NEW GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SOLVING FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

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by

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LIST OF ABBREVIATIONS

BVPs	Boundary value problems
ODEs	Ordinary differential equations
IVPs	Initial value problems
OFDEs	Ordinary fractional differential equations
FDs	Fractional derivatives
FDEs	Fractional differential equations
RLD	Riemann-Liouville derivative
DTM	Differential transform method
BCM	Bessel collocation method
FDTM	Fractional differential transform method
LT-HPM	Laplace transform homotopy perturbation method
GDTM	Generalized differential transform method
SRKM	Simplified reproducing kernel method
GFDTM	Generalized fractional differential transform method
ADM	Adomian decomposition method
NGDTM	New generalized differential transform method

KAEDAH PENJELMAAN PEMBEZAAN TERITLAK BAHARU UNTUK PENYELESAIAN PERSAMAAN PEMBEZAAN PECAHAN BIASA

ABSTRAK

Persamaan pembezaan pecahan (FDE) telah mendapat perhatian yang signifikan dalam beberapa tahun kebelakangan ini kerana kebolehan mereka untuk memodelkan fenomena bukan tempatan dan bergantung kepada ingatan dalam pelbagai bidang sains dan kejuruteraan. Berbeza dengan persamaan pembezaan klasik, FDE melibatkan pembezaan dengan orde bukan integer, yang menghasilkan sifat matematik yang rumit dan baru. Ciri unik ini membolehkan FDE untuk berjaya diterapkan dalam bidang seperti teori kawalan, kimia, ekonomi, fizik, kewangan, perubatan, dan biologi. Oleh itu, terdapat keperluan mendesak untuk inovasi dan pengembangan teknik penyelesaian yang tepat dan berkesan untuk jenis persamaan pembezaan ini. Walau bagaimanapun, salah satu cabaran utama dalam menangani FDE adalah kekurangan penyelesaian tepat, terutamanya disebabkan oleh sifat kompleks masalah dunia nyata yang mereka wakili. Untuk menangani cabaran ini, tujuan utama tesis ini adalah untuk memperkenalkan kaedah novel yang dipanggil Kaedah Transform Pembezaan Umum Baru (NGDT). Kaedah NGDT menggunakan Pembezaan Riemann-Liouville (RLD) dan direka untuk menawarkan penyelesaian analitik atau yang dihampiri secara numerik untuk keadaan tertentu FDE biasa, sama ada dalam bentuk linear atau bukan linear. Selain daripada pengembangan teoritisnya, kaedah NGDT dinilai melalui beberapa contoh dan dibandingkan dengan kaedah lain yang sedia ada, seperti Kaedah Kolokasi Bessel (BCM), Kaedah Pembezaan Adomian (ADM), Kaedah Kernel Menghasilkan yang Dipermudahkan (SRKM), Kaedah Transform Pembezaan Pecahan (FDTM), Kaedah Perturbasi

Homotopi Transform Laplace (LT-HPM), dan Kaedah Transform Pembezaan Umum (GDTM). Kaedah NGDT membolehkan penyelidik dan pengamal untuk mengatasi had pendekatan sebelumnya, menghasilkan penyelesaian yang lebih berkesan dan tepat. Analisis perbandingan menunjukkan bahawa NGDT memberikan penyelesaian yang dihampiri dengan lebih mudah, dengan ketepatan yang lebih baik dan pengurangan usaha pengiraan berbanding dengan kaedah sedia ada. Keberkesanannya ditunjukkan melalui aplikasi kepada pelbagai persamaan, termasuk persamaan Riccati pecahan, persamaan jenis Bratu pecahan, dan persamaan Bagley-Torvik. Kajian kes ini menonjolkan kemampuan kaedah NGDT untuk menangani masalah dunia nyata dengan boleh dipercayai. Secara keseluruhan, kajian ini memajukan kaedah analitik yang dihampiri untuk menyelesaikan FDE dan menekankan kaedah NGDT sebagai alat yang berkuasa untuk mengatasi cabaran yang berkaitan, memudahkan kemajuan dalam pelbagai bidang sains dan kejuruteraan.

NEW GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SOLVING FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

ABSTRACT

Fractional differential equations (FDEs) have garnered significant attention in recent years due to their ability to model nonlocal and memory-dependent phenomena in various scientific and engineering domains. Unlike classical differential equations, FDEs involve derivatives of non-integer order, leading to intricate and novel mathematical properties. This unique characteristic has allowed FDEs to find successful applications in fields such as control theory, chemistry, economics, physics, finance, medicine, and biology. Consequently, there is a pressing need for innovation and the development of accurate and efficient solution techniques for these types of differential equations. However, one of the main challenges in dealing with FDEs is the scarcity of exact solutions, mainly due to the complex nature of the real-world problems they represent. To address these challenges, the main goal of this thesis is to introduce a novel method called the New Generalized Differential Transform Method (NGDT). The NGDT method utilizes the Riemann-Liouville Derivative (RLD) and is designed to offer analytical or numerically approximated solutions for particular instances of ordinary FDEs, whether in linear or nonlinear form. In addition to its theoretical development, the NGDT method is evaluated through several examples and benchmarked against other existing methods, such as the Bessel Collocation Method (BCM), the Adomian Decomposition Method (ADM), the Simplified Reproducing Kernel Method (SRKM), the Fractional Differential Tansform Method (FDTM), the Laplace Transform Homotopy Perturbation Method (LT-HPM), and the Generalized Differential Transform

