ENHANCED OPTIMAL HOMOTOPY ASYMPTOTIC MULTIPLE PARAMETERS METHOD FOR SOLVING FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

ABDALLAH ESHBEEL ABDALLAH ALSHBEEL

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ABDALLAH ESHBEEL ABDALLAH ALSHBEEL

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KAEDAH ASIMPTOT HOMOTOPI OPTIMAL MULTI PARAMETER TERTINGKAT BAGI MENYELESAIKAN PERSAMAAN PEMBEZAAN KABUR.

ABSTRAK

Dalam beberapa tahun kebelakangan ini, persamaan pembezaan pecahan (PPP) telah mendapat perhatian yang semakin meningkat kerana keupayaannya untuk memodelkan sistem dunia sebenar yang kompleks yang menunjukkan kesan memori, kesan warisan, dan ketidakpastian. Namun begitu, dalam banyak situasi praktikal, nilai parameter fizikal selalunya tidak tepat disebabkan oleh kesilapan pengukuran, variasi operasi, atau maklumat yang tidak lengkap. Ketidakpastian ini mendorong penggunaan logik kabur untuk mewakili parameter sedemikian, yang membawa kepada pembangunan persamaan pembezaan pecahan kabur (PPPK). Walaupun berkesan dalam memodelkan sistem yang tidak pasti, PPPK, terutamanya yang tidak linear, jarang mempunyai penyelesaian analitik yang tepat, justeru memerlukan kaedah analitik hampiran dan kaedah berangka. Tesis ini menangani cabaran-cabaran tersebut dengan menerapkan dan memperluaskan Kaedah Asimptotik Homotopi Optimum (KAHO) untuk menyelesaikan PPPK linear dan tidak linear. Sumbangan utama adalah pembangunan kerangka OHAM umum yang menggabungkan bentuk termaju bagi pembezaan pecahan, khususnya terbitan Caputo-Katugampola (CK) dengan dua parameter dan terbitan Atangana–Baleanu–Caputo (ABC) dengan kernel Mittag–Leffler (ML) yang melibatkan tiga parameter. Penggunaan umum ini membolehkan pemodelan yang lebih fleksibel dan tepat terhadap sistem dinamik dengan tingkah laku memori dan ketidakpastian yang kompleks. Selain itu, parameter kawalan penumpuan diperkenalkan dan dioptimumkan untuk meningkatkan ketepatan dan kestabilan penyelesaian. Eksperimen berangka mengesahkan keberkesanan kaedah yang dicadangkan, menunjukkan kemampuannya menghasilkan penyelesaian hampiran yang sangat tepat dengan usaha pengiraan yang lebih rendah berbanding teknik analitik tradisional. Hasil kajian menonjolkan kebolehlaksanaan dan keteguhan pendekatan KAHO yang ditambah baik, menjadikannya alat yang bernilai untuk menyelesaikan sistem tidak pasti dalam sains

gunaan dan kejuruteraan.

ENHANCED OPTIMAL HOMOTOPY ASYMPTOTIC MULTIPLE PARAMETERS METHOD FOR SOLVING FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT

In recent years, fractional differential equations (FDEs) have gained growing attention for their ability to model complex real-world systems that exhibit memory, hereditary effects, and uncertainty. However, in many practical situations, the values of physical parameters are often imprecise due to measurement errors, operational variability, or incomplete information. This uncertainty motivates the use of fuzzy logic to represent such parameters, leading to the development of fuzzy fractional differential equations (FFDEs). Despite their effectiveness in modeling uncertain systems, FFDEs, especially nonlinear ones, rarely admit exact solutions, necessitating the use of approximate analytical and numerical methods. This thesis addresses these challenges by applying and extending the Optimal Homotopy Asymptotic Method (OHAM) to solve both linear and nonlinear FFDEs. The main contribution is the development of a generalized OHAM framework that incorporates advanced forms of fractional derivatives, specifically the Caputo-Katugampola (CK) derivative with two parameters and the Atangana–Baleanu–Caputo (ABC) derivative with Mittag–Leffler (ML) kernels involving three parameters. These generalizations enable more flexible and accurate modeling of dynamic systems with complex memory and uncertainty behavior. Additionally, convergence-control parameters are introduced and optimized to improve solution accuracy and stability. Numerical experiments validate the effectiveness of the proposed method, demonstrating its ability to produce highly accurate approximate solutions with less computational effort compared to traditional analytical techniques. The results highlight the feasibility and robustness of the improved OHAM approach, making it a valuable tool for solving uncertain systems in applied science and engineering.

