MULTI-STAGE DIFFERENTIAL TRANSFORM METHOD FOR SOLVING CLASSICAL AND FRACTIONAL DIFFERENTIAL ALGEBRAIC SYSTEMS

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by

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LIST OF ABBREVIATIONS

ADM	Adomian Decomposition Method
APM	Adomian Polynomial Method
DTM	Differential Transform Method
DEs	Differential Equations
EMsDTM	Efficient Multi-stage Differential Transform Method
Eq	Equation
Eqs	Equations
DAEs	Differential Algebraic Equations
FDAEs	Fractional Differential Algebraic Equations
FDTM	Fractional Differential Transform Method
HPM	Homotopy Perturbation Method
MDTM	Modified Differential Transform Method
MsDTM	Multi-stage Differential Transform Method
MsFDTM	Multistage Fractional Differential Transform Method
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
HAM	Homotopy Analysis Method

LIST OF SYMBOLS

h	Step size
K	Order of approximation
N	The number of sub-interval
T	The end of interval time
Ω	A small constant between 0 and 1
f'	Derivative of f
$\int$	Integral
$\|.\|$	Norm
D^{α}	Fractional derivative of order α

KAEDAH TRANSFORMASI PEMBEZAAN BERBILANG PERINGKAT UNTUK MENYELESAIKAN SISTEM ALGEBRA PEMBEZAAN KLASIK DAN PECAHAN

ABSTRAK

Persamaan pembezaan dikenali sebagai suatu kaedah matematik bagi memodelkan suatu permasalahan yang wujud dalam sains, kimia, fizik dan ekonomi. Umumnya, sistem persamaan pembezaan boleh dikelaskan kepada sistem persamaan pembezaan algebra pecahan dan sistem persamaan pembezaan algebra. Persamaan pembezaan algebra pecahan dan persamaan pembezaan algebra telah digunakan untuk memodelkan sistem kompleks dalam pelbagai bidang, tetapi kaedah penyelesaian berangka tradisional yang digunakan sering bergelut dengan tingkah laku penyelesaiannya. Objektif penyelidikan ini adalah untuk membangunkan dua teknik untuk menyelesaikan sistem persamaan pembezaan algebra pecahan serta sistem persamaan pembezaan algebra berdasarkan masalah nilai awal melalui kaedah penjelmaan pembezaan piawai dan kaedah penjelmaan pembezaan pecahan. Analisis perbandingan antara penyelesaian yang diperolehi oleh kaedah yang dicadangkan dan kaedah lain seperti kaedah analisis homotopi, juga akan dibincangkan. Contoh berangka telah disimulasikan menggunakan perisian Maple 19 untuk menilai kaedah yang dicadangkan. Keputusan berangka menunjukkan bahawa kaedah penjelmaan pembezaan berbilang peringkat dan kaedah penjelmaan pembezaan pecahan berbilang peringkat memberikan anggaran yang tepat berbanding dengan penjelmaan pembezaan dan skema penjelmaan

pembezaan pecahan untuk penyelesaian sistem persamaan pembezaan algebra pecahan dan sistem persamaan pembezaan pecahan. Selain itu, teknik yang dicadangkan mengatasi had penumpuan pada selang masa yang besar, memastikan kebolehpercayaan dan kecekapan skim. Tambahan pula, teknik yang dicadangkan menawarkan penyelesaian baharu untuk pelbagai fenomena kehidupan sebenar dan menangani banyak masalah dunia sebenar yang dimodelkan sebagai sistem persamaan pembezaan algebra pecahan dan sistem persamaan pembezaan algebra.

MULTI-STAGE DIFFERENTIAL TRANSFORM METHOD FOR SOLVING CLASSICAL AND FRACTIONAL DIFFERENTIAL ALGEBRAIC SYSTEMS

ABSTRACT

Differential equations are known as a mathematical method to model a problem that exists in science, chemistry, physics and economics. Generally, systems of differential equations can be classified into systems of fractional differential algebraic equations and systems of differential algebraic equations. Fractional differential algebraic equations and differential algebraic equations have been used to model complex systems in various fields, but the traditional numerical solution methods used often struggle with their solution behavior. The objective of this research is to develop two techniques to solve a system of fractional differential algebraic equations as well as a system of differential algebraic equations based on initial value problems through the standard differential transformation method and the fractional differential transformation method. A comparative analysis between the solutions obtained by the proposed method and other methods such as the homotopy analysis method, will also be discussed. Numerical examples were simulated using Maple 19 software to evaluate the proposed method. Numerical results show that the multilevel differential transformation method and the multilevel fractional differential transformation method provide accurate approximations compared to the differential transformation and the fractional differential transformation scheme for solving systems of fractional differential alge-

