NUMERICAL SOLUTION OF THE GOURSAT PROBLEM
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Abstract
The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of applications. Several finite difference schemes have been proposed to solve the Goursat problem. Amongst these schemes is a scheme which implements harmonic mean averaging of function values. A comparative study which has been carried out concluded that harmonic mean averaging yielded more accurate results than arithmetic mean averaging. However, there seemed to be discrepancies between the conclusions and the displayed results. In this paper, we present the results of a comparative study which we have conducted on three Goursat problems over a range of grid sizes. Our results indicate that arithmetic mean averaging is more accurate than harmonic mean averaging. We also show that arithmetic mean averaging has an advantage when applied to linear Goursat problems.

Key Words
Goursat problem, finite difference schemes, arithmetic mean, harmonic mean

1. Introduction
Many mathematical models in science and engineering necessitate the solution of a partial differential equation. As analytical solutions are often difficult to obtain, numerical methods (such as the finite difference or finite element method) are frequently used.

The Goursat problem, associated with hyperbolic partial differential equations, arises in several areas of physics and engineering. Frisch and Chao [1], Cheung [2], Kaup and Newell [3], An and Hua [4], Hillion [5], McClauughlin et al. [6], Chen and Li [7], Kaup and Steudel [8] describe in detail areas of applications where a Goursat problem arises. Several finite difference schemes have been proposed to solve the Goursat problem. Amongst these schemes is a scheme which implements harmonic mean averaging of function values. A comparative study which has been carried out [9] concluded that this approach yielded more accurate results compared with the use of the standard method of arithmetic mean averaging.

In this paper we study the accuracy of a finite difference scheme based on harmonic mean averaging and a finite difference scheme based on arithmetic mean averaging when applied to three (one linear, one nonlinear and one with a derivative term) Goursat problems.

2. The Goursat Problem and Finite Difference Schemes
The Goursat problem is of the form [9]:
\[ u_{xy} = f(x, y, u_t u_x, u_y) \]
\[ u(x, 0) = \phi(x), u(0, y) = \psi(y), \phi(0) = \psi(0) \] \hspace{0.5cm} (1)
\[ 0 \leq x \leq a, 0 \leq y \leq b \]

The established finite difference scheme is based on arithmetic mean (AM) averaging of function values and is given by (Wazwaz, 1993):
\[ u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1} \]
\[ \frac{h^2}{h} = \frac{1}{4}(f_{i-1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}) \] \hspace{0.5cm} (2)

Wazwaz (1993) presented a new scheme for the Goursat problem. This scheme is based on harmonic mean averaging of function values and is given by:
\[ u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1} \]
\[ \frac{h^2}{h} = \frac{4f_{i-1,j+1}f_{i,j} + f_{i+1,j}f_{i,j+1} + f_{i+1,j+1}}{f_{i+1,j+1}f_{i,j} + f_{i+1,j+1}f_{i,j} + f_{i,j}} \] \hspace{0.5cm} (3)

\[ \frac{f_{i-1,j+1}f_{i,j} + f_{i,j}f_{i+1,j+1} + f_{i,j+1}}{f_{i+1,j+1}f_{i,j} + f_{i+1,j+1}f_{i,j} + f_{i,j}} \]
Henceforth, we shall refer to the finite difference scheme (4) as the HM scheme. The harmonic mean (HM) of any two real numbers \(a\) and \(b\) is \(\frac{2ab}{a+b}\). The function value at location \((i+1/2, j+1/2)\), i.e. the r.h.s of equation (4), is obtained from:

\[
\text{HM(HM of } f_{i,j+1} \text{ and } f_{i-1,j} \text{; HM of } f_{i-1,j+1} \text{ and } f_{i,j})
\]

(5)

Wazwaz [9] stated that he investigated the application of the AM and HM scheme over a wide range of examples and concluded that the HM scheme appears to give better results (in terms of accuracy). However, results were only presented for the non-linear Goursat problem (with \(h=0.05\)):

\[
\begin{align*}
\frac{\partial u}{\partial y} &= u \\
\frac{\partial u}{\partial x} &= e^x \\
\frac{\partial u}{\partial y} &= -1 + y + u \\
\end{align*}
\]

(6)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= 1 + y + u \\
\frac{\partial u}{\partial y} &= -1 + y + u \\
0 \leq x \leq 2, 0 \leq y \leq 2 \\
0 \leq x \leq 2.4, 0 \leq y \leq 4
\end{align*}
\]

(7)

(8)

In this way we hope to draw firmer conclusions regarding the accuracy of the AM and HM schemes. Analytical solutions for (6), (7) and (8) can be found in Wazwaz [10].

