(x)=1-\frac{x}{L},\quad N_2(x)=\frac{x}{L}, \qquad (5)$$

for $x\in[0,L]=e_i$.

(a) Show that

$$\int_e\left(\dot{\underset{\sim}{N}}\times\dot{\underset{\sim}{N}}\right)dx=\frac{1}{L}\begin{pmatrix}1&-1\\-1&1\end{pmatrix},$$

$$\int_e\left(\underset{\sim}{N}\times\underset{\sim}{N}\right)dx=\frac{L}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix},$$

$$\int_e\underset{\sim}{N}\,dx=\frac{L}{2}\begin{pmatrix}1\\1\end{pmatrix}.$$

...5/-

(b) Show that (4) can be reduced to

$$\sum_{i=1}^{M} \int_{e_i} \left\{ a\left(\dot{\underset{\sim}{N}} \times \dot{\underset{\sim}{N}} \right) - b\left(\underset{\sim}{N} \times \underset{\sim}{N} \right) \right\} dx . \{ \underset{\sim}{Y} \} =$$
$$\sum_{i=1}^{M} \int_{e_i} \left(C \underset{\sim}{N} \right) dx + a \underset{\sim}{N}^{(M)} \dot{y} \Big|_{x_m} - a \underset{\sim}{N}^{(1)} \dot{y} \Big|_{x_o} \qquad (6)$$

Now consider the heat equation

$$\ddot{y} - 4y = 0, \quad x \in (0.0, \quad 0.3), \qquad (7)$$
$$y(0) = 1.0, y(0.3) = e^{-0.6} = 0.5488116 \qquad (8)$$

(c) Obtain the exact solution $y = e^{-2x}$ for (7) and (8).

(d) Solve (7) and (8) by the finite element method with 3 uniform linear elements with element length $L = 0.1$ for each element $e_i, i = 1,2,3$, by means of the software HEAT04.

(e) Repeat question (d) above by forming the system of (4×4) equations

$$A \underset{\sim}{x} = \underset{\sim}{b} . \qquad (9)$$

Then use LEQ to solve the system (9)

(f) Compare the solutions for (c), (d) and (e) in a single table.

(g) Use HEAT04 with 10 linear elements with $L = 0.03$ to solve (7) and (8). Compare the solution with the exact solution in a table. Discuss the accuracy with clarity.

[100 marks]

3. *Persamaan sebaran pada keadaan mantap diberi oleh*

$$a\ddot{y} + by + c = 0, \quad x \in J = [x_o, x_m] . \qquad (3)$$

Persamaan (3) dapat diselesaikan dengan kaedah unsur terhingga seperti berikut

$$\int_{e_i} \underset{\sim}{N}(a\ddot{y} + by + c)\, dx = \underset{\sim}{0}, \text{ untuk } i = 1,2,3,..,M . \qquad (4)$$

Di sini J telah dibahagikan kepada M unsur e_i *supaya* $J = \bigcup_{i=1}^{M} e_i$. *Tambahan pula* $\underset{\sim}{N}$ *ialah fungsi uji yang ditakrifkan sebagai*

$$\underset{\sim}{N} = (N_1, N_2)^T, \quad N_1(x) = 1 - \frac{x}{L}, \quad N_2(x) = \frac{x}{L}, \qquad (5)$$

untuk $x \in [0, L] = e_i$.

(a) *Tunjukkan bahawa*

$$\int_e \left(\underset{\sim}{\dot{N}} \times \underset{\sim}{\dot{N}} \right) dx = \frac{1}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$\int_e \left(\underset{\sim}{N} \times \underset{\sim}{N} \right) dx = \frac{L}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix};$$

$$\int_e \underset{\sim}{N}\, dx = \frac{L}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) *Tunjukkan bahawa persamaan (4) dapat diturunkan kepada*

$$\sum_{i=1}^{M} \int_{e_i} \left\{ a \left(\underset{\sim}{\dot{N}} \times \underset{\sim}{\dot{N}} \right) - b \left(\underset{\sim}{N} \times \underset{\sim}{N} \right) \right\} dx. \left\{ \underset{\sim}{Y} \right\} =$$

$$\sum_{i=1}^{M} \int_{e_i} \left(C \underset{\sim}{N} \right) dx + a \underset{\sim}{N}^{(M)} \dot{y} \Big|_{x_m} - a \underset{\sim}{N}^{(1)} \dot{y} \Big|_{x_o} \qquad (6)$$

Sekarang pertimbangkan persamaan haba

$$\ddot{y} - 4y = 0, \quad x \in (0.0,\ 0.3), \qquad (7)$$

$$y(0) = 1.0, y(0.3) = e^{-0.6} = 0.5488116 \qquad (8)$$

(c) *Dapatkan penyelesaian tepat* $y = e^{-2x}$ *bagi (7) dan (8).*

(d) *Selesaikan persamaan (7) dan (8) melalui kaedah unsur terhingga dengan 3 unsur linear seragam dengan panjang* $L = 0.1$ *untuk setiap unsur* $e_i, i = 1,2,3$, *dengan menggunakan perisian HEAT04.*

(e) *Ulangi soalan (d) di atas dengan membentukkan sistem persamaan linear* (4×4)

$$A \underset{\sim}{x} = \underset{\sim}{b}. \qquad (9)$$

Kemudian gunakan perisian LEQ untuk menyelesaikan sistem (9)

(f) *Bandingkan jawapan (c), (d) dan (e) dalam satu bentuk jadual.*

(g) *Gunakan HEAT04 melalui 10 unsur linear dengan* $L = 0.03$ *untuk mendapat penyelesaian bagi (7) dan (8).*
Bandingkan jawapan ini dengan penyelesaian tepat dalam satu jadual.
Bincangkan kejituan dengan jelas.

[100 markah]

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