braic equations and systems of differential algebraic equations. Moreover, the proposed technique overcomes the convergence limit at large time intervals, ensuring the reliability and efficiency of the scheme. Furthermore, the proposed technique offers new solutions for various real-life phenomena and addresses many real-world problems modeled as systems of fractional differential algebraic equations and systems of differential algebraic equations.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Differential-algebraic equations (DAEs) arise naturally in many mathematical models that combine differential and algebraic constraints. Unlike ordinary differential equations (ODEs), DAEs include variables that are not all associated with derivatives, making their structure more complex and requiring specialized analytical and numerical techniques for their solution. Several physical problems are described using DAEs. Electrical networks, mechanical systems, chemical process models, and incompressible fluids all require DAEs. An key characteristic that distinguishes.The structure and index of a DAE influence the difficulty of solving it, especially for higher-index systems. The index is a DAE that has an essential role in the processing of these equations. There is one more definition for an index of DAE. Günther and Wagner (2001); Martinson and Barton (2000), however the most commonly used is the index of distinction. The index refers to the smallest number of times the DAE must be derived with respect to time to generate an ordinary differential equation Martinson and Barton (2000). Higher-index DAEs (Differential index more than one) arise in many major application challenges. They simulate multibody systems with constraints. Benhammouda and Vazquez-Leal (2015); Simeon (1993), vehicle system dynamics Simeon (1993, 1996), space shuttle simulation Brenan (1985) and incompressible fluids, among other things. For a variety of reasons, DAEs present significant challenges in both analytical and numerical solution processes, primarily due to their structural complexity arising from the interaction between differential and algebraic components. The next reason is that higher-index DAE solutions are always restricted by some implicit algebraic equation. Consequently, initial conditions for all solution components cannot be described arbitrarily. Another issue is that in order to begin numerical integration, some consistent initial conditions must be computed in order to identify these initial conditions, which must satisfy all of the restrictions. This is due to the use of inconsistent initial conditions or inaccurate estimations. As a result of the inconsistencies in the initial conditions, a nonphysical solution emerges. Numerical integration of higher-index DAEs is particularly challenging, with index reduction being a common and necessary step to make such problems tractable. Before employing the numerical integration approach, this procedure converts the DAEs to ordinary differential systems (index zero) or index-one DAEs. The index-reduction technique is time-consuming and can alter the properties of the original problem's solution. As a result, the most critical application problems in chemistry, science, physics, and engineering frequently lead to higher-index DAEs, necessitating the development of novel methodologies to efficiently solve these DAEs. Crucial development has been made in the treatment of DAEs over the last few decades. Several studies have focused on numerical solutions and have used backward differentiationBrenan (1985) Runge Kutta method Hairer, Roche, and Lubich (1989), pseudospectral method Hosseini (2005) and finite difference method Wu and White (2004). Other approaches for solving DAEs exist, such as mixed implicit methods Brugnano, Magherini, and Mugnai (2006), implicit Euler Sand (2002), Tchebychev polynomials Jaradat (2008).Several analytical approximation methods for solving DAEs have been developed in recent years. One of these approaches is the adomian decomposition method (ADM), Hosseini (2006), the homotopy perturbation method (HPM) Soltanian, Dehghan, and Karbassi (2010) the variational iteration method (VIM) Karta and Celik (2012) the homotopy analysis method (HAM) Awawdeh, Jaradat, and Alsayyed (2009), the pade' method Çelik and Bayram (2003) and the differential transform method Ayaz (2004); Benhammouda and Vazquez-Leal (2015). The DTM approach is utlized to solve linear and nonlinear DAEs-higer index. Adomain polynomials are used in this technique, Duan (2010a, 2010b); Rach (1984); Rudall and Rach (2008); Wazwaz (2000) and the DTM Al Ahmad and Abdullah (2020); Benhammouda, Vazquez-Leal, and Sarmiento-Reyes (2014); El-Zahar (2015); Fatoorehchi and Abolghasemi (2013); Gökdoğan, Merdan, and Yildirim (2012b); Odibat, Bertelle, Aziz-Alaoui, and Duchamp (2010). The DTM is used to solve DAE first, with Adomian polynomials being used to find the differential transform of nonlinear terms and the power series coefficients of the system. The index condition is then used to derive and solve a nonsingular linear recursion system. The most essential feature of this technique is that it does not need an index reduction or linearization, and it does not depend on complex tools such as perturbation parameters, trial functions, or Lagrangian multipliers like the perturbation method, HPM, or VIM. To cover the entire domain of convergence of the power series solution, multi-stage DTM (MsDTM) is applied. On the other hand our research is highlighted on Fractional DAEs (FDAEs). A generalization of DAEs is fractional differential-algebraic equations (FDAEs), in which the standard derivatives are replaced by fractional derivatives, often in the Caputo sense. These fractional operators allow for the modeling of memory and hereditary properties, which are not captured by integer-order models. FDAEs extend the modeling capabilities of DAEs and are useful in capturing complex dynamical behaviors in systems with non-local effects. FDAEs had been interesting during the last two decades is due to its applications in many different real life problems which are related to physics, chemistry, biology, engineering and etc Ahmed and Melad (2018); Arikoglu and Ozkol (2007); Krishnasamy, Mashayekhi, and Razzaghi (2017); Odibat and Shawagfeh (2007); Shiri and Baleanu (2019). many semi-analytical methods and numerical methods are applied to find the solutions of FDAEs such as ADM Momani and Al-Khaled (2005), HAM Hashim, Abdulaziz, and Momani (2009), VIM Odibat and Momani (2006) and DTM Liu, Yin, Wang, Zhao, and Cai (2013); Raslan and Sheer (2013). The DTM is used the first time to solve fractional differential equations by Pukhov Pukhov (1981). Fractional differential transform method FDTM is developed by Arikoglu and Ozkol (2007). The MsDTM is employed to overcome the basic disadvantages of the DTM and reduced DTM (RDTM), which are both giving the power series solution often converges in a very small intervals. The FMsDTM is an effective tool to generate approximate solutions approaches quickly to the exact solutions Abuasad, Yildirim, Hashim, Abdul Karim, and Gómez-Aguilar (2019). FMsDTM is applied in this work to find analytical approximate solutions of FDAEs in large range and without any changes of the behaviour of the approximate solution of the original problem such other schemes which use reduction technique as a tool to solve FDAEs. Although DAEs and FDAEs are widely used to model real-world systems such as electrical circuits, constrained mechanical systems, biological networks, and control systems. This thesis focuses on the mathematical development and application of numerical methods for solving a class of nonlinear FDAEs of index 2. Specifically, the study applies the fractional differential transform method (FDTM) and a multi-stage approach (MsFDTM) to improve the convergence and accuracy of solutions. While the real-world implications of such equations are well recognized, the current work is theoretical in nature, aiming to advance the solution techniques that can later be applied to practical problems.

