NUMERICAL APPROXIMATIONS OF TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BASED ON B-SPLINES COLLOCATION METHODS

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by

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LIST OF SYMBOLS

a	Initial point in space dimensiom
b	End point in space dimensiom
(a, b)	Open interval , $\{a < t < b\}$
$[a, b]$	Close interval , $\{a \leq t \leq b\}$
$B_{q,j}(t)$	j^{th} B-spline basis function of degree q
${}_0^C D_t^\alpha$, $\frac{\partial^\alpha \underline{u}(r,t)}{\partial t^\alpha}$	Caputo's time fractional derivative of order α
e_j^n	Unknown coefficients for $j = -1, 0, 1, \cdots, N + 1$ and $n = 0, 1, \cdots, M$
$f(r, t)$, $g(r, t)$	Forcing terms
h	Grid size in r-dimension
j, i	Index for space
L_2	Euclidean norm
L_∞	Maximum absolute norm
M	Number on interval in time-dimension
N	Number on interval in space-dimension
O	Order of convergence
r	Space variable
r_j	Grid point in r-dimension
t	Time variable
T	Total time
t_n	Grid point in t-dimension
$w(r, t)$, $p(r, t)$, $\underline{u}(r, t)$	Exact solutions
$W(r, t)$, $P(r, t)$, $\underline{U}(r, t)$	Approximate solutions

α	Fractional order derivative
β	Free parameter for ExCBS
$\mathbb{C}$	Complex numbers
$\mathbb{N}$	Natural numbers
$\Re$	Real numbers
$\Re^+$	Set of positive real numbers
$\mathbb{Z}$	Integers
Γ	Gamma function
$\triangle t$	Grid size in t-dimension
$\triangle$	Laplace operator (Laplacian)
θ	Weight function
φ_1	Initial condition
φ_2, φ_3	Boundary conditions

LIST OF ABBREVIATIONS

ABD	Atangana–Baleanu derivative
ADM	Adomian decomposition method
BVPs	boundary value problems
CBS	cubic B-spline
CCM	Chebyshev collocation method
CDE	convection-diffusion equation
CDRE	convection-diffusion-reaction equation
CFD	Caputo's fractional derivative
CFDM	compact finite difference method
CHBS	cubic hybrid B-spline
CTBS	cubic trigonometric B-spline
ExCBS	extended cubic B-spline
FC	Fractional calculus
FCDE	fractional convection-diffusion equations
FCDRE	fractional convection-diffusion-reaction equation
FDEs	fractional differential equations
FDM	finite difference method
FEM	finite element method
FNSE	fractional Navier-Stokes equation
FODIs	Fractional-order derivatives and integrals
FPDE	fractional partial differential equation
FPM	finite point method

GFEM	Galerkin finite element method
G-L	Grünwald-Letnikov
HAM	homotopy analysis method
HPETM	homotopy perturbation Elzaki transform method
HPM	homotopy perturbation method
LDG	local discontinuous Galerkin
LFPs	Lucas and Fibonacci polynomials
LRPSM	Laplace residual power series method
MExCBS	modified extended cubic B-spline
MHBS	modified hybrid B-spline
MHGIM	modified hybrid group iterative method
NSE	Navier-Stokes equation
PDE	partial differential equation
q-HATM	q-homotopy analysis transform method
RBFM	Radial basis functions method
RExCBS	redefined extended cubic B-spline
TFACE	time-fractional Allen–Cahn Equation
TFADE	time-fractional advection-diffusion equation
TFBE	time-fractional Burger's equation
TFCDE	time-fractional convection-diffusion equation
TFCDRE	time-fractional convection-diffusion-reaction equations
TFDE	time-fractional diffusion equation
TFFE	time-fractional Fisher equation
TFKGE	time–fractional Klein–Gordon equation
TFMBE	time-fractional modified Burger's equation

TFNSE time-fractional Navier-Stokes equation

TFTE time-fractional telegraph equation

LIST OF APPENDICES

ANGGARAN BERANGKA BEBERAPA PERSAMAAN PEMBEZAAN SEPARA PECAHAN MASA BERDASARKAN KAEDAH KOLOKASI SPLIN-B

ABSTRAK

Baru-baru ini, terdapat lonjakan dalam kajian melibatkan persamaan pembezaan separa pecahan (FPDE). Hal ini kerana FPDE mempunyai banyak aplikasi dalam bidang kejuruteraan dan sains kerana sifatnya yang tidak lokal. Sebagai contoh, FPDE telah digunakan untuk menyelesaikan masalah dalam bidang fizik, kimia, biologi, biokimia, perubatan, pengangkutan elektron, data kewangan, aliran air bawah tanah dan geo-hidrologi. FPDE ialah generalisasi bagi persamaan pembezaan separa tradisional. Penyelesaian analitikal tepat kepada FPDE umumnya tidak dapat ditentukan kerana sifatnya yang rumit. Jadi, motivasi yang kuat wujud untuk meneroka kemungkinan untuk membangunkan kaedah berangka dan analitikal anggaran yang cekap dan boleh dipercayai untuk menyelesaikan FPDE. Kajian ini memberi tumpuan kepada pembangunan kaedah berangka anggaran yang cekap. Dua teknik kolokasi splin B yang berbeza dibangunkan untuk menyelesaikan FPDE tak linear. Teknik ini ialah penghampiran splin B kubik (CBS) dan splin B kubik lanjutan (ExCBS). Operator Caputo digunakan untuk menerangkan terbitan pecahan masa. Kaedah pembezaan terhingga tradisional digunakan untuk mendiskretkan terbitan pecahan Caputo manakala pendiskretan ruang dilakukan oleh CBS dan ExCBS dengan skema berwajaran θ. Teknik ExCBS digunakan untuk menyelesaikan persamaan perolakan-peresapan pecahan masa dan persamaan perolakan-pecahan-tindakbalas tak linear dengan pekali pembolehubah. CBS pula digunakan untuk menyelesaikan persamaan Navier-Stokes pecahan masa. Teknik-teknik ini dibuktikan menumpu dan stabil tanpa syarat manakala beberapa uji kaji berangka mengesahkan penemuan berkenaan. Tambahan pula, kerumitan pengiraan skim dikurangkan dengan ketara disebabkan ketepatan tertib tingginya (penumpuan spatial tertib kedua dan penumpuan temporal tertib $(2-\alpha)$, membolehkan simulasi yang cekap pada grid sambil mengekalkan ketepatan. Keputusan berangka yang diperoleh adalah lebih tepat berbanding kaedah sedia ada, menunjukkan kebolehpercayaan dan kecekapan pengiraan skim yang dicadangkan untuk menyelesaikan FPDE.