Method (GDTM). The NGDT method enables researchers and practitioners to surpass the limitations of previous approaches, yielding solutions that are more efficient and accurate. Comparative analysis reveals that NGDT provides approximate solutions more straightforwardly, with improved accuracy and reduced computational effort compared to existing methods. Its effectiveness is demonstrated through applications to various equations, including the fractional Riccati equation, fractional Bratu-type equations, and the Bagley-Torvik equation. These case studies highlight the NGDT method's capability to address real-world problems reliably. Overall, this study advances approximate analytical methods for solving FDEs and underscores the NGDT method as a powerful tool for overcoming associated challenges, facilitating progress in diverse scientific and engineering fields.

CHAPTER 1

INTRODUCTION

1.1 General Introduction

Fractional calculus extends the principles of classical calculus, which were introduced by Leibniz and L'Hôpital in 1695, by addressing integrals and derivatives of any order. Since then, many eminent mathematicians have worked on these topics and related disciplines, which have attracted much interest in recent decades. Since the 19th century, fractional calculus has substantially developed, and numerous contributors have provided a number of definitions of fractional derivative and integrals. Specifically, this field encompasses the concepts and techniques for solving differential equations that involve fractional derivatives of an unknown function, referred to as fractional differential equations (FDEs).

FDEs serve as a tool to depict physical models for diverse phenomena in both engineering and applied sciences. However, obtaining exact solutions for many FDEs is often challenging due to the inherent difficulty in accurately modeling real-world problems. Therefore, we often opt for approximate analytical or numerical solutions to the equations representing the problems. Several recent studies on numerical and analytical approximate solutions to FDEs have been developed. While numerical methods are applicable in numerous practical scenarios, several authors express a preference for analytical approximate methods. This preference stems from the fact that analytical methods offer a representation of the solution in an analytical form, providing more detailed insights over specific intervals of interest. In contrast, numerical methods present solutions in a discretized form, making the attainment of a continuous representation somewhat intricate. Academics have focused a lot of attention recently on fractional differential equations. These equations involve fractional-order derivatives, which make their mathematical formulation and implementation highly challenging. The study of these equations will enhance our understanding of complex phenomena and enable the development of more accurate modeling in diverse scientific disciplines. Although some numerical or analytical methods have been suggested for solving FDEs, there are numerous questions that arise. There is therefore a need to develop new numerical methods.

Hence, the objective of this thesis is to introduce an advanced approximate analytical technique tailored for the resolution of both initial and boundary value problems. This approach strikes a balance between accuracy and computational feasibility and provides reliable solutions within a reasonable amount of time. Through comprehensive numerical tests and comparisons with existing methodologies, the efficacy and reliability of the suggested technique will be thoroughly investigated, showcasing its better performance in terms of accuracy and computational effectiveness.

1.2 Motivation

In most cases, the FDEs do not have exact analytical solutions. As a result, the equations are solved through several approaches, such as the FDTM (Arikoglu & Ozkol, 2007), BCM (Tohidi & Nik, 2015), GFDTM (Odibat et al., 2008), and ADM (Kumar & Umesh, 2022), all of which offer approximate analytical solutions. Further, many researchers developed methods for solving linear and nonlinear FDEs, as discussed in (Abuteen et al., 2014).

Although approximate analytical methods are extensively used in solving various types of FDEs, since the solutions are derived through series, the techniques have some disadvantages, such as their effectiveness being limited in some applications. One of the major drawbacks reported by numerous researchers is that obtaining satisfactory approximate solutions requires a substantial number of iterations with these methods. To address these limitations, minimize computational effort, and streamline the calculations, we are inspired to propose and develop new techniques built upon existing methods.

1.3 Objective

The objectives of this dissertation are:

- To analyze a novel technique for solving nonlinear and linear FDEs of initial value problems based on a generalized Taylor series, called the NGDTM.
- To develop a new technique based on the NGDTM to investigate different linear and nonlinear boundary value problems with fractional derivatives (FDs).
- To solve linear and nonlinear FDEs associated with both initial and boundary value problems using combined NGDTM with ADM.
- To investigate the efficacy of the NGDTM by comparing it with the established FDTM, GDTM, and several methods, as well as known exact solutions.

1.4 Methodology

This study is conducted using the following methodology:

1. The literature pertaining to methods for solving boundary and initial value problems, such as problems related to fractional differential equations (FDEs), will be comprehensively reviewed. Emphasis will be placed on the FDTM, ADM, and the GDTM, with in-depth discussions of their general conceptual details.

2. Afterwards, amine focus will be placed on the Taylor series and its generalizations. This step will serve as the basis of this study.

3. Subsequently, the new method NGDTM will be constructed and formulated to investigate boundary and initial value problems of initial and boundary value problems of ordinary FDEs.

4. Afterward, the NGDTM will be integrated with the ADM to explore and address both initial and boundary value problems.

5. To demonstrate the effectiveness of NGDTM, numerical tests will be conducted. The results will be compared to the results found from the FDTM, ADM, BCM, LT-HPM, and GDTM, as well as known solutions or exact results. Through this comparison, the reliability and accuracy of the NGDTM approach will be assessed.