1.2 Motivation of Research

The search for a credible and effective scheme to find approximate solutions to DAEs and FDAEs makes it a difficult task. This goes back to its complex structure Benhammouda (2015); Mehrmann and Unger (2023); Mortezaee, Ghovatmand, and Nazemi (2024); Moya and Lin (2023). DAEs are a mixture of ordinary differential equations and algebraic equations. Particularly when these DAEs are of higher index which makes it difficult to obtain analytical solutions. Many researchers have presented analytical methods for solving systems of algebraic differential equations in a variety of methods, such as the finite difference method, Li and Zeng (2012), the HAM method,Al-Masaeed and Jaradat (2012) and the DTM method. Most of these methods depended on reducing the index of the DAEs to a lower index and thus transforming them into an ODE system, Bauchau and Laulusa (2008); Benhammouda and Vazquez-Leal (2015). Reducing the index of a higher index DAEs can introduce several complications. It may result in initial conditions that no longer align with the original system, leading to inconsistency. More importantly, index reduction can change the qualitative behavior of the solution series. Such distortions are especially problematic in applications involving physical, mechanical, or engineering systems, where the original high-index formulation is crucial for accurately representing the underlying phenomena. Therefore, in order to overcome all these shortcomings that depend on the approach of reducing the index of the DAEs and thus the change in the behavior of the analytical solution and the initial conditions which do not meet the solutions of the DAEs. On the other hand, FDAEs has occupied important position in calculus is due to it's various applications in different fields such as engineering, physics and others. To solve problems that are modelled as FDAEs without linearization and random elements, FMsDTM is chosen to avoid these defects. A novel technique is proposed that avoids the need for index reduction in DAEs. Instead, it relies on transforming the system of DAEs into an equivalent system of algebraic equations using the DTM method in combination with the MsDTM method. This approach eliminates the drawbacks associated with index reduction and the introduction of arbitrary elements. Moreover, the use of the multi-stage strategy extends the convergence interval of the analytical solution series, enabling the construction of highly accurate and effective approximate solutions. Finding approximate solutions for DAEs and FDAEs that satisfy the initial conditions and keep the behavior of the solution without changing is the main motivation of our thesis. For this reason, MsDTM and FMsDTM were chosen to avoid these defects. The proposed technique is designed to be applied directly to DAE and FDAE systems without the need for index reduction or the introduction of arbitrary auxiliary components. This direct application presents a distinct advantage over existing methods that rely on index reduction, as such procedures often lead to modifications in the qualitative behavior of the approximate solution, potentially affecting the accuracy and consistency of the results.

1.3 Problem Statement

Differential algebraic equations (DAEs), particularly those of higher index, present significant analytical and numerical challenges due to the presence of algebraic constraints and the complex coupling between differential and algebraic variables. Traditional solution techniques often require index reduction, which may lead to inconsistent initial conditions, loss of accuracy, or alterations in the behavior of the solution. These issues become more pronounced in the context of fractional differential-algebraic equations (FDAEs), where memory effects and nonlocal behavior further complicate the solution process. This study addresses the problem of solving nonlinear, high-index DAEs and FDAEs without resorting to index reduction or introducing arbitrary components. The proposed method transforms the original system into an algebraic system using the DTM and enhances convergence using a MsDTM. This approach aims to preserve the structure of the original equations, maintain consistency with initial conditions, and provide accurate and efficient approximate solutions over a broader interval.