NUMERICAL APPROXIMATIONS OF TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BASED ON B-SPLINES COLLOCATION METHODS

ABSTRACT

Recently, there has been an upsurge in the study of fractional partial differential equations (FPDEs). This is because FPDEs have numerous applications in engineering and science owing to their non-local properties. For instance, FPDEs have been employed to solve problems in the fields of physics, chemistry, biology, biochemistry, medicine, electron transportation, financial data, groundwater flow, and geo-hydrology. FPDEs are the generalizations of traditional partial differential equations. Exact analytical solutions for FPDEs are generally undetermined owing to their complicated nature. Thus, there is intense motivation to explore possibilities for developing efficient and reliable approximate analytical and numerical methods for solving FPDEs. This study focused on the development of efficient approximate numerical methods. Two different B-spline collocation techniques are developed to solve nonlinear FPDEs. These techniques are cubic B-spline (CBS) and extended cubic B-spline (ExCBS) approximations. The Caputo operator is used to describe the time-fractional derivative. The traditional finite difference method is engaged to discretize the Caputo fractional derivative while the spatial discretization is performed by CBS and ExCBS with the θ-weighted scheme. The ExCBS technique is applied to solve time-fractional convection-diffusion and convection-diffusion-reaction equations with variable coefficients, whereas CBS is used for the time-fractional Navier-Stokes equation. These techniques are proven to be convergent and unconditionally stable, with numerical experiments confirming their findings. Furthermore, the computational complexity of the schemes is significantly reduced due to their high-order accuracy (second-order spatial convergence and $(2 - \alpha)$-order temporal convergence), allowing for efficient simulations on the grid while maintaining precision. The obtained numerical results are more accurate compared to existing methods, demonstrating the reliability and computational efficiency of the proposed schemes for solving FPDEs.

CHAPTER 1

INTRODUCTION

1.1 Research Background

The notion of the derivative of a line segment is old, probably dating back to 300 BC whereas differential calculus development is usually credited to Gottfried Leibniz and Isaac Newton, who simultaneously formulated the basis of the subject (Debnath, 2004). Fractional calculus (FC) is an old and yet refreshing concept. The subject is over 329 years of age and is currently undergoing development. It is a branch of mathematical analysis that involves the integration and differential of non-integer orders (rational, irrational, or even complex) known as differintegrals in engineering and sciences.

The first mention of FC was in the historical letter conversation of September 30, 1695, between Leibniz and Marquis de L'H$\hat{o}$pital. In the said communication, L'H$\hat{o}$pital put forward the query to Leibniz *"What would the outcome be ($n-th$ order derivative) if $n = \frac{1}{2}$?"*. In reply, Leibniz prophetically said: *"This is an apparent paradox from which, one day, useful consequences will be drawn ..."*. This letter can be regarded as the beginning of FC. After this, FC was built on a formal foundation by several well-known mathematicians. Detailed features on the contributions of the FC theory could be found in (Milller & Ross, 1993; Oldham & Spanier, 1974; Ross, 1977; Samko et al., 1993). Also, Machado et al. (2019) has provided an extensive summary list of documents and significant events in the progress of FC up till the last five decades. Further, Diethelm et al. (2022) highlighted the trend, directions for onward research include some open problems of FC.

One of the leading areas of FC is fractional differential equations (FDEs). The study of FDEs covers a wide area of pure and applied mathematics. Pure mathematicians are concerned with the properties of FDEs such as the existence and uniqueness of their solutions. At the same time, the applied mathematicians underscore FDEs from the perspective of the application's existence and uniqueness questions, in addition to the difficult credible methods for approximation solutions. Many books and monographs have dealt with the subject of FC. From the record, the first book exclusively concerned a systematic exposition of the idea approaches, and applications of FC were written by Oldham and Spanier (1974). Among the outstanding and notable contributions to an excellent introduction to FC is the monograph by Milller and Ross (1993). The volume by Podlubny (1998) deals principally with the theory and some applications of FDEs. An encyclopedic-type extraordinary monograph (i. e. the 'Bible' of FC) was published by Samko et al. (1993).

There are many contending definitions of the fractional derivative of order $\alpha > 0$, such as, the Riemann, Liouville, Weyl, Marchaud, Riemann-Liouville, Caputo, Coimbra, Hadamard, Gr$\ddot{u}$nward-Letnikov, Sonin-Letnikov, Erdlyi-Kober, Beta, Riesz-Feller, Atangana-Baleanu, Caputo-Fabrizio, local and conformable fractional derivatives and integrals (Teodoro et al., 2019). These definitions differ in their mathematical formulations and properties, where each has advantages and disadvantages depending on the specific application. However, the most frequently used definition are those of Riemann-Liouville and Caputo. Equations containing fractional derivatives of the form $\frac{d^\alpha}{dx^\alpha}$ are known as FDEs and are defined for $\alpha > 0$, where α may undoubtedly be an integer. These equations could be linear or nonlinear, as well as ordinary or partial FDEs. FDEs are a generalization of traditional differential equations. They have been extensively used in physical, biological, chemical, and engineering sciences. For example, the system

$$\frac{\partial^2 w(r,t)}{\partial t^2} + \mu \frac{\partial^2 w(r,t)}{\partial r^2} = g(r,t) \tag{1.1}$$

and

$$\frac{\partial^\alpha w(r,t)}{\partial t^\alpha} + \beta \frac{\partial^2 w(r,t)}{\partial r^2} = f(r,t) \tag{1.2}$$

are the traditional partial differential equation (PDE) and fractional partial differential equation (FPDE), respectively, where $\alpha > 0$ stand for the fractional order of the stated PDE.

FPDEs expeditiously model various complex phenomena and plays significant role in mathematical physics (Kilbas et al., 2006; Oldham & Spanier, 1974; Podlubny, 1998; Samko et al., 1993). Prominent applications of FPDEs are in electron transportation (Scher & Montroll, 1975), image processsing (Jacobs & Harley, 2018), ion acoustic wave (Goswami et al., 2019), HIV/AIDS infection model (Babaei et al., 2019; Boukhouima et al., 2017), electric circuit (Ahmadova & Mahmudov, 2021), baroclinic atmosphere (Yang et al., 2019), inventory system (A. Kumar et al., 2023), population growth (Asl et al., 2022), oil pollution (Mohan & Prakash, 2023; Patel et al., 2023), and geometries (Terpák, 2023). The FPDEs are also suitable and appropriate tools in modeling and description of certain natural and non-natural phenomena like diffusion processes, COVID-19 spread, groundwater flow and Geo-Hydrology, fluid mechanics, fractal network, astrophysics, cybernetics, signal processes, fractal media, rheology, and energy infrastructure among others (Abuasbeh et al., 2023; Ahmad et al., 2023; Atangana & Vermeulen, 2014; Benson et al., 2013; Diethelm & Freed, 1999; Hilfer, 2000; Luchko & Punzi, 2011; Mainardi, 2010; Okposo et al., 2021; Owolabi et al., 2020; Rashidinia et al., 2024; Sun et al., 2018).