6. The numerical examples in this study will be computed using Maple 2016.

1.5 The Thesis outline

The flowchart in Figure 1.1 depicts the overall structure of this study. The thesis is organized into Chapters 2–7, which are outlined as follows:

Chapter 2 provides a comprehensive review of the fundamental concepts and techniques essential for understanding fractional calculus. Additionally, this chapter offers a concise overview of the key principles underlying the approximate analytical methods investigated in the context of this thesis.

Chapter 3 of the thesis provides an overview of recent studies conducted by various researchers, focusing on the search for approximate solutions to various types of ordinary FDEs.

In Chapter 4, an advanced method called the NGDTM is introduced, and a set of theorems is presented along with their corresponding proofs. The effectiveness of the proposed method is evaluated through various numerical examples involving initial conditions. Moreover, a comparative analysis is conducted, comparing the approximate solutions obtained by the FDTM and GDTM with the exact solutions.

In Chapter 5, we present a significant modification of the conditional boundary setting method along with the utilization of NGDTM to solve specific types of ordinary FDEs, including the Bagley-Torvik equation and fractional Bratu-type equations. This chapter also includes a comparative analysis, comparing the approximate solutions obtained by the NGDTM with those obtained by other methods such as the BCM, LT-HPM, SRKM, and FDTM. The solutions obtained are further compared with the exact solutions for assessment and evaluation.

In Chapter 6, we combine the NGDTM with the ADM and applied this hybrid method to solve the same types of FDE as addressed in the preceding chapters. A comparative analysis is conducted, comparing the approximate solutions obtained by these combined methods with the exact solutions.

Chapter 7 functions as the concluding chapter, offering a summary of the key findings and main results derived from this study. Furthermore, it includes recommendations for future research directions and potential areas of exploration within this field.

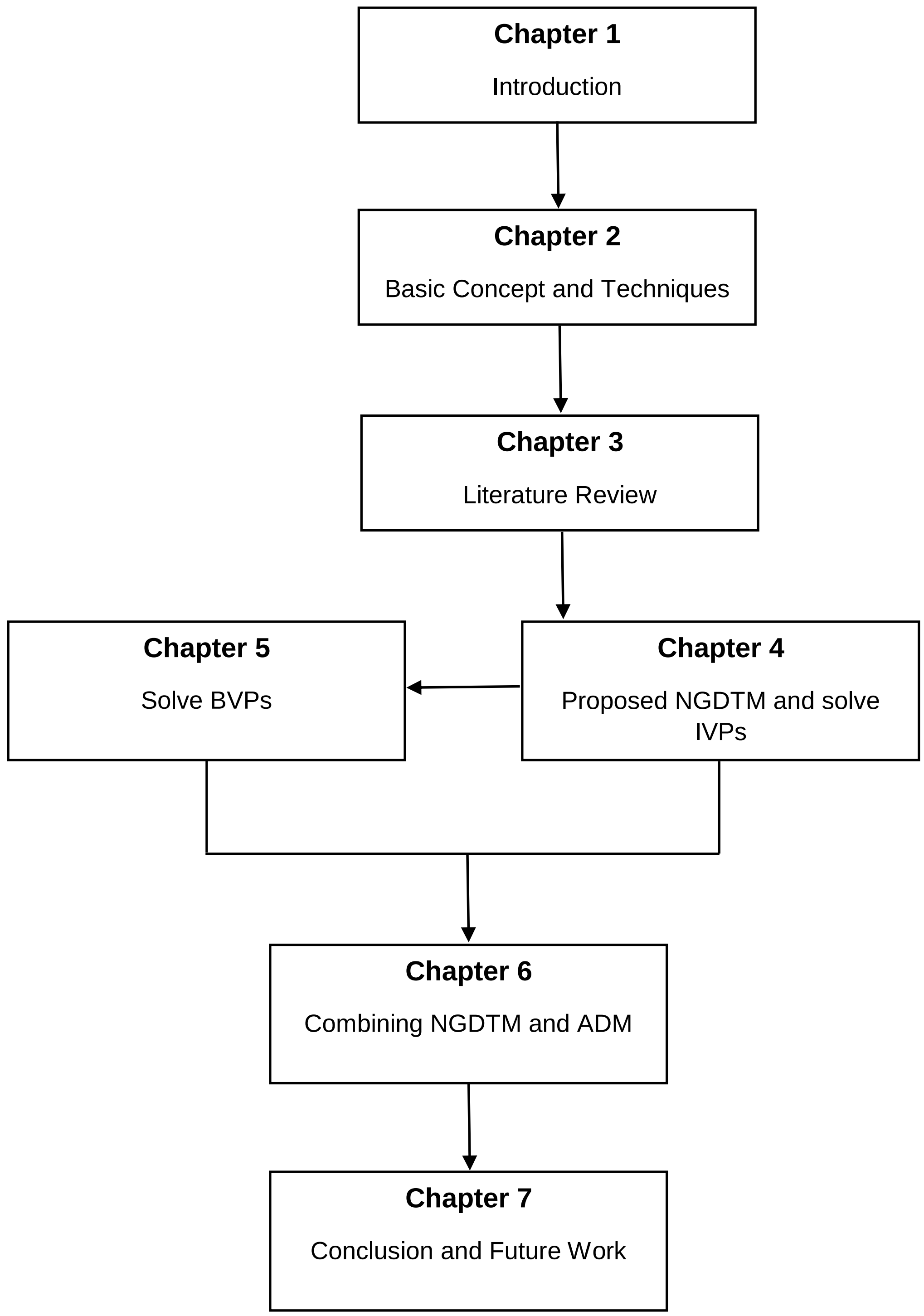

Figure 1.1: Flow Chart of the Chapters .