CHAPTER 1

INTRODUCTION

1.1 General Background

Mathematical models are fundamental tools for representing diverse phenomena in science and engineering. While classical differential equations—both ordinary and partial—have been widely employed for this purpose, fractional differential equations (FDEs) provide a more versatile and comprehensive framework. These equations involve derivatives and integrals of arbitrary (non-integer) order, as established by pioneering mathematicians (Gorenflo and Mainardi, 1997; Hilfer, 2000; Machado et al., 2011). Fractional-order models introduce additional degrees of freedom, enabling more accurate and adaptable descriptions of dynamic systems, especially those exhibiting memory and hereditary effects. The growing interest in fractional calculus is driven by its capacity to extend classical models and more effectively capture complex real-world behaviors. FDEs are particularly beneficial for modeling non-local dynamics and systems with memory, such as viscoelastic materials, biological processes, and anomalous diffusion. In control systems, they offer improved stability and robustness. However, the mathematical complexity and computational intensity of fractional models can be limiting, particularly in systems where traditional models suffice. The decision to use classical or fractional models depends on the specific requirements of the problem. For instance, in mechanical engineering, viscoelastic materials exhibit stress–strain responses dependent on deformation history, which FDEs model effectively. In biomedical applications, fractional models more accurately describe anomalous diffusion in tissues and the electrical behavior of neurons. Similarly, electrical circuits with non-ideal components and financial systems with long-term memory in volatility are better modeled using fractional dynamics. In control theory, fractional-order controllers enhance performance and robustness in uncertain environments.

In his seminal work, Zadeh (Zadeh, 1965) introduced the concept of fuzzy sets. Later developments included the fuzzy derivative (1972) and the formal definition of fuzzy numbers and their arithmetic operations by Chang and Zadeh (1996). FFDEs provide a robust mathematical framework for modeling real-world systems under uncertainty by incorporating vague or imprecise parameters (Buckley and Feuring, 2001). FFDEs have been applied in diverse fields, including civil engineering, medicine, population dynamics, and particle systems.

The first significant application of FFDEs was presented by Agarwal in 2010 (Agarwal et al., 2010), which marked a turning point in the field. His work encouraged further research into applications in fluid dynamics (Salahshour et al., 2015), physics (Salahshour and Allahviranloo, 2013), and other domains. As the use of FFDEs in physical models expanded, various analytical methods were developed to solve them.

Most FFDEs, particularly nonlinear ones, do not have exact analytical solutions. This has led to the development and application of several approximate analytical and numerical techniques. These include the power series expansion (Shahidi and Khastan, 2018), Laplace transform (Abdollahi et al., 2019; Das and Roy, 2017), Sumudu transform (Rahman and Ahmad, 2015, 2017), and semi-analytical methods such as the Variational Iteration Method (VIM) (Khodadadi et al., 2015; Panahi, 2017), Adomian Decomposition Method (ADM) (Esmailzadeh et al., 2019), and Modified Homotopy Perturbation Method (MHPM) (Khan et al., 2014).