1.4 Research Objectives

The following are the study's goals:

1. To develop a direct numerical technique for solving nonlinear high-index DAEs and FDAEs using the DTM and the MsDTM, without requiring index reduction or linearization.
2. To evaluate the computational accuracy of the proposed method by comparing its results with existing numerical techniques and calculating the absolute error.
3. To perform a convergence analysis of the proposed numerical scheme, which is based on the DTM method combined with the MsDTM method, when applied to nonlinear high-index DAEs and FDAEs.

1.5 Research Questions

Based on the stated objectives, this study seeks to answer the following research questions:

1. How can a direct numerical approach be developed using the DTM method and the MsDTM method to solve nonlinear high-index DAEs and FDAEs without requiring index reduction or linearization?

2. How accurate is the proposed DTM and MsDTM-based numerical method when compared with existing numerical techniques in terms of absolute error?

3. What are the convergence properties of the proposed method when applied to nonlinear high-index DAEs and FDAEs?

1.6 Significance of the Study

The study's findings will be extremely beneficial to the following:

- The research will improve novel strategies for solving higher-index system of DAEs. As a result, it will offer novel answers to a wide range of real-world problems.

- Solving real-world problems in electrical circuit analysis, such as RLC circuits with constraints, which are commonly modeled using systems of DAEs.

- Addressing real-world problems in viscoelastic material modeling and anomalous diffusion, which involve memory effects and are often described by systems of FDAEs.

- The outcomes of this research provide a foundation for extending the proposed technique to more complex systems in the future, such as fractional partial differential-algebraic equations used in fluid dynamics, biological systems modeling, and multi-physics simulations.

1.7 Methodology

In this research, Standard DTM and MsDTM are applied to find approximate solutions for a system of DAEs and a system of FDAEs. The two methods are utilized to study and understand the behavior of the approximate solutions of two systems of DAEs and FDAEs equations. Productive solutions are compared with exact solutions. Convergence analysis is worked out by the proposed method.All computations are done using Maple 19 software. The two flow charts of the proposed methods are presented as follows:

Figure 1.1 presents the algorithm for solving the system of DAEs, while Figure 1.2 illustrates the procedures for solving the system of FDAEs.