Researchers have observed that problems occurring in the theory of fractional derivatives are of great importance, very challenging, and require hard mathematical techniques. The exact analytical solutions of FPDEs are generally unavailable due to their complicated nature (W. Chen et al., 2010). This refers to the inherent mathematical challenges posed by FPDEs. These equations involve fractional derivatives, which are non-local operators that depend on the entire history of the function, making them significantly more complex than standard integer-order PDEs. Additionally, FPDEs often model real-world phenomena with coupled effects, variable coefficients, or non-linearities, further complicating the derivation of exact solutions and necessitating numerical methods for practical problem-solving. Hence, numerical methods provide powerful alternative ways of solving FPDEs under prescribed conditions. In the last three to four decades, considerable efforts have been made to develop stable and efficient numerical techniques for solving FPDEs. However, the formation and execution of precise numerical steps with rigorous analysis of FPDEs are quite difficult. Nevertheless, researchers have suggested some analytical and numerical methods for solving FPDEs.

The numerical techniques like the Adomian decomposition method (ADM) (Momani & Odibat, 2006), finite point method (FPM) (J. Li & Qin, 2019; Liu et al., 2021), finite difference method (FDM) (Anley & Zheng, 2020; H. Wang & Zheng, 2019), finite element method (FEM) (Arifeen et al., 2022; X. Li et al., 2017), Radial basis functions method (RBFM) (Kamran et al., 2021), Chebyshev collocation method (CCM) (Saw & Kumar, 2021), and homotopy analysis method (HAM) (Dehghan et al., 2010), among others, have been well used to solve FPDEs. Further improvements to some of the methods are homotopy perturbation Elzaki transform method (HPETM) (Jena & Chakraverty, 2019), q-homotopy analysis transform method (q-HATM) (A. Prakash et al., 2019), non-polynomial quintic spline (Amin et al., 2019), implicit difference scheme (U. Ali et al., 2022), modified hybrid group iterative method (MHGIM) (Salama et al., 2022), and so forth.

Furthermore, a large number of studies have shown the use of B-splines to solve FPDEs. For example, time-fractional diffusion equation (TFDE) was examined using modified extended cubic B-spline (MExCBS) (Khalid et al., 2019) and cubic trigonometric B-spline (CTBS) (Yaseen, Abbas, & Riaz, 2021); time-fractional advection-diffusion equation (TFADE) was solved using redefined extended cubic B-spline (RExCBS) (Khalid, Abbas, Iqbal, Singh, & Ismail, 2020) and cubic B-spline (CBS) (Shafiq et al., 2021); time-fractional Burger's equation (TFBE) was solved using ExCBS (Akram, Abbas, Riaz, et al., 2020), CTBS (Yaseen & Abbas, 2020), modified hybrid B-spline (MHBS) (Hashmi et al., 2021), and CBS (Shafiq et al., 2022); time–fractional Klein–Gordon equation (TFKGE) was solved using RExCBS (Amin et al., 2020) and CTBS (Yaseen, Abbas, & Riaz, 2021); time-fractional modified Burger's equation (TFMBE) was solved using ExCBS (Majeed, Kamran, & Rafique, 2020). Also, the time-fractional telegraph equation (TFTE) was solved using MExCBS (Akram, Abbas, Iqbal, et al., 2020) and RExCBS (Amin et al., 2021); time-fractional Fisher equation (TFFE) was solved using extended cubic B-spline (ExCBS) (Akram, Abbas, & Ali, 2021); and time-fractional Allen–Cahn Equation (TFACE) solved using ExCBS (Fatima et al., 2024). The main advantages of the B-splines method are its simplicity, high accuracy, and computational efficiency (less effort). The produced algorithm can also be carried out on a computer (Akram, Abbas, Riaz, et al., 2020).

The convection-diffusion equation (CDE) describes the transport of a quantity such as mass, heat or concentration, within a physical system. It combines two processes: convection (transport by bulk motion) and diffusion (transport due to random molecular motion) (Nyaupane et al., 2024). The convection-diffusion-reaction equation (CDRE) extends the CDE by including the reaction term, which accounts for some processes such as biological growth, decay, or chemical reactions (Ho & Yang, 2019). The Navier-Stokes equation (NSE) is the fundamental equation that describes the motion of viscous fluid substances in fluid mechanics. It has applications in various areas such as biology, physics, geology, medicine, oceanography, and civil and mechanical engineering (Łukaszewicz & Kalita, 2016).

Meanwhile, all the three equations are second-order PDEs. They are all used to model physical phenomena involving transport processes even though in different contexts (e.g., heat transfer, chemical reactions, and fluid flow). The NSE is naturally nonlinear while the other two can be linear or nonlinear depending on their specific formulations (e.g., reaction term in CDRE). All involve diffusion and convection terms; only the CDRE includes an explicit reaction term, while the NSE includes fluid dynamic terms like pressure and external forces. There is therefore a relationship between them.

This research will concentrate on the fractional forms of the equations mentioned above.

1.2 Motivation of the Study

Over the last few years, there has been an overwhelming interest in studying FPDEs owing to their diverse applications in science and engineering. These equations have a nonlocal property in time or space as their primary features. However, the central problem with all fractional differential operators is the difficulty of computation. In addition to difficult mathematical techniques, they require high computational effort and storage requirements. Despite this, different analytical and numerical methods have been used to solve the FPDEs. Nonetheless, apart from the simplified initial and boundary conditions, exact analytical solutions of these FPDEs are generally not available (W. Chen et al., 2010).

Hence, there exists a strong motivation for exploring the possibilities of formulating efficient and reliable numerical techniques for the solutions of FPDEs. In this study, we aim to examine the use of appropriate B-splines for the numerical solutions of FPDEs. Generally, a cubic B-Spline can be used to solve second-order differential equations. The main features of the B-spline are its flexibility and high localization. They are numerically stable and can perform large numbers of calculations with sufficient accuracy (Akram, Abbas, Abualnaja, et al., 2021; Caglar & Caglar, 2009; Shafiq et al., 2023). There are many research studies based on B-Splines for the solution of FPDEs (Adetona et al., 2024; Akram, Abbas, & Ali, 2021; Amin et al., 2021; Arqub et al., 2022; Hashmi et al., 2021; Hepson, 2018; Majeed, Kamran, & Rafique, 2020; Roul et al., 2021; Shafiq et al., 2021; Shi et al., 2023; Vivas-Cortez et al., 2024; Yaseen, Abbas, & Riaz, 2021). These studies will be surveyed in more detail in Chapter 2.