CHAPTER 2

SOME PREMILECES

2.1 Introduction

In this chapter, we present some special functions related to the theory of fractional calculus and discuss the fundamental concepts and theories that are useful and Play a substantial role in the theory of fractional calculus and FDEs. Furthermore, a brief description of the analytical methods required in this study will also be given. These methods are the DTM, ADM, FDTM, and GDTM, which will be used throughout this thesis.

2.2 Some Special Functions

This section presents specific functions that prove beneficial in the theory of fractional calculus.

2.2.1 Euler's Gamma Function

The gamma function, denoted as $\Gamma(z)$, serves as an extension of the factorial to encompass both complex and real numbers. and it is one of the most basic functions of fractional calculus; it was first introduced by the mathematician Leonhard Euler (1707-1783) (Sebah & Gourdon, 2002). It extends the concept of the factorial $n!$ (Das, 2011), enabling n to take any positive integer, non-integer, or complex value except the non-positive integers.The gamma function is defined by the following expression:

Definition 2.1. *The gamma function $\Gamma(z)$ is defined through the integral(Podlubny, 1998).*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.1}$$

or

$$\Gamma(z) = 2\int_0^\infty t^{2z-1}\, e^{-t} dt. \tag{2.2}$$

Euler's gamma function fulfills the following properties (Sebah & Gourdon, 2002):

1. The gamma function is analytic for all $z \in \mathbb{C}$ except for non-positive integers.

2. For $z \neq 0, -1, -2, \ldots$, the gamma function satisfies the relation $\Gamma(z) = \frac{\Gamma(z+1)}{z}$.

3. For $n = 2, 3, 4, \ldots$, the expression $\Gamma(nz)$ is given by:

$$\Gamma(nz) = \frac{n^{nz-0.5}}{(2\pi)^{0.5(n-1)}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right).$$

4. For non-negative integers n, we have $\Gamma(n+1) = n!$.

5. $\Gamma(1-z) = -z\Gamma(-z),\ z \in \mathbb{C}\{0, 1, 2, ...\}$.

6. For $z \in \mathbb{C}$, we have the relation $\Gamma(1-z) = -z\Gamma(-z)$.

2.2.2 Mittag-Leffler Function

Mittag-Leffler introduced a significant function that made a valuable contribution to the field of fractional calculus (Mittag-Leffler, 1903):

Definition 2.2. *For $z \in \mathbb{C}$ and $\alpha > 0$, the Mittag-Leffler Function $E_\alpha(z)$ is (Arshad & Mubeen, 2018):*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{2.3}$$

2.3 Fractional Calculus

Fractional calculus is a branch of mathematical analysis concerned with integrals and derivatives of orders that are not whole numbers. This field arose after Leibnitz invented the notation $\frac{y^n y}{dx^n}$, where n is a positive integer. L'Hôpital asked Leibnitz in 1695 "What if n is a fractional number (n=$\frac{1}{2}$)" and he replied, "Now it is a paradox, from which one-day useful consequences will be drawn." The subject has endured for more than 350 years, and many mathematicians have been interested in and studied this area over the years.

2.3.1 Riemann-Liouville Fractional Integral

The Riemann-Liouville integral is attributed to Bernhard Riemann and Joseph Liouville, who in 1832 were the first to formulate the possibility of fractional calculus. The definition of the Riemann-Liouville integral is given by:

Definition 2.3. *For $\alpha \in R^+$, $x > 0$, and $n \in N$, where N are the positive integers, the Riemann-Liouville integral is defined as (C. Li et al., 2011):*

$$I_x^y f(y) = \frac{a}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}\, dy. \tag{2.4}$$

Here are several significant properties of the fractional integral that are helpful in solving equations incorporating fractional-order integrals.

Proposition 2.1. *For any real numbers α and β, the Riemann-Liouville fractional integral satisfies (Weilbeer, 2006):*

1. $J^{\alpha+\beta} f(t) = I^{\alpha} + I^{\beta} f(t)$.

2. $I^0 f(t) = f(t)$.

2.3.2 Riemann-Liouville Fractional Derivative

Property 2.1. *If $f(x) = x^{\zeta}$ for some $\zeta > -1$ and $\alpha > 0$ then (Gorenflo & Mainardi, 1997):*

$$D^{\alpha} f(x) = \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta - \alpha + 1)} (x)^{\zeta - \alpha}, \quad t > 0. \tag{2.5}$$

Property 2.2. *If $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and the function $f(t)$ is defined on the interval $[a, b]$, then the following composition property holds for the fractional derivatives (Kilbas & Trujillo, 2006):*

$$D_a^{\alpha_1} D_a^{\alpha_2} f(t) = D_a^{\alpha_1 + \alpha_2} f(t). \tag{2.6}$$

Property 2.3. *If $\alpha \in \mathbb{R}^+$, and the function $f(t)$ belongs to the interval $[a, b]$, then (Kilbas & Trujillo, 2006):*

$$D^{\alpha} \sum_{n=0}^{\infty} f(t) = \sum_{n=0}^{\infty} D^{\alpha} \Big(f(t) \Big). \tag{2.7}$$

2.4 Taylor Series

A Taylor series is a mathematical expression representing a function as an infinite sum of terms obtained from the derivatives of the function evaluated at a particular point. This concept is credited to Brook Taylor, who introduced it in 1715, as cited in (Ghosh, 2018).