Marinca introduced the OHAM for solving nonlinear problems without relying on small or large parameters (Herişanu et al., 2008a; Marinca et al., 2009; Marinca et al., 2008a). OHAM has since been widely used in engineering and physics (Hamarsheh et al., 2015; Riccati et al., 2013). Liao (Liao, 1992) developed the related Homotopy Analysis Method (HAM), which offers parameter independence and flexibility in selecting base functions for nonlinear problems. One of HAM's strengths is the ability to control convergence through a tunable parameter h (Liao, 2012). Similarly, OHAM does not require an initial guess or discretization, avoids round-off errors, and delivers high accuracy without increasing computational cost. Unlike traditional methods, it includes built-in convergence control and is parameter-free.

The CK fractional derivative, which unifies the Riemann–Liouville and Hadamard approaches, is increasingly used to solve generalized FDEs (Katugampola, 2011a, 2014b). Numerical techniques based on CK derivatives, including adaptive predictor–corrector schemes (Odibat and Baleanu, 2020) and theoretical analyses of existence and uniqueness (Van Hoa et al., 2019), have further expanded its applications.

This thesis addresses the challenge of finding accurate analytical solutions to FFDEs, which are essential for modeling complex systems affected by uncertainty. While FFDEs of arbitrary order are powerful, their exact solutions are rare. To overcome this, the research proposes novel approximate analytical techniques using convergence-controlled OHAM. The study extends OHAM by incorporating generalized fractional operators, specifically the CK (with two parameters) and the ABC fractional derivative (with ML kernels involving three parameters). This integration enhances both flexibility and accuracy in solving linear and nonlinear FFDEs, offering an effective balance between solution precision and computational efficiency.

Atangana and Baleanu (Atangana and Baleanu, 2016) introduced a fractional derivative with a non-singular kernel based on the Mittag–Leffler (ML) function. This AB derivative simplifies numerical modeling while ensuring the existence of solutions. The generalized fractional operators with ML kernels, involving three parameters α, μ, and γ, have been further developed (Abdeljawad, 2019). These operators are being actively explored through both numerical and analytical approaches for solving a wide range of fractional problems.

1.2 Motivation

FDEs extend classical differential equations to describe complex phenomena (Bonilla et al., 2007) better. In real-world systems, uncertainty and imprecision are often present, which can be effectively captured using fuzzy quantities. FFDEs thus provide a powerful framework for modeling such systems, where crisp quantities may be vague or uncertain.

FFDEs are widely applied in science and technology to model systems affected by uncertainty. However, obtaining exact solutions is often not possible, and finding analytical solutions remains a major challenge. Traditional methods—such as numerical techniques and approximate analytical methods like the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM), face several limitations, including high computational cost, slow convergence, and difficulties in handling fuzzy parameters and complex fractional operators.

To address these challenges, this study proposes a modified OHAM tailored for FFDEs. This approach integrates generalized fractional derivatives—specifically, the CK derivative with multiple parameters and the ABC fractional derivative with an ML kernel—to enhance flexibility, accuracy, and solution quality. While existing literature primarily focuses on standard Caputo derivatives (with a single parameter) (Hashim et al., 2022), this work extends the scope by incorporating more generalized forms of fractional derivatives for solving FFDEs.

The proposed OHAM approach offers several advantages over conventional methods. Unlike power series techniques, which are limited by a narrow radius of convergence, OHAM significantly broadens the convergence domain. It also overcomes limitations in ADM, which cannot guarantee convergence, and includes the Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) as special cases. Moreover, OHAM eliminates the need for identifying an initial guess or tracing the h-curve and allows direct control over the convergence region.

Furthermore, unlike ADM and HPM, which typically require a large number of iterations to achieve acceptable accuracy (Pandey et al., 2012), OHAM achieves highly accurate results with fewer iterations. This makes it an efficient and robust approach for solving FFDEs involving generalized fractional operators, particularly over large domains.