However, to our uttermost knowledge, no research has been conducted employing B-splines to solve time-fractional convection-diffusion equation (TFCDE), time-fractional convection-diffusion-reaction equations (TFCDRE) and nonlinear time-fractional Navier-Stokes equation (TFNSE), respectively. We noticed that numerical approaches have advantages and disadvantages. Some techniques use the strengths that are lacking in others. Motivated by the reputation of spline approaches in determining the numerical solutions of FPDEs, we consider using CBS and ExCBS in this study to solve second-order FPDEs. The accuracy of the methods will be determined, in addition to performing stability and convergence analyses.

1.3 Problem Statement

Analytical solutions of FPDEs are often difficult to obtain. Consequently, diverse numerical methods, such as ADM, FDM, FPM, HAM, RBFM, CCM, q-HATM, and HPETM have been extensively used to solve FPDEs. These methods involve high computational effort, complexity, and mathematical techniques, which affect the accuracy and stability of the results. Researchers have suggested using different B-splines to address the issues of simplicity, accuracy, and stability. B-spline methods like CBS, ExCBS, RExCBS, MExCBS, CTBS, MHBS, and others have been proposed to solve FPDEs, such as TFDE, TFADE, TFBE, TFTE, TFFE, and TFACE.

Meanwhile, we have observed that no work has been done on using B-splines to solve TFCDEs, TFCDREs and TFNSE. Therefore, there should be an investigation on the application of B-splines to numerically solve TFCDEs, TFCDREs and TFNSEs. To this end, we aim to fill this gap in this study. This venture would contribute to the development of additional numerical techniques for solving TFCDEs, TFCDREs, and TFNSEs. Given the significance of these equations in modelling complex physical and engineering phenomena, such as fluid dynamics, pollutant transport, and anomalous diffusion, where fractional derivatives capture non-local and memory effects inherent in real-world systems. Thus, enhancing their numerical solutions will provide valuable tools for researchers and practitioners working in these fields.

1.4 Research Questions

The research questions pertinent to the study are listed as follows:

1. What is the method of ExCBS and CBS for solving FPDEs and their differences in methodology between the two methods?

2. Are the results of the methods stable and consistent?

3. Do the results of the schemes converge numerically?

4. Are the methods computationally more accurate when compared to other existing methods?

1.5 Objectives of the Study

The objectives of this study are listed as follows:

1. To develop a numerical scheme for solving TFCDEs by ExCBS, TFCDREs by ExCBS, and TFNSEs by CBS, respectively.

2. To perform the stability analysis of the numerical schemes developed in objective no 1.

3. To determine the order of convergence of the numerical schemes developed in objective no 1.

4. To compare the accuracy of the results of the developed numerical schemes with other existing methods.

1.6 Methodology of the Study

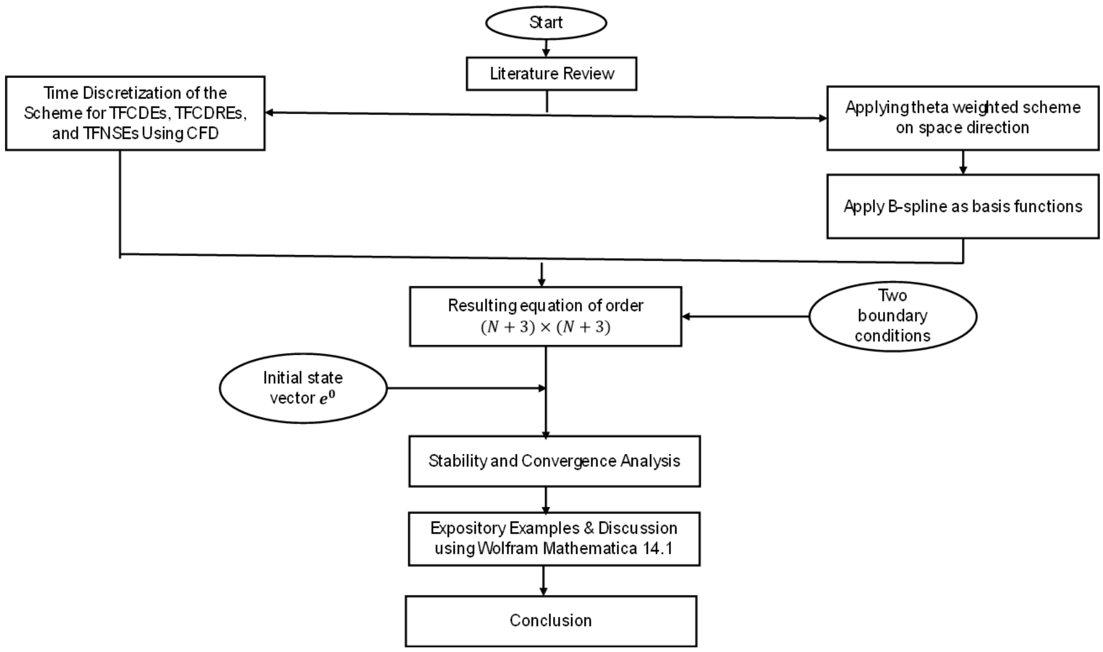

Figure 1.1: Flowchart of the research methodology

Figure 1.1 presents the flowchart of the research methodology. Consequently, to achieve objectives 1 to 5, the literature on techniques for solving the initial and boundary value problems of FPDEs is examined. The focal point was on CBS and ExCBS. The structures obtained using these methods were investigated. Subsequently, these methods were constructed and formulated to solve nonlinear second-order FPDEs.

This procedure provides a background for the study. In this research, the fractional derivatives were determined using Caputo's fractional derivative (CFD). The FDM was employed to discretize the Caputo operators. Caputo derivatives are suitable fractional operators for real–world phenomena. It allows the initial and boundary conditions to take the same form as the traditional PDEs, and its derivative of a constant is zero. The ExCBS and the CBS are proposed and applied to solve TFCDEs, TFCDREs and TFNSEs. Further, the stability analysis is performed by engaging the von Neumann approach and convergence analysis is also discussed. Numerical experiments are performed to establish the stability, convergence, and accuracy of the suggested methods. The results obtained using these methods are presented in tables and graphically represented to show agreement between exact and approximate solutions and compared with existing results in the literature.

The numerical examples in this study will be illustrated using Wolfram Mathematica 14.1 with HP Laptop (i7-7500 CPU 2.70GHz, 16.00 GB RAM).