Definition 2.4. *Suppose $f(x)$ is a function such that $f^{n+1}(x)$ exists for all x in an open interval containing $b_\circ$ (Goodwillie, 2003). Then:*

$$
\begin{aligned}
f(x) &= f(b_\circ) + f'(b_\circ)(x - b_\circ) + f''(b_\circ)\ \frac{(x - a_\circ)^2}{2!} + f'''(b_\circ)\ \frac{(x - b_\circ)^3}{3!} + \ldots \\
&= \sum_{k=0}^{\infty} c_k\ (x - b_\circ)^k. \qquad (2.8)
\end{aligned}
$$

where $c_k = \frac{f^{(k)}(b_\circ)}{k!}$*, and $f^{(k)}(b_\circ)$ denotes the k^{th} derivative of f at the point $b_\circ$.*

2.5 Generalized Taylor Series

This section introduces the generalized Taylor formula, including the Riemann-Liouville fractional derivative (Trujillo, 1999).

Theorem 2.1. *(Generalized Taylor's formula): Suppose $n \in N$, f be a continuous function in $(a, b]$, and $\alpha \in [0, 1]$. satisfying the following condition (Trujillo, 1999):*

(i) $D_a^{(n-1)\alpha} f$ is continuous on [a,b].

(ii) $\forall l$ = 1,...,n, $D_a^{j\alpha} f \in$ C((a,b]) and $D_a^{j\alpha} f \in {}_aI_\alpha$.

(iii) If $a < \frac{1}{2}$, then for each $l \in N$, $1 \leq l \leq n$, such that $(l + 1)\alpha < 1$, $D_a^{(l=1)\alpha}$f(x) is ζ-continuous at x = a for some ζ, $1 - (l + 1)\alpha \leq \zeta \leq 1$, or a-singular of order α .

Then, $\forall x \in (a, b]$,

$$f(x) = \sum_{l=0}^{n} \frac{c_l (x - a)^{(l+1)\alpha - 1}}{\zeta((l + 1)\alpha)} + R_n(x, a), \qquad (2.9)$$

with

$$R_n(x, a) = \frac{D^{(n+1)\alpha - 1} f(\varphi)}{\Gamma((l + 1)\alpha)}(x - a^{(n+1)\alpha - 1}), \qquad a \leq \varphi \leq x, \qquad (2.10)$$

and for each $l \in N$, $1 \leq l \leq n$,

$$c_l = \Gamma(\alpha)[(x-a)^{1-\alpha} D_a^{l\alpha} f(x)](a^+) = I_a^{1-\alpha} D_a^{l\alpha} f(a^+). \tag{2.11}$$

Proof. See (Trujillo, 1999). □

Theorem 2.2. *Naturally, any function $f(x)$ that fulfills the condition outlined in the previous theorem can be expressed through a generalized Taylor series, as follows (Trujillo, 1999):*

$$f(x) = (x-a)^{\alpha-1} \sum_{l=0}^{n} \frac{c_l (x-a)^{l\alpha}}{\Gamma((l+1)\alpha)}, \tag{2.12}$$

Given the previously defined c_l, this expression applies to all $x \in (a, b]$, where the series converges and

$$\lim_{n \to \infty} R_n(x, a) = 0. \tag{2.13}$$

A function that can be expanded as (2.12) is referred to as an α-analytic at $x = a$.

Theorem 2.3. *Suppose that $D_{x_\circ}^{l\alpha} f \in C[x_\circ, y_\circ]$, where $0 < \alpha \leq 1$, $l = 0, 1..., n+1$, and If $x \in [x_\circ, y_\circ]$. Then:*

$$g(x) \cong \sum_{l=0}^{N} \frac{C_l}{\Gamma(\alpha l + \alpha)} (x - x_\circ)^{l\alpha + \alpha - 1}, \tag{2.14}$$

where

$$c_l = \Gamma(\alpha)[(x - x_\circ)^{1-\alpha} D_{x_\circ}^{l\alpha} f(x)](x_\circ^+). \tag{2.15}$$

2.6 Fractional Differential Transform Method

Within this section, we present the Fractional Differential Transform Method (FDTM). This methodology was formulated by Arikoglu and Ozkol (Arikoglu & Ozkol, 2007). The definition of fractional differentiation in the Riemann-Liouville sense is as follows:

$$D_{a_\circ}^{p} f(x) = \frac{1}{\Gamma(m-p)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left[\int_{a_\circ}^{x} \frac{f(t)}{(x-t)^{1+p-m}} dt \right], \tag{2.16}$$