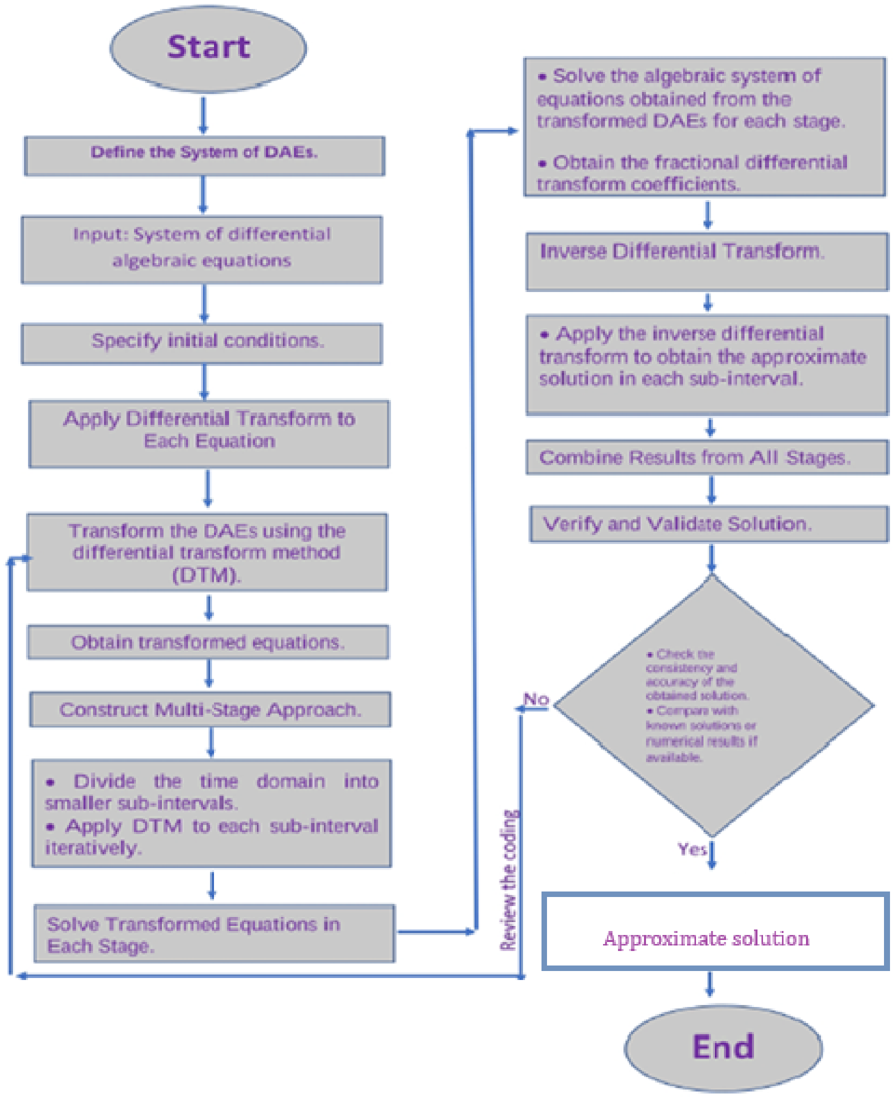

Figure 1.1: Flow Chart of Solving a System of DAEs by MsDTM Method

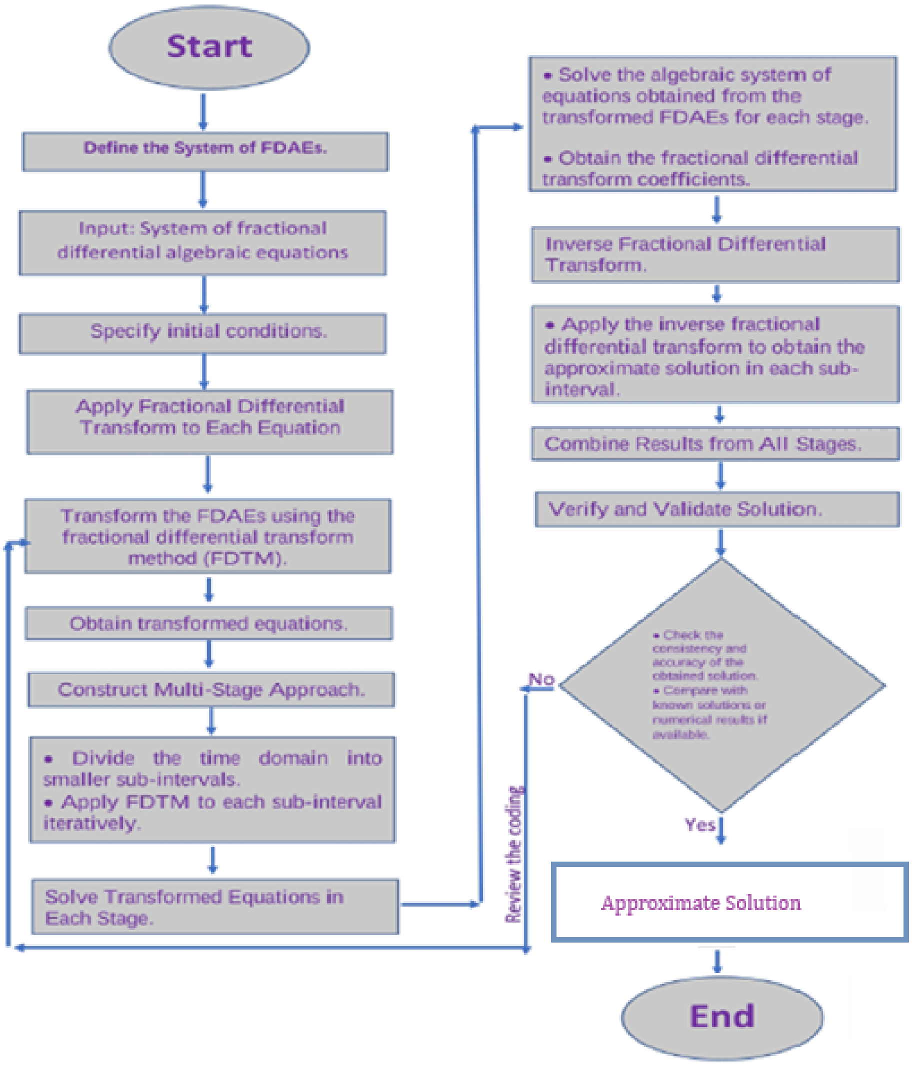

Figure 1.2: Flow Chart of Solving a System of FDAEs by MsFDTM Method

Flowcharts for Solving DAEs and FDAEs

Figures 1.1 and 1.2 illustrate the step-by-step procedure for solving DAEs and FDAEs using the DTM method and MsDTM method. Below is a detailed description of each step:

1. **Define the System of DAEs:** Establish the governing system that includes both differential and algebraic equations.