1.3 Problem Statement

Fractional fuzzy differential equations (FFDEs) have become essential tools for modeling complex systems that involve uncertainty, memory, and hereditary effects in fields such as finance, epidemiology, and engineering. However, the nonlinear nature of many FFDEs, combined with uncertain parameters, often prevents the derivation of exact analytical solutions. This poses major challenges for practical implementation, including increased computational demands due to reliance on numerical methods, difficulty in parameter identification from experimental data, limited real-time applicability, and an incomplete understanding of system behavior.

To address these challenges, this study proposes a modified framework based on the OHAM tailored specifically for FFDEs. The aim is to enhance convergence behavior, improve solution accuracy, and reduce computational costs. By incorporating generalized fractional operators such as the CK and ABC derivatives with ML kernels, the proposed approach is expected to offer superior performance in solving complex FFDEs. This framework will provide significant improvements over existing analytical and approximate methods in terms of efficiency, robustness, and flexibility.

1.4 Significance of the Study

OHAM is a powerful method for solving linear and nonlinear differential equations, particularly FFDEs. This study extends OHAM by integrating generalized fractional definitions, specifically the CK derivative (with two parameters) and the ABC derivative (with ML kernels involving three parameters). The research develops convergence-controlled, approximate analytical algorithms that offer the following benefits:

1. **Improved Convergence Control:** The introduction of convergence-control parameters allows precise tuning of solution accuracy and stability.

2. **Higher Accuracy and Efficiency:** Compared to traditional approximation methods, the enhanced OHAM produces more accurate results with reduced computational complexity.

3. **Application to Real-World Problems:** The developed methods are applicable to a wide range of scientific and engineering problems, including heat equations, diffusion processes, and dynamic systems.

4. **Flexibility in Handling Uncertainty:** The integration of fuzzy logic enables the modeling of systems with uncertain or imprecise parameters, expanding the applicability of differential equation models.

By overcoming the limitations of existing methods, this study contributes meaningfully to the fields of applied mathematics, computational techniques, and engineering modeling.

1.5 Research Questions

1. How does the choice of parameters in the generalized CK fractional derivative affect the accuracy and computational efficiency of OHAM solutions for linear and nonlinear FFDEs?

2. What are the optimal strategies for selecting ML kernels in the generalized ABC definition to improve convergence and accuracy in OHAM-based solutions for FFDEs?

3. How feasible is the analytical approximation of FFDEs using OHAM when multiple parameters are involved?

4. How can convergence-control parameters be systematically applied within the OHAM framework to ensure and enhance the precision and stability of solutions for FFDEs with generalized fractional derivatives?

5. To what extent does the proposed OHAM approach—incorporating generalized fractional derivatives and optimized convergence control—outperform existing analytical and approximate methods in terms of accuracy, computational efficiency, and stability for solving FFDEs?

1.6 Objective

The primary objective of this research is to develop novel, convergence-controlled, approximate analytical techniques for solving FFDEs. This goal will be achieved through the following specific objectives:

1. To develop an enhanced OHAM-based framework for solving linear and nonlinear FFDEs using the generalized CK fractional derivative involving two parameters.

2. To extend the OHAM approach for application to FFDEs governed by the ABC fractional derivative with ML kernels involving three parameters.

3. To evaluate the applicability and robustness of the proposed OHAM-based methods in addressing the complexity and uncertainty associated with multi-parameter FFDEs.

4. To assess the accuracy, efficiency, and convergence behavior of the proposed approach through rigorous comparative analysis with existing analytical and approximate methods.

5. To validate the performance of the modified OHAM by solving benchmark linear and nonlinear FFDEs and demonstrating its computational advantages over conventional techniques.

1.7 Scope of the Study

This study develops four effective algorithms for solving FFDEs, including both ordinary and partial differential equations that involve the generalized CK and ABC fractional derivatives. These methods are constructed using the OHAM and are based on the careful selection of parameters that ensure convergence of the solution in the examples presented.