3. Numerical Experiments

Computer programs for problems (6), (7) and (8) were developed. For the non-linear Goursat problem (6) with \(h = 0.05\), we obtained:

| Table 1: Relative errors for the AM scheme, \(h=0.05\) |
|---|---|---|---|
| \(y\) | 1.0 | 2.0 | 3.0 | 4.0 |
| 1.0 | 7.2890713e-005 | 9.707691e-005 | 6.497523e-005 | 4.0385671e-005 |
| 2.0 | 7.290791e-005 | 9.707691e-005 | 6.497523e-005 | 4.0385671e-005 |
| 3.0 | 6.497523e-005 | 3.4128209e-004 | 8.0424181e-004 | 9.5132232e-004 |
| 4.0 | 4.0385671e-005 | 2.5890833e-004 | 9.5132232e-004 | 2.1149666e-003 |
Table 2: Relative errors for the HM scheme, h=0.05

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>9.0306034e-005</td>
<td>1.8110830e-004</td>
<td>2.0081184e-004</td>
<td>1.7570651e-004</td>
</tr>
<tr>
<td>2.0</td>
<td>1.8110830e-004</td>
<td>5.3529258e-004</td>
<td>7.9519142e-004</td>
<td>7.6182887e-004</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>2.0081184e-004</td>
<td>7.9519142e-004</td>
<td>1.9192810e-003</td>
<td>2.5539158e-003</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>1.7570651e-004</td>
<td>7.6182887e-004</td>
<td>5.3529258e-004</td>
<td>7.9519142e-004</td>
<td></td>
</tr>
</tbody>
</table>

We also computed that the:

Number of grid points where the AM scheme is superior = 6400

Number of grid points where the HM scheme is superior = 0

Average relative error of the AM scheme = 2.5268253e-004

Average relative error of the HM scheme = 6.2442268e-004

For grid sizes h = 0.025, 0.1, we obtained the following results:

Table 3: Results for Problem (6) with h= 0.025, 0.1

<table>
<thead>
<tr>
<th>h = 0.025</th>
<th>h = 0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of grid points for which AM scheme is superior</td>
<td>25600</td>
</tr>
<tr>
<td>No. of grid points for which HM scheme is superior</td>
<td>0</td>
</tr>
<tr>
<td>Average relative error of the AM scheme</td>
<td>6.2298718e-005</td>
</tr>
<tr>
<td>Average relative error of the HM scheme</td>
<td>1.5377398e-004</td>
</tr>
</tbody>
</table>

For the linear Goursat problem (7), we obtained the following results:

Table 4: Results for Problem (7) with h=0.025, 0.05, 0.1

<table>
<thead>
<tr>
<th>h = 0.025</th>
<th>h = 0.05</th>
<th>h = 0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of grid points for which the AM scheme is superior</td>
<td>6400</td>
<td>1600</td>
</tr>
<tr>
<td>No. of grid points for which the HM scheme is superior</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Average relative error of the AM scheme</td>
<td>4.3506679e-005</td>
<td>1.7771664e-004</td>
</tr>
<tr>
<td>Average relative error of the HM scheme</td>
<td>8.6977635e-005</td>
<td>3.5484552e-004</td>
</tr>
</tbody>
</table>

For the Goursat problem (8), we obtained the following results:

Table 5: Results for Problem (8) with h = 0.003, 0.006, 0.03

<table>
<thead>
<tr>
<th>h = 0.003</th>
<th>h = 0.006</th>
<th>h = 0.03</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of grid points for which the AM scheme is superior</td>
<td>637647</td>
<td>158706</td>
</tr>
<tr>
<td>No. of grid points for which the HM scheme is superior</td>
<td>2353</td>
<td>1294</td>
</tr>
<tr>
<td>Average relative error of the AM scheme</td>
<td>1.0024714e-3</td>
<td>2.0028735e-3</td>
</tr>
<tr>
<td>Average relative error of the HM scheme</td>
<td>5.1360557e-2</td>
<td>5.1946317e-2</td>
</tr>
</tbody>
</table>

From the above results (and the results for h values not displayed in this paper) it is clear that the AM scheme is more accurate than the HM scheme.
4. Implementation Aspects

Consider the linear Goursat problem (7) as an example. If the AM scheme is used, we obtain the finite difference scheme:

\[ u_{i+1,j} = \frac{1}{4} (u_{i+1,j+1} + u_{i,j} + u_{i-1,j} + u_{i,j-1}) \]  \hspace{1cm} (9)

Equation (7) is a linear equation which can easily be solved for the unknown \( u_{i+1,j} \).

If the harmonic mean scheme is used we obtain:

\[ u_{i+1,j} = \frac{1}{h^2} \left( \frac{4u_{i+1,j-1}u_{i,j}u_{i,j+1}u_{i+1,j}}{u_{i+1,j}u_{i,j} + u_{i,j-1} + u_{i+1,j-1} + u_{i,j+1}} \right) \]  \hspace{1cm} (10)

This is a non-linear equation in the unknown \( u_{i+1,j} \) and would require iteration, with its associated computational costs, for its solution. We thus see that the AM scheme has an advantage in that it preserves the linearity of a linear Goursat problem and consequently the straightforward solution procedure.

5. Conclusions

In this paper we have studied the AM and HM finite difference schemes for the solution of the Goursat problem. A previous comparative study concluded that the HM scheme was more accurate. However, the displayed results indicated otherwise. Our investigations, involving the computation of the number of points at which one scheme was more accurate than the other and the comparison of the average relative error for three Goursat problems, have found that the AM scheme is more accurate. We further make the observation that for linear problems the AM scheme has an advantage in that it preserves the linearity of linear Goursat problems.

Acknowledgements

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