2. **Input: System of Differential-Algebraic Equations:** Formally input the equations to be solved.
3. **Specify Initial Conditions:** Provide the required initial values for the dependent variables.
4. **Apply Differential Transform to Each Equation:** Convert each equation into a series form using DTM.
5. **Transform the DAEs using the Differential Transform Method (DTM):** Reformulate the original system into a set of algebraic recurrence relations.
6. **Obtain Transformed Equations:** Collect the resulting system of algebraic equations from the DTM process.
7. **Construct Multi-Stage Approach:** Introduce a multi-stage strategy to divide the domain and improve the convergence of the solution.
8. **Divide the Time Domain into Smaller Sub-Intervals:** Break the domain into sub-intervals to increase accuracy and handle stiff behavior.
9. **Apply DTM to Each Sub-Interval Iteratively:** Implement DTM in each sub-interval using updated initial values.
10. **Solve Transformed Equations in Each Stage:** Compute the DTM coefficients for every stage.
11. **Solve the Algebraic System of Equations for Each Stage:** Solve the transformed system algebraically in each sub-interval.
12. **Obtain the Fractional Differential Transform Coefficients:** (If dealing with FDAEs) Obtain the necessary coefficients for fractional-order derivatives.

13. **Inverse Differential Transform:** Apply the inverse DTM to convert the solution back to the time domain.

14. **Apply Inverse Transform to Each Sub-Interval:** Perform the inverse transform for each stage.

15. **Combine Results from All Stages:** Assemble all sub-interval solutions into a continuous piecewise solution.

16. **Verify and Validate Solution:** Check consistency, accuracy, and convergence. Compare with analytical or numerical results if available.

17. **Decision: Is the Solution Valid?**

    - **Yes:** Proceed to accept the solution.
    - **No:** Re-check the steps, modify initial conditions, increase sub-interval resolution, or change the transform order.

18. **Approximate Solution:** The final solution obtained using the MsDTM is a reliable approximation over the entire time domain.

19. **End:** This marks the conclusion of the solution process.

1.8 Thesis Outline

This thesis has six chapters. Chapter 2 includes the literature reviews of past studies. The basic formulas and definitions that are required for this research are presented in Chapter 3. Chapter 4 offers multistage approximate solutions for a system of DAEs and a system of FDAEs. The numerical examples are checked in this chapter.Convergence analysis of the proposed technique is presented in Chapter 5. Moreover, the numerical results are discussed and compared to other schemes. Finally, the conclusion and the future work are included in Chapter 6.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Differential Transform Method (DTM) has been widely used to solve various types of differential equations, including DAEs and FDAEs, due to its simplicity and efficiency in converting differential systems into algebraic forms. However, its limitations in convergence range and its sensitivity to high-index DAEs have led to the development of improved or hybrid methods. Several of the developed approximate and numerical schemes are proposed in this chapter. Most of the recent studies are focused on solving a system of DAEs and a system of FDAEs using DTM and MsDTM, which are briefly described. Advantages and amendments of these methods are presented in this chapter.

2.2 Differential Transform Method

The Differential Transform Method (DTM) is a semi-analytical technique widely applied to solve differential equations, including DAEs and FDAEs, by transforming them into simpler algebraic forms. This section reviews the development, applications, and limitations of DTM, particularly in handling high-index or complex systems.

2.2.1 Development and Early Applications

The DTM was first introduced by Pukhov (1981) and later rediscovered and applied by Zhou (1986) for solving linear and nonlinear initial value problems in electrical circuit analysis. These foundational works laid the groundwork for applying DTM to broader problem classes. Ayaz (2004) extended DTM to solve DAEs with singular matrices, demonstrating reduced computational effort compared to traditional methods like the implicit Runge-Kutta scheme. Arikoglu and Ozkol (2007) expanded the method further by applying it to fractional differential equations (FDEs), highlighting its effectiveness but also pointing out limitations in convergence over large time intervals.

2.2.2 DTM in Solving High-Index or Complex DAEs

Several studies investigated the use of DTM for DAEs with higher index or complex structure. Osmanoğlu, Kurulay, and Bayram (2008) applied DTM to solve index-3 DAEs with singular matrices and found it efficient in producing reliable numerical results. Attili, Tabash, Dhaheri, Salhi, and Hassan (2009) applied DTM to linear index-2 Hessenberg DAEs, but their approach required index reduction prior to using DTM, which may alter the behavior of the solution a key concern addressed in the present work. Similarly, Biazar and Eslami (2010) applied DTM to quadratic Riccati equations, noting its advantage in avoiding Adomian polynomials and offering fast convergence. However, convergence limitations persisted in broader applications.

2.2.3 Application to FDAEs and Hybrid Approaches

To extend DTM to fractional systems, İbiş, Bayram, and Ağargün (2011) successfully used it to solve FDAEs and compared results with the Homotopy Analysis Method (HAM), showing high accuracy. Benhammouda et al. (2014) further applied a hybrid Reduced DTM (RDTM) with Laplace–Padé techniques to solve partial DAEs

(PDAEs) of index-2 and index-3 without index reduction, aligning more closely with the goals of the present thesis.