For $m-1 \leq p < m$, where $m \in Z^+$ and$x > a_\circ$, we proceed to represent the analytical and continuous function $f(x)$ using a fractional power series expansion as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-a_\circ)^{\frac{k}{\theta}}, \tag{2.17}$$

where θ is the order of the fraction to be selected and $F(x)$ is the fractional differential transform of $f(x)$. Given that practical applications typically utilize integer-order derivatives to implement initial conditions, the transformation of these initial conditions is outlined as follows (Arikoglu & Ozkol, 2007):

$$F(k) = \begin{cases} If\ \frac{k}{\theta} \in Z^+ & \frac{1}{(\frac{k}{\theta})!} \left[\frac{\mathrm{d}^{k/\theta}}{\mathrm{d}x^{k/\theta}} \right]_{x=a_\circ} \\ If\ \frac{k}{\theta} \notin Z^+ & 0, \end{cases} \tag{2.18}$$

where k ranges from 0 to $(\theta \cdot \alpha - 1)$, where α represents the order of the considered Fractional Differential Equation (FDE). Consequently, n needs to be selected such that $n\alpha$ yields a positive integer.

The subsequent theorem outlines fundamental properties of FDTM, where $H(x)$, $G(x)$, and $F(x)$ denote the fractional differential transforms of the functions $h(x)$, $g(x)$, and $f(x)$, respectively.

Theorem 2.4. *Suppose $H(x)$, $G(x)$, and $F(x)$ represent the fractional differential transforms of the functions $h(x)$, $g(x)$, and $f(x)$, respectively, as stated in (Arikoglu & Ozkol, 2007):*

(i) If $h(x) = g(x) \pm f(x)$, then $H(k) = G(k) \pm F(k)$.

(ii) If $h(x) = af(x)$, then $H(k) = aF(k)$.

(iii) If $h(x) = g(x)f(x)$, then $H(k) = \sum_{i=0}^{k} G(i)F(k-i)$.

(iv) If $h(x) = f_1(x)f_2(x)...f_{n-1}(x)f_n(x)$, then

$H(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1}^{k_2} F_1(k_1)F_2(k_2-k_1)...F_{n-1}(k_{n-1}-k_{n-2})F_n(k-k_{n-1})$.

(v) If $h(x) = (x-a_\circ)^p$, then $H(k) = \delta(k-\alpha p)$, where $\delta(k) = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$.

(vi) If $h(x) = D_{a_\circ}^{q}\left[f(x)\right]$, then $H(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)}F(k+\alpha q)$.

It seems that the (FDTM) offers a simple approach to solving various types of FDEs by converting them into algebraic equations (Ibis et al., 2011). This simplification makes it particularly useful for dealing with linear and nonlinear equations, whether they have constant or variable coefficients, and can handle initial or boundary conditions (Khudair et al., 2017; Nazari & Shahmorad, 2010). However, its effectiveness might be limited when dealing with highly complex or nonlinear problems. It seems that the method's efficiency can vary depending on the specific equation at hand. Additionally, applying boundary conditions might be tricky, especially in scenarios involving multiple conditions or intricate geometries (Rysak & Gregorczyk, 2021). As with any numerical method, the suitability of FDTM depends on the problem's nature. It is valuable for its strengths, but it is equally important to be aware of its limitations and potential challenges, especially when applying it to real-world problems.

2.7 Generalized Differential Transform Method

In this section, we introduce the Generalized Differential Transform Method (GDTM) formulated by Vedat Suat Erturk, Shaher Momani, and Zaid Odibat (Odibat et al., 2008). The method is structured as follows:

We define the GDTM of the k^{th} derivative of a function $f(x)$ as follows:

$$F_\alpha(k) = \frac{1}{\Gamma(\alpha k + 1)} \left[(D^\alpha)^k f(x) \right], \tag{2.19}$$

and the differential inverse transform of $F_\alpha(k)$ is define as follows:

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x - x_\circ)^{\alpha k}. \tag{2.20}$$

If we substitute (2.19) into (2.20), it is obtained

$$f(x) = \sum_{k=0}^{\infty} \frac{(x - x_\circ)^{\alpha k}}{\Gamma(\alpha k + 1)} \left((D^\alpha)^k f(x) \right). \tag{2.21}$$

Some basic properties of (2.5) are provided in the following theorem. The fractional differential transforms of the functions $h(x)$, $g(x)$, and $f(x)$ are shown here by the symbols $H_\alpha(x)$, $G_\alpha(x)$, and $F_\alpha(x)$, respectively.

Theorem 2.5. *Suppose $H(x)$, $G(x)$, and $F(x)$ represent the fractional differential transforms of the functions $h(x)$, $g(x)$, and $f(x)$, respectively, as stated in (Odibat et al., 2008):*

(1) If $h(x) = g(x) \pm f(x)$*, then* $H(k) = G(k) \pm F(k)$.

(2) If $h(x) = af(x)$, *then* $H(k) = aF(k)$.

(3) If $h(x) = g(x)f(x)$, *then* $H(k) = \sum_{i=0}^{k} G(i)F(k-i)$.

(4) If $h(x) = f_1(x)f_2(x)...f_{n-1}(x)f_n(x)$, *then*

$H(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1}^{k_2} F_1(k_1)F_2(k_2-k_1)...F_{n-1}(k_{n-1}-k_{n-2})G_n(k - k_{n-1})$.

(5) If $h(x) = (x - a_\circ)^{n\alpha}$, *then* $H(k) = \delta(k-n)$ *where,* $\delta(k) = \begin{cases} 0 & \textit{if } k \neq 0 \\ 1 & \textit{if } k = 0 \end{cases}$.