1.7 Organization of the Thesis

This thesis is segmented into seven chapters. A brief component of each chapter is highlighted as shown:

- Chapter 1 presents the research background and the key factors of the research namely motivation, problem statement, research questions, objectives, and methodology.
- Chapter 2 contains a literature survey of numerical techniques for solving PDEs and FPDEs with many references on those based on B-splines basis functions. An extensive analytical and numerical methods which have been used for solving fractional convection-diffusion, fractional convection-diffusion–reaction, and fractional Navier-Stokes equations are also discussed.
- In Chapter 3, the basic concepts of FC, the special functions, and fractional order derivatives and integrals used in solving FPDEs are discussed. This chapter also presents the recursive formula of B-spline with some important properties including CBS and ExCBS.
- In Chapters 4 and 5, numerical methods based on CFD and ExCBS technique in combination with the θ-weighted scheme for solving TFCDR and TFCDRE, each with variable coefficients are presented. The stability analysis and convergence analysis of the ExCBS approximation are to be discussed in these chapters.
- Chapter 6 presents a combination of CFD and CBS with the θ-weighted scheme for solving TFNSE. The stability analysis and convergence analysis of the scheme approximation are also discussed in this chapter.

- Meanwhile, to substantiate the performance of the developed schemes, comparisons with selected results are shown in Chapters 4, 5, and 6.
- Finally, we present the conclusion of the research and possible future work of this study in Chapter 7.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The theory of spline functions originated in ship-making and aeroplane industries. It is accepted that the first mathematical documentation of spline functions was in 1946 by Schoenberg in his outstanding research paper (Schoenberg, 1946a, 1946b), where for the first time the word "spline" was used in relationship with smooth, piecewise polynomial approximation. Before the advent of computers, research on spline functions typically depended on the description of spline, which expresses it as a piecewise polynomial function. Mathematical scientists and engineers have recently developed tremendous interest in using spline functions to study approximation theory and numerical analysis. These interests are due to spline function abilities, which are helpful in applications in diverse fields. Different degrees of splines, such as linear, quadratic, cubic, quartic, quintic, sextic, septic, etc., are used for solving diverse mathematical problems; however, cubic spline functions are the most generally used in science, engineering, and mathematical physics.

Figure 2.1 depicts the flowchart of the literature review consisting of five sections. In the first section, the spline functions are briefly discussed. The second covers the application of splines in solving differential equations. The rest of the sections are related to techniques for solving FPDEs. Firstly, the literature concerning FCDEs is presented. The second explores the methods regarding the FCDREs and a brief history of the methods of the nonlinear FNSEs. Reasonable information regarding methods for solving the FPDEs is presented to demonstrate the originality of this research.

2.2 B-Spline Techniques for Solving Differential Equations

The term “B-splines” (shortened form for basis splines) was originally proposed by I. J. Schoenberg in 1946 (Schoenberg, 1946a) for statistical data smoothing. A

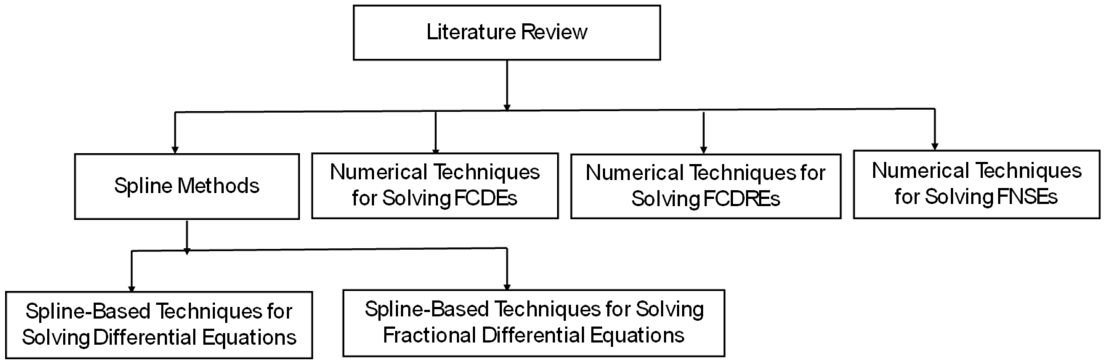

Figure 2.1: Flowchat of literature review

compendium of many algebraic properties of B-splines was covered by Curry and Schoenberg (1966). In the early 1970s, Cox (1972) and de Boor (1972) presented recursive equivalent B-splines. Subsequently, numerous studies have been conducted to develop the theory of spline functions and their applications to several practical problems (Ahlberg et al., 1967; de Boor, 1978; Greville, 1969; Rubin & Graves Jr, 1975).

Abd Hamid et al. (2010) solved linear two-point boundary value problems (BVPs) by adopting an ExCBS interpolation method. The authors obtained the numerical results and claimed that it gives better results when compared with CBS results. Mittal and Jain (2012) solved CDE with Dirichlet's boundary conditions using RExCBS collocation method. The authors discussed the stability of the proposed method to be unconditional and used different numerical experiments to confirm the efficiency and accuracy of the technique compared with the extended trapezoidal formula scheme. Khalid et al. (2019) suggested CBS for solving linear sixth-order BVPs arising in astrophysics. The authors claimed that the method has a second-order convergent and lesser computational cost when compared with the HPM and non-polynomial splines method, respectively. The CTBS based on the Hermite formula was employed to solve CDE by Yousaf et al. (2020). The CBS and the Hermite formula were combined to discretize the space derivative and FDM was applied for the time discretization. Stability was performed and numerical results obtained indicate a better accuracy when compared with a semi-discrete and a padé approximation method.

Iqbal et al. (2020) solved the coupled nonlinear Schrödinger equation using CBS Galerkin method. This technique includes obtaining numerical values of one, two, and three single solitary waves which show better accuracy than a delta-shaped basis functions scheme. A new CBS approximations method was employed to solve CDE by Tassaddiq et al. (2021). The FDM was used for the time derivative. Stability and convergence analysis were discussed. The numerical results show a better accuracy than CTBS from the literature. A second-order Volterra partial integrodifferential equation was solved by George et al. (2022) using ExCBS. They discretized the time derivative with FDM and the space derivative on the ExCBS basis. Numerical results were obtained and the accuracywas compared with that of CBS, CTBS, and quartic B-spline collocation methods. A linear two-point BVPs were solved by S. N. K. A. Khan and Misro (2024) using B-spline collocation methods. The B-spline method comprises the extended trigonometric cubic hybrid B-spline (CHBS) and extended CHBS collocation methods. The authors compared their results with CBS, CTBS, and CHBS collocation methods and claimed it demonstrated a better approximation.