Recent work by Christopher, Magesh, and Tamil Preethi (2020) demonstrated the practical relevance of DTM in modeling real-world phenomena such as COVID-19, while Al-Ahmad, Sulaiman, Nawi, Mamat, and Ahmad (2022) introduced a modified DTM incorporating Laplace transforms and Padé approximants to overcome convergence issues in Volterra integro-differential equations.

Benhammouda (2023) proposed a DTM-based algorithm capable of directly solving nonlinear index-3 DAEs without reduction - marking a significant advance over earlier work, but still lacking the convergence extension offered by multistage approaches.

2.2.4 Summary and Limitations

In general, DTM has been shown to be an effective method for converting complex differential equations into simpler algebraic forms. Its main advantages include ease of implementation, avoidance of secular terms, and reduction in computational complexity. However, challenges remain in:

- Handling high-index DAEs directly without modification,
- Ensuring convergence over extended intervals,
- Maintaining solution behavior integrity without index reduction.

These limitations motivate the integration of DTM with multi-stage techniques, which is the focus of the next section.

2.3 Multi-stage Differential Transform Method

The Multi-Stage Differential Transform Method (MsDTM) was developed to overcome the convergence limitations of the traditional DTM by dividing the interval into sub-intervals and applying DTM stepwise.

2.3.1 Convergence Improvement and Real-World Applications

Odibat et al. (2010) introduced MsDTM to improve convergence in chaotic and non-chaotic systems such as Lorenz and Chen systems. Gökdoğan, Merdan, and Yildirim (2012a) further demonstrated its effectiveness on nonlinear problems like Duffing and Riccati equations, showing improved accuracy with minimal computation. In applied models, Ndii, Anggriani, and Supriatna (2018) used MsDTM to model dengue transmission dynamics, confirming its alignment with RK4 results and showing the method's strength in handling nonlinear real-world systems. Khader and Megahed (2014) translated PDEs into ODEs and applied DTM to model Newtonian fluid flow and heat transfer, further supporting its flexibility in handling physical systems.

2.3.2 Enhanced Multistage and Adaptive Methods

El-Zahar (2015) introduced an adaptive MsDTM for singular perturbation initial value problems, achieving improved convergence and reduced computation size. Similarly, Nourifar, Sani, and Keyhani (2017) proposed an Efficient MsDTM (EMsDTM) to solve systems such as Van der Pol and Rayleigh equations, demonstrating accuracy comparable to finite difference methods. More recently, Al Ahmad, Abdullah, and Zainuddin (2024); Al Ahmad and Abdullah (2020) validated MsDTM for non-autonomous nonlinear systems of ODEs. They demonstrated that MsDTM extends convergence over the full interval, outperforming traditional DTM which is limited to short intervals. Benhammouda and Vazquez-Leal (2025) proposed an enhanced Ms-DTM that integrates Adomian polynomials to solve fully implicit nonlinear ODEs. Their approach expands the convergence domain and simplifies computation without requiring system rearrangement. This recent advancement underscores the potential of multistage techniques but differs from the present study in that it is limited to ODEs, whereas this thesis extends MsDTM directly to nonlinear high-index DAEs and FDAEs, without index reduction.

2.3.3 Summary and Relevance to Present Work

The MsDTM improves upon traditional DTM by:

- Extending the convergence interval,
- Preserving computational efficiency,
- Providing accurate solutions for highly nonlinear or real-world systems.

However, most existing MsDTM studies focus on ODEs or low-index systems. This thesis fills a gap by applying the MsDTM framework to nonlinear high-index DAEs and FDAEs without index reduction, which is a novel contribution compared to the reviewed literature.

2.4 Summary

The literature review of the DTM and MsDTm methods is presented in this chapter. All the advantages and disadvantages of the DTM and MsDTM are mentioned. Several specific issues are treated based on the past research in this chapter as follows:

- Many authors used DTM and MsDTM methods to solve different systems of various differential equations, such as ODEs, PDEs, DAEs, and FDAEs equations. In this light, a new technique will be proposed and explained to solve two systems of two different differential equations like DAEs and FDAEs.

- A new technique that will be presented in Chapter 4 will handle index reduction of DAEs, address FDAEs, and maintain consistent initial conditions for the solution. As a result, the approximate solution remains valid and accurate, exhibiting no changes in behavior compared to other methods that rely on index reduction.

CHAPTER 3

BASIC CONCEPTS AND TECHNIQUES

3.1 Introduction

DAEs and FDAEs are powerful tools for modeling real-world systems in physics, engineering, chemistry, and beyond. These systems often involve processes where some variables evolve over time while others are constrained by algebraic relationships, such as conservation laws or equilibrium conditions. This chapter introduces the foundational concepts required to solve such systems. It begins with the definitions of DAEs and FDAEs and explains their key properties. It then presents the DTM method and its fractional variant, which are used to construct approximate solutions efficiently. Throughout the chapter, effort is made to bridge theoretical definitions with intuitive insights and real-world analogies, helping the reader grasp not just the mathematics, but the underlying ideas and motivations. Visual aids and step-by-step explanations are included where possible to clarify each concept.