(6) If $h(x) = D^{\alpha}_{x_\circ} f(x)$, *then* $H_\alpha(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} F_\alpha(k+1)$.

(7) If $h(x) = D^{\beta} f(x)$, *then* $H_\alpha(k) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} F_\alpha(k+\frac{\beta}{\alpha})$.

The Generalized Differential Transform Method (GDTM) is known for its simplicity and ability to be applied to various types of equations, making it a popular numerical technique. One of its advantages is that it can handle both nonlinear equations, including fractional-order equations, and systems of equations without relying on grids or meshes (Di Matteo & Pirrotta, 2015; Erturk et al., 2008). However, it faces challenges with convergence, applicability to certain equations, sensitivity to parameters, and dependence on initial conditions (S. Kumar et al., 2017). When deciding whether GDTM is appropriate, it is crucial to carefully consider the characteristics of the problem, as its effectiveness varies according to the complexity and nature of the equation (Bansal & Jain, 2015). Sensitivity to parameter choices and a dependency on initial conditions necessitate careful consideration and calibration for optimal results. Assessing its suitability based on the complexity, behavior, and boundary conditions of the equation is crucial to effectively leverage its strengths while navigating its limitations(Odibat et al., 2022).

2.8 Adomian Decomposition Method

In this section, we introduce the Adomian Decomposition Method (ADM) by George Adomian in the 1980s (Adomian, 1988). The ADM is a numerical technique used to solve nonlinear differential equations. ADM aims to approximate the solutions of differential equations without the need for discretization, linearization, or perturbation methods (Noeiaghdam et al., 2021; Yunus et al., 2023). In order to understand the workings of ADM, we need to consider a fractional differential equation.

$$D^{\beta} y(x) = f(x) - Ly(x) - Ny(x), \tag{2.22}$$

where $n - 1 < \beta \leq n, n \in N, x > 0$,

with the following boundary conditions:

$$\zeta(y, \frac{dy}{dx}), \tag{2.23}$$

D^{β} denotes a fractional derivative, N represents a nonlinear operator, L signifies a linear operator, f is a specified function, and ζ acts as a boundary operator within the equation.

Using the Riemann-Liouville integral (J^{β}) on both sides of Equation (2.22) yields:

$$J^{\beta}\left(D^{\beta} y(x)\right) = J^{\beta}\Big(f(x) - Ly(x) - Ny(x)\Big), \tag{2.24}$$

where

$$J^{\beta}\left(D^{\beta} y(x)\right) = y(x) - \sum_{m=0}^{k-1} \frac{D^m u(0)}{m!} x^m. \tag{2.25}$$

Replace Eq. (2.25) within Eq. (2.24) to obtain:

$$y(x) - \sum_{m=0}^{k-1} \frac{D^m u(0)}{m!} x^m = J^{\beta}\Big(f(x) - Ly(x) - Ny(x)\Big), \tag{2.26}$$

or

$$y(x) = J^{\beta}\Big(f(x) - Ly(x) - Ny(x)\Big) + \sum_{m=0}^{k-1} \frac{D^m u(0)}{m!} x^m, \tag{2.27}$$

Expressing the linear term $Ly(x)$ can be done in the following manner:

$$Ly(x) = \sum_{n=0}^{\infty} y_n. \tag{2.28}$$

Expressing the nonlinear term $Ny(x)$ can be achieved in the following manner:

$$Ny(x) = \sum_{n=0}^{\infty} A_n. \tag{2.29}$$

The term A_n, which represents Adomian polynomials, is defined for $n \geq 0$ as:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left(F \left[\sum_{i=0}^{n} \lambda^i y_i \right] \right). \tag{2.30}$$

Put differently, if we assume that the nonlinear function is denoted as $F(y(x))$, then the Adomian polynomials can be represented as:

$$\begin{aligned}
A_0 &= F(y_0). \\
A_1 &= y_1 F'(y_0). \\
A_2 &= y_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0). \\
A_3 &= y_3 F'(y_0) + y_1 y_2 F''(y_0) \frac{1}{3!} y_1^3 F'''(y_0). \\
&\dots
\end{aligned} \tag{2.31}$$

The expression that results from substituting the Adomian decomposition series for the solution $y(x)$ and the series of Adomian polynomials corresponding to the nonlinearity Nu from Eqs. (2.28) and (2.29) into Eq. (2.27) is:

$$\sum_{n=0}^{\infty} y_n = J^{\beta} \left(f(x) - \sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n \right) + \sum_{m=0}^{k-1} \frac{D^m y(0)}{m!} x^m. \tag{2.32}$$

The analysis of the equation above reveals that:

$$
\begin{aligned}
y_0 =& \sum_{m=0}^{k-1} \frac{D^m y(0)}{m!} x^m. \\
y_1 =& J^\beta[f(t) - Ly_0 - A_0]. \\
y_2 =& J^\beta[f(t) - Ly_1 - A_1]. \\
y_3 =& J^\beta[f(t) - Ly_2 - A_2]. \\
&\dots \\
y_{n+1} =& J^\beta[f(t) - Ly_n - A_n].
\end{aligned}
\tag{2.33}
$$