2.3 B-Spline Techniques for Solving Fractional Differential Equations

X. Li (2012) presented the application of the CBS wavelet collocation method to solve FDE with Caputo's sense. The author claimed that the technique produced many advantages, such as high accuracy and computational efficiency, when experimental results were obtained and compared with others, such as implicit algorithms and operational matrix methods. Siddiqi and Arshed (2015) employed a quintic B-spline collocation technique in solving time-fractional fourth-order PDEs with Caputo's operator. The CFD was discretized using a backward Euler scheme and the quintic B–spline basis function for space discretization. The proposed method is shown to achieve 4th-order accuracy in space and order 2 accuracy in time. The authors discussed the stability and convergence of the scheme and validated their approach through several test problems, showcasing its efficiency and reliability. The linear B-spline wavelet operational method was utilized by Kargar and Saeedi (2017) for solving time-space

FPDEs. B-spline functions are employed to discretize the spatial domain and the CFD was used to handle the time-fractional component. They also used numerical results to demonstrate the efficiency and accuracy of the method and claimed to have better results when compared with a new wavelet operational method from the literature.

A third-degree MExCBS was suggested for solving TFDE that includes reaction and damping terms by Khalid et al. (2019). The CFD was approximated by FDM and the MExCBS functions were used for spatial discretization. The modification was done in such a way that the ExCBS preserved the diagonal dominance property. They carried out the stability analysis for the force-free case and found it to be unconditionally stable. The numerical results were established to be better than that of RBHM, CTBS, and class of FDM based on the Hermite formula. Akram, Abbas, Riaz, et al. (2020) adopted the ExCBS technique in solving TFBE. The authors applied Caputo's formulation for the time derivative and FDM to discretize the CFD. The stability analysis using the von Neumann method was shown to be unconditionally stable and convergence analysis using the L_2 error norm was shown to be second-order convergence. The approximate and exact solutions were compared and showed strong agreement while the error norms compared with the quadratic B-spline Galerkin method from the literature indicate better results. An ExCBS was suggested by Akram, Abbas, Ali, et al. (2020) for solving a time fractional reaction-diffusion model with a non-singular kernel. The time derivative was discretized by Caputo-Fabrizio operator while ExCBS was engaged for the space direction. The stability analysis was proved to be unconditionally stable and second-order convergence in time and space dimensions.

A redefined CBS technique was employed by Khalid, Abbas, Iqbal, and Baleanu (2020) for solving the time fractional Allen-Cahn equation of Caputo's sense. The CFD was used for time discretization while redefined CBS was used for space discretization. The stability and convergence formulations were also discussed. A computational technique has been developed for solving TFADE by Khalid, Abbas, Iqbal, Singh, and Ismail (2020). RExCBS was employed to discretize space dimension and CFD for time derivative. The stability was shown to be unconditional and convergence order of two. A numerical method based on MExCBS functions was proposed to solve nonlinear TFTEs by Akram, Abbas, Iqbal, et al. (2020). The authors discussed the stability and convergence analysis and also carried out experimental results to affirm the accuracy and efficiency of the method. Majeed, Kamran, Abbas, and Singh (2020) employed the CBS collocation technique for solving time fractional generalized Fisher's equation of CFD. The time derivative was discretized in Caputo's form based on the L_1 formulation that captures the memory effect with first-order time accuracy and the CBS functions used for the spatial derivative as part of a collocation method.

An efficient numerical scheme based on CTBS and FDM has been proposed by Yaseen and Abbas (2020) to solve TFBE. In this study, the scholars also performed stability analysis that obtained global unconditional stability and three experimental results to confirm the accuracy and efficiency. Furthermore, TFKGE was solved using RExCBS by Amin et al. (2020). The finite difference formula and RExCBS were used to discretize Caputo's derivative and spatial dimension. Stability and convergence analyses of the technique were also performed and experimental results show agreement with theoretical results. Roul and Goura (2021) suggested a high-order B-spline for the numerical solution of nonhomogeneous TFDE. They used the backward Euler scheme to discretize the time-fractional derivative in the sense of Riemann-Liouville while the fourth–order optimal CBS collocation was used for the space dimension. The stability analysis was obtained to be unconditionally stable and the convergence analysis was forth-order. Akram, Abbas, and Ali (2021) developed a new extended CBS to solve TFFE. Caputo's operator was used to approximate the time-fractional derivative and the new extended CBS for the space dimension. The scheme is shown to be unconditionally stable and convergent.

Akram, Abbas, Abualnaja, et al. (2021) developed a numerical model based on ExCBS to solve the time-fractional Black-Scholes model. The model is based on the European option pricing model. Caputo's derivative and ExCBS were used for the discretization of time and space dimensions. A CBS collocation method was suggested by Hadhoud et al. (2021) to solve the time-fractional Huxley–Burgers' equation utilizing the mean value theorem for integrals and CBS functions for the approximate solution. The scholars engaged von Neumann's technique in discussing the stability of the scheme. Three numerical experiments were performed to confirm the accuracy and efficiency of the method. A modified hybrid B-spline basis function was suggested by Hashmi et al. (2021) to solve TFBE. The authors employed CFD and differential quadrature method based on B-spline for the time and space discretization.

Kamran et al. (2021) utilized CBS to solve the time-fractional Phi-four equation numerically. Caputo's time derivative and spatial derivatives were discretized by the L_1 and B-spline basis functions respectively. Stability was proven to be unconditional and error norms were obtained to quantify the accuracy. Majeed, Kamran, Abbas, and Misro (2021) solved the Benjamin-Bona-Mahony-Burger model of fractional order using the CBS basis functions. The spatial and temporal derivatives were by FDM and Crank-Nicolson scheme, respectively. Experimental results show unconditional stability and agreement between the exact and approximate solutions. Majeed, Kamran, Asghar, and Baleanu (2021) suggested a CBS collocation scheme to solve an in-homogeneous time-fractional Burgers–Huxley equation. The authors used CFD for the temporal derivative based on L_1 formula and the CBS basis function for the spatial derivative. Unconditional stability and convergence analysis were proven and numerical results show agreement with the exact solutions. Roul et al. (2021) adopted the quintic B-spline basis function to solve the neutron diffusion equation of fractional order. They used L_1 approximation method for the CFD and the quintic B-spline for the space discretization.