3.2 Definitions and Properties of Differential Algebraic and Fractional Differential Algebraic Equations

Some basic concepts, definitions, and properties of FDAEs and DAEs are presented in this section. In addition, the definitions and properties of DTM and FDTM are also explained.

Definition 3.1

A system of equations that is defined as follows :

$F(t,u,u') = 0$, is called a differential algebraic equation (DAE) if the Jacobian matrix

$\frac{\partial F}{\partial u'}$ is singular (non-invertible); where each t, $u(t) \in \mathbb{R}^n$ and

$$F(t,u(t),u'(t)) = \begin{bmatrix} F_1(t,u(t),u'(t)) \\ F_2(t,u(t),u'(t)) \\ \vdots \\ F_n(t,u(t),u'(t)) \end{bmatrix},$$

März (1992). The key condition that makes this a DAE is that the Jacobian matrix $\partial F/\partial u'$ is **singular** (non-invertible). This means you cannot simply solve for the derivatives directly. Instead, part of the system enforces constraints between variables rather than governing their evolution.

Analogy: Imagine a train where some wagons (variables) are pulled by the engine (dynamics), while others are rigidly connected and follow based on constraints (algebraic conditions). DAEs represent both types.

Definition 3.2

Suppose $F(t,u,u') = A(t)u' + B(t)u + d(t)$.

Hence, for the system $F(u,u',t) = 0$ the Jacobian will be $\frac{\partial F}{\partial u'} = A(t)$

If $A(t)$ is a non-singular (an invertible), then

$[A(t)]^{-1}(A(t)u'(t) + B(t)u(t) + d(t)) = [A(t)]^{-1}.0,$

$u'(t) + [A(t)]^{-1}B(t)u(t) + [A(t)]^{-1}d(t) = 0,$

$u'(t) = -[A(t]^{-1}B(t)u(t) - [A(t)]^{-1}d(t).$

This is an ordinary differential equationMärz (1992).

Definition 3.3

The minimum number of differentiation steps required to transform a DAE into an ODE is known as the (differential) index of the DAE Martinson and Barton (2000).

Analogy: Think of peeling layers from an onion. A higher-index DAE has more hidden constraints that need to be unwrapped step by step.

3.2.1 General Form of Differential Algebraic Equations)

The general form of a DAE as follows:

Consider the DAEs of order m .

$$u^{(m)} = f(u, u', ..., u^{(m-1)}, t) - B(u, u', ..., u^{(m-1)}, t)v, \tag{3.1}$$

$$0 = g(u, t), \tag{3.2}$$

which is index $m+1$ if GB is nonsingular, where $G = \frac{\partial g}{\partial u}$ Ascher and Lin (1997).

Analogy: Think of a robotic arm where motors drive some joints (differential equations), while the links and mechanical constraints ensure the structure remains intact (algebraic equations).

Worked Example: Pendulum System

Consider a simple pendulum of length ℓ and mass m, swinging under gravity. Let $x(t)$ and $y(t)$ be the Cartesian coordinates of the pendulum mass. The motion is subject to the geometric constraint:

$$x(t)^2 + y(t)^2 = \ell^2$$

This constraint must hold at all times. Meanwhile, Newton's second law gives the equations for the accelerations $\ddot{x}(t)$ and $\ddot{y}(t)$, resulting in a system of second-order differential equations. By rewriting this system in first-order form using auxiliary variables, one obtains a DAE system:

$$\begin{aligned}
\dot{x} &= v_x, \\
\dot{y} &= v_y, \\
m\dot{v}_x &= \lambda x, \\
m\dot{v}_y &= \lambda y - mg, \\
x^2 + y^2 &= \ell^2
\end{aligned}$$

where $\lambda(t)$ is a Lagrange multiplier enforcing the constraint.

3.2.1(a) Differential Transform Method

Some definitions related to DTM method are presented in this section.

Definition 3.4

If a function $u(t)$ is analytical with respect to t in the domain of interest, then

$$U(k) = \frac{1}{k!}\frac{d^k u(t)}{dt^k}\Big|_{t=t_0} \quad , \tag{3.3}$$

is the transformed function of $u(t)$ Arikoglu and Ozkol (2007); Zhou (1986).

Definition 3.5

The differential inverse transforms of the *set* $\{U(k)\}_{k=0}^{n}$ is defined by:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k \ . \tag{3.4}$$

Substituting (3.3) into (3.4), we deduce that

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k u(t)}{dt^k}\Big|_{t=t_0}(t-t_0)^{\mathrm{k}} \tag{3.5}$$

Arikoglu and Ozkol (2007); Zhou (1986). It is clear from the previous definitions (3.4) and (3.5) that the DTM principle is derived from the power series expansion. Consider