Therefore, by computing all the terms of y, the general solution is derived in accordance with the ADM as:

$$
y(x) = \sum_{n=0}^{\infty} y_n(t). \tag{2.34}
$$

The Adomian Decomposition Method (ADM) has benefits when used to solve differential equations. It proves effective in handling equations without the need for linearization, which is often a requirement in other methods (Nhawu et al., 2016). The ADM presents solutions in a series format and, under certain circumstances, can converge quickly and efficiently, providing results with fewer terms in the series expansion (Abdelrazec & Pelinovsky, 2011; Daoud & Khidir, 2018). However, it's important to acknowledge that the ADM also has its limitations and may not always be the best approach. The convergence of the ADM can be influenced by the nature of the terms. There are instances where convergence may be slow or even nonexistent for certain equations or conditions (Turkyilmazoglu, 2019). Moreover, the accuracy of the ADM solutions might vary depending on the characteristics of the problem at hand. Like any method, understanding these limitations and assessing the suitability of using the ADM in scenarios is crucial to effectively applying it to solve differential equations. In applications, combining the ADM with numerical techniques or validating against known solutions can significantly enhance its reliability and accuracy (Az-Zo'bi et al., 2020; Elsaid, 2012; Hassan, 2008).

CHAPTER 3

LITERATURE REVIEW

3.1 Introduction

A natural extension of the classical differential equations is the fractional differential equations (FDEs), which have extensive applications in a variety of disciplines, including physics, biology, finance, control theory, engineering, etc. By allowing non-integer differentiation orders, FDEs provide a powerful mathematical tool to model and analyze complicated processes that cannot be adequately described by conventional integer-order differential equations. Due to its adaptability, FDEs serve as a useful framework for comprehending and resolving real-world problems in various disciplines.

The order of differentiation in FDEs is typically represented by a non-integer number, denoted by the symbol α. A fractional derivative of order α can be defined using various mathematical formulations, such as the Riemann-Liouville, Caputo (C. Li et al., 2011), Grunwald-Letnikov (Cioc, 2018), conformable (Fleitas et al., 2021), beta derivative (Yusuf et al., 2019), etc. Each formulation has its own advantages and limitations and is suitable for different types of problems.

FDEs can be classified into two main types: linear and nonlinear. Linear FDEs can be solved using techniques similar to those used for solving ODEs, such as Laplace transforms (Kazem, 2013), eigenfunction expansions (Provencher, 1976), and numerical methods. Nonlinear FDEs, on the other hand, are more difficult to solve and require specialized techniques such as the Adomian decomposition method (Nhawu et al., 2016), the homotopy perturbation method (Ghazanfari & Sepahvandzadeh, 2015), or numerical methods.

FDEs have gained a lot of attention recently. This is due to the development of different formulae and theorems of fractional calculus and the realization of its extensive applications in numerous fields. Fractional calculus has found widespread application in diverse fields such as applied mathematics, physics, mechanics, medicine, biology, finance, and others. The concepts of FDEs are currently used to drive research in fractional calculus and how they can be applied in different fields. There is a lot of literature regarding the different aspects of FDEs and much of the research is driven by the practical application of these equations in solving current engineering and scientific problems in the fields identified above. A review of the literature on FDEs is necessary to map out the current knowledge and development in regards to fractional calculus, its origin, and applications in contemporary development and research fields.

Fractional calculus is not a new concept, but its use and deployment have been developed largely in recent times. Calculus as a subject can be traced back to Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century (Daftardar-Gejji, 2019b). Starting with these scientists, calculus has always been defined as a study of continuous change, with the two main approaches being differential and integral calculus. The German philosopher used dx and dy as notations of infinitesimal increments of x and y. The Leibniz notation came to be defined as: $\frac{d^n y}{dx^n}$; for the nth derivative of $f(x) = x$; under the condition that n is an integer number. This notation and its exploration have been the basis for differential calculus over the centuries and are the standard notation. Calculus, as initially defined, dealt with integer values, but while challenging its approach, questions of a non-integer derivative were raised. In a letter to Leibniz, Guillaume de l'Hôpital, a French mathematician, questioned how the equation would be solved if instead of an integer, n was a fraction such as ½. To this, Leibniz is quoted as answering that it would be "an apparent paradox, from which one day useful consequences will be drawn" (Oldham & Spanier, 1974). This is the origin of the idea of fractional calculus and the basis of today's emerging calculus.

The main challenge identified in studying fractional calculus was that of the physical and geometric interpretations of the derivatives and integrals. In integer-based derivative and integral system, the geometric and physical properties of the applications are clear and defined but in fractional calculus, they are not. Nevertheless, the subject has been extensively studied, starting with the 1819 paper by Lacroix, the first time fractional calculus was used in a scientific paper, although in short detail. Euler and Fourier mentioned fractional calculus with arbitrary derivatives but did not present its applications. The first application of the theory, therefore, is credited to(Abel, 1823) when solving the Tautochrone problem. Based on Abel's solution, Liouville developed equations for defining fractional calculus and gave applications in potential theory (Daftardar-Gejji, 2019a). Most of the current definitions of fractional calculus have since been variations of Liouville's equations. The most popular definition of the fractional derivative is a combination of the Riemann-Liouville definition.

The applications of fractional calculus and derivatives are numerous with more solutions being researched. The simplest, for instance, is the calculation of payments using months as opposed to annual payments. Calculation of decay and growth prob-