The TFADE involving Atangana–Baleanu derivative (ABD) was solved by Shafiq et al. (2021) using CBS. The discretization of the ABD was by FDM and the space dimension was by the CBS. The study shows unconditional stability and the convergence of order two. Yaseen, Abbas, and Riaz (2021) adopted a CTBS collocation technique to solve the TFDE of the Caputo formulation. Stability and convergence analyses were performed to confirm the error does not magnify. Numerical solutions were acquired as a piecewise polynomial smooth continuous curve. Experimental results were secured and compared with existing results to confirm the method's accuracy and efficiency. A CTBS was used by Yaseen, Abbas, and Ahmad (2021) to solve a nonlinear generalized TFKGE. The scholars employed the FDM in discretization of the CFD and CTBS for the space direction. Experimental results were obtained including absolute maximum errors and error norms to confirm the accuracy and efficiency of the scheme. B. Khan et al. (2022) proposed a numerical scheme based on new CBS approximations to solve TFADE involving ABD. The FDM is used to discretize the time derivative with a non-singular kernel, while the spatial derivative is discretized using the new CBS functions.

A numerical solution of TFBE with ABD was obtained by Shafiq et al. (2022) using CBS basis functions and θ-weighted scheme. The discretization of the temporal and spatial dimensions was down by FDM and CBS functions, respectively. They also proved the stability to be unconditionally stable and second order of convergence in both time and space dimensions. Arifeen et al. (2022) solved multiterm time–FDE using CBS FEM. Here, the temporal fractional part was defined by CFD while B-spline FDE was engaged for space approximation. The scholars also used four–point Gauss–Legendre quadrature to obtain the source term. L_2 and L_∞ were used to confirm the accuracy and efficiency of the method. Benjamin–Bona–Mahony–Burger equation of CFD was solved by M. Kamran et al. (2022) using CBS functions. The temporal derivative was discretized using the L_1 formula while the space derivative was by the B-spline functions. Stability and convergence were also discussed and error norms were obtained to confirm the accuracy of the suggested method. Abbas et al. (2022) suggested using CBS functions to solve third-order time–FDEs. The authors formulated the approximations based on Crank-Nicolson FDM by CBS basis functions including Caputo's operators. They provided numerical results to affirm the accuracy of the method.

Yaseen et al. (2022) presented a comparative numerical solution of time-fractional Cattaneo Equation with Caputo-Fabrizio derivative using CBS, CTBS, and ExCBS. The scholars employed the FDM in discretizing the Caputo-Fabrizio derivative and B–spline functions for the space direction. Stability and convergence were also explored including experimental results to compare the efficiency of the three methods. A numerical method combined CBS and FDM was developed by Arqub et al. (2022) to solve nonlinear TFDE with reaction and damping terms. The CFD was estimated using FDM while uniform CBS functions were used to achieve spatial discretization. A numerical technique has been suggested for solving a generalized Burger-Huxley equation of fractional order (Akram et al., 2023). The scholars used the θ-weighted scheme and the ABD to discretize the equation while the ExCBS was used for the space dimension. The stability and second-order convergence were also proved with numerical simulations. Shafiq et al. (2023) proposed a numerical technique to solve TFDE with an exponential kernel involving Caputo-Fabrizio derivative. The temporal and spatial discretization was performed using FDM and CBS with a θ-weighted scheme. Stability and convergence were also proved with experimental results. Shi et al. (2023) employed CBS functions to solve fractional Painlevé and Bagley–Trovik equations in three fractional derivatives: Caputo, Caputo–Fabrizio, and conformable. The scholars used a piecewise spline of degree three polynomial to discretize the fractional derivatives.

Akram et al. (2024) developed an ExCBS collocation method to solve the time-fractional coupled-Burgers equation involving exponential kernel and Caputo-Fabrizio derivative. The stability was proven to be unconditionally stable and second-order convergence was obtained. A numerical technique based on quadratic B-spline collocation has been proposed by Adetona et al. (2024) to solve the space-fractional diffusion equation. The method was proven to be unconditionally stable and converged through experimental results. An ExCBS collocation was developed by Vivas-Cortez et al. (2024) to solve a generalized nonlinear TFKGE in mathematical physics. The temporal and spatial discretization was performed using FDM and ExCBS functions. Fatima et al. (2024) suggested a new ExCBS technique for solving TFACE in mathematical physics. Caputo's operator was used to discretize the time derivative while the new ExCBS were for spatial dimension. Both stability and convergence analysis were also discussed. Ghafoor et al. (2024) proposed the CBS and Crank–Nicolson FDM for solving coupled Viscous Burgers equation of fractional order. Unconditional stability was proven and experimental results were used to confirm the accuracy of the scheme.

2.4 Numerical Methods for Solving Fractional Convection-Diffusion Equation (FCDE)

Metzler and Klafter (2000) presented a comprehensive review of the application of fractional dynamics to anomalous diffusion processes. The authors discuss the fundamental concepts of FC and its connection to the random walk theory. They explain how FDEs can be used to model subdiffusive and superdiffusive transport phenomena observed in various scientific fields, like physics, chemistry and biology. Analytical solutions were obtained using Laplace transform and the Mittag-Leffler function while numerical solutions were secured using FDM and Monte Carlo simulations. Gracia et al. (2018) applied FDM to investigate the convergence properties of TFCDEs. A standard FDM on a uniform mesh was used to discretize the problem and establish convergence in positive time. Theoretical error estimates were derived and numerical results were generated to demonstrate that the method converges under certain conditions. A meshless finite point method (FPM) was introduced by J. Li and Qin (2019) in solving 1D and 2D TFCDEs. This method combines the moving least square shape function with the collocation method. The L_1 interpolation approximation formula was used to discretize the CFD, and the stability was theoretically proved. The numerical results show high accuracy.

A new numerical technique, the sinc-Galerkin method was suggested by L. J. Chen et al. (2020) for solving the TFCDE with variable coefficients. The L_1 formula was used for the time discretization while the sinc-Galerkin method was employed for the spatial derivative discretization. The convergence analysis was discussed and experimental results were obtained to validate the method's accuracy. A space-fractional convection-diffusion equations (FCDE) coupled with space variable coefficients was solved by Anley and Zheng (2020) using FDM. The Crank–Nicolson difference scheme with the right-shifted Gr$\ddot{u}$nwald-Letnikov derivative was discussed and the stability was found to be unconditionally stable. L. Li et al. (2020) used a compact finite difference method (CFDM) for solving 2D TFCDEs commonly used to model groundwater pollution problems. They employed Crank-Nicolson difference scheme for temporal discretization and a fourth-order CFDM for spatial discretization. Unconditional Stability and convergence of the proposed method were established while experimental results were used to demonstrate the efficiency and accuracy.

A pure meshfree method was developed by Jiang et al. (2020) to solve the 1D and 2D TFCDE of CFD. The CCM was employed by Saw and Kumar (2021) to solve the TFCDE of Caputo's sense. The finite-difference approximation was used for the time discretization and the CCM for the space direction. The error and convergence analysis of the problem were illustrated. The efficiency and accuracy were examined with some experimental results. The RBFM were employed by Kamran et al. (2021) to determine the numerical solution of a 3D TFCDE. The stability and convergence analyses of the method were discussed. Two experimental results were presented to validate the technique. Adibmanesha and Rashidiniab (2021) used Sinc and B-Spline scaling functions to solve TFCDEs. The double exponential transformation was used for the spatial discretization and the B-Spline scaling functions for the temporal discretization. They obtained numerical results to justify the accuracy and efficiency of the method.

Dong et al. (2021) rigorously solved a 2D TFCDE using a combination of compact difference approach for spatial discretization and alternating direction implicit method in the time direction. Liu et al. (2021) introduced a novel FPM for solving time-fractional convection-dominated diffusion equations. The authors employed the tailored FPM for the spatial discretization while the Gr$\ddot{u}$nwald-Letnikov (G-L) discretization method and the L_1 approximation were used for the time direction. The

G-L approximation is based on the Grünwald-Letnikov definition of fractional derivatives while the tailored FPM is based on the local exponential basis function. The stability of the scheme was theoretically analysed. Numerical examples of 1D and 2D were presented to validate the method's efficiency. A FMD was adopted by H. Qiao and Cheng (2021) to obtain the numerical solution for TFCDE. They discretize the fractional derivative in L_1 format on graded meshes and used the meshless method for spatial discretization. A new modified technique which combines the Laplace transform with the Adomian decomposition method was used by H. Khan, Kumam, et al. (2022) to obtain the analytical solution of TFCDE with variable coefficients. The authors claimed their methods can be uniquely applied to solve other FPDEs with variable coefficients including initial and boundary conditions.

Roul and Rohil (2022) proposed using an alternating direction implicit approach to solve a 2D TFCDE with Caputo's derivative with a weak singularity at the initial time. They adopted a non-uniform graded mesh in discretizing the time derivative and a high-order compact operator for the space direction. Stability and convergence analysis were discussed and experimental results were performed to affirm the method. The local discontinuous Galerkin (LDG) scheme was used by Z. Wang (2022) to obtain the numerical solution of TFCDE of Caputo's sense. The $L2 - 1_\sigma$ method was used to discretize the fractional derivative while the spatial derivative was by the LDG method. Their method produced conditional stability and convergent and numerical results to affirm the effectiveness of the algorithm. The paper by Ahmad et al. (2023) presents a computational analysis of time-fractional models applied to energy infrastructure systems. The study focuses on the TFCDE of Caputo's derivative. The authors used a hybrid method that combines Lucas and Fibonacci polynomials (LFPs) to obtain the numerical results of the 1D and 2D problems and compared them with existing results in the literature. Basha et al. (2023) adopted a linearized Crank-Nicolson difference method to solve the nonlinear Riesz space-FCDE over a finite domain in 2D. The Crank-Nicolson difference method is used for the temporal discretization and weighted–shifted Grünwald-Letnikov difference method for the spatial discretization.

The order of convergence is confirmed to be 2 in space and time while unconditional stability and convergence analysis were also investigated theoretically.

Aniley and Duressa (2023) proposed a uniformly convergent numerical method for solving the TFCDEs of Caputo's sense with variable coefficients. They used the implicit Euler scheme for temporal discretization and a compact difference scheme for spatial discretization. They proved the unconditional stability and uniform convergence of the method. The error analysis shows second-order time and fourth-order space accuracy, respectively. Rashidinia et al. (2024) presented an advanced numerical approach for addressing groundwater pollution. The study focuses on solving the CDE of fractional order, a pivotal model for simulating contaminant transport in groundwater. The spectral collocation method is centralized on orthogonalized Bernoulli polynomials and was employed to obtain the numerical results.

2.5 Numerical Methods for Solving Fractional Convection-Diffusion-Reaction Equation (FCDRE)

The TFCDRE with variable coefficient was solved using compact exponential FDM by Cui (2015). The authors discretized the equation by applying the compact exponential FDM with a high-order approximation for the CFD. The stability was discussed using the Fourier method while the error estimate was by matrix analysis. Two experimental results were obtained to confirm the accuracy and efficiency of the method. G. Wang and Chen (2019) suggested an efficient CFDM for solving a class of TFCDREs with variable coefficients. The authors developed and used the $L2-1_\sigma$ approximation to discretize the time-fractional derivative and the fourth-order compact finite difference scheme for the spatial derivatives. The suggested techniques were shown to be second-order accurate in time and space and exhibit high accuracy and stability, confirmed through a series of numerical experiments. Y. Qiao et al. (2019) employed the compact integrated RBFM to solve the TFCDRE. The authors discretized the spatial derivatives by coupling the integrated RBF approximation and the compact difference method. Furthermore, the second order shifted Grünwald scheme was also used for the time discretization. Numerical experiments were performed on some 2D and 3D TFCDREs to demonstrate the effectiveness and efficiency of the method. The stability and convergence were discussed theoretically and numerically.

C. Li and Wang (2021) reviewed and developed several numerical schemes to solve TFCDRE focusing on their stability and convergence. They used the Gr$\ddot{u}$nwald-Letnikov and L1 schemes for temporal discretization and cubic spline FDM for spatial discretization. The stability and convergence of the suggested method were analyzed to achieve second-order accuracy in time and fourth-order accuracy in space. Moreover, the existence and uniqueness of the analytical solution were obtained using the Sumudu decomposition method and the maximum-minimum principle. Toprakseven (2021) proposed a weak Galerkin finite element method (GFEM) for solving TFCDRE with variable coefficients. Here, they used the L_1 approximation scheme to discretize the time derivative and GFEM for the spatial derivative. The authors earnestly examined the stability and convergence properties of the numerical scheme and demonstrated its effectiveness through various numerical experiments. A two-level fourth-order numerical scheme was developed by Ngondiep (2022) to solve the TFCDREs subject to some initial and boundary conditions. They discretized the time derivative employing a fourth-order accurate L_1-type scheme and the spatial derivatives engaging a fourth-order compact FDM. The authors investigated the stability to be unconditional and the convergence order of four. Lastly, numerical experiments were used to confirm the computational efficiency and accuracy, and compare results with existing results.

A second-order numerical scheme was employed by Choudhary et al. (2022) to solve time-FPDEs with a time delay extended to FCDREs. They used the Crank-Nicolson technique for the temporal discretization and the spline functions with a tension factor for the spatial discretization. The authors obtained conditional stability and convergence of the proposed method and demonstrated its effectiveness through two numerical experiments. Naeem et al. (2022) explored the numerical solution of the fractional convection-diffusion-reaction equation (FCDRE) using the CFD and