By combining fractional operators with multiple parameters—specifically the CK and ABC definitions—and an algorithm that offers convergence control, the proposed approach guarantees accurate solutions that align with real-world data. This study focuses on the CK and ABC derivatives due to their modeling advantages. The Riemann–Liouville derivative, though foundational, poses challenges regarding initial conditions and yields a non-zero derivative for constant functions. The Caputo derivative resolves initial condition issues but still involves a singular kernel. The Hadamard derivative, defined in logarithmic terms, is best suited to problems with logarithmic singularities.

In contrast, the CK derivative unifies the Riemann–Liouville and Hadamard forms, enabling more flexible modeling through two tunable parameters. The ABC derivative, on the other hand, incorporates a non-singular ML kernel, simplifying numerical modeling and avoiding singularities. These properties make the CK and ABC definitions particularly suitable for integration with OHAM in modeling systems involving memory effects and uncertainty.

Numerical simulations will be performed to illustrate the effectiveness of the proposed methods. The results will include the presentation of the solutions, adjustments made, and comparisons with known exact solutions. All numerical experiments in this research will be conducted using *Mathematica 11.3*.

1.8 Methodology

This research introduces an enhanced semi-analytical approach based on the OHAM for solving linear and nonlinear FFDEs. The methodology focuses on two generalized fractional derivatives: the CK derivative (with two parameters) and the ABC derivative (with three parameters via ML kernels). This approach is designed to address limitations in existing methods when dealing with fuzzy parameters, convergence control, and complex fractional dynamics.

The modified method, referred to as Fractional Fuzzy OHAM (FFOHAM), is motivated by the need for a reliable, efficient, and accurate tool for FFDEs. While OHAM already avoids linearization and discretization, applying it directly to FFDEs with generalized derivatives and fuzzy uncertainty requires further adaptation. **The FFOHAM framework is designed to address several key challenges associated with FFDEs effectively. It accounts for memory and hereditary effects that are intrinsic to systems modeled using fractional operators, capturing the system's history-dependent behavior. Additionally, it integrates fuzzy uncertainty, allowing the modeling of systems without requiring precisely defined initial conditions. By employing auxiliary convergence-control parameters, the method enhances the accuracy and stability of the obtained solutions. Furthermore, FFOHAM generates series-based approximate solutions, which are especially valuable in the context of nonlinear FFDEs where closed-form solutions are often unattainable.**

1.8.1 Theoretical Work: FFDEs for ODEs Using the Generalized CK Definition

The theoretical development of FFDEs for ordinary differential equations (ODEs) using the generalized CK derivative proceeds through three main phases. **First, the process begins with the defuzzification of fuzzy fractional initial value problems (IVPs) that involve the CK derivative, transforming the fuzzy problem into a crisp formulation suitable for analytical treatment. Next, the CK based FFOHAM is constructed and analyzed for these IVPs, enabling the formulation of an iterative solution method that leverages the structure of the CK operator. Finally, a rigorous convergence analysis of the FFOHAM series solution is carried out to ensure the method's reliability and accuracy in approximating the underlying system's true behavior.**

1.8.2 Theoretical Work: FFDEs for ODEs Using the ABC Derivative with ML Kernels

The theoretical framework for FFDEs using the ABC fractional derivative with ML kernels consists of three primary phases. **The first phase involves the defuzzification of fuzzy fractional initial value problems (IVPs) that incorporate ABC derivatives and ML kernels, thereby transforming the fuzzy model into a crisp form suitable for analytical and numerical treatment. In the second phase, the ABC-based FFOHAM is systematically constructed and analyzed for these IVPs, enabling the development of a robust solution method tailored to the nonlocal and nonsingular nature of the ABC operator. The final phase focuses on the convergence analysis of the resulting FFOHAM solution series, ensuring the reliability, stability, and applicability of the method in handling complex fuzzy fractional models.**

1.8.3 Theoretical Work: FFDEs for PDEs Using CK and ABC Definitions

The approach to FFDEs for partial differential equations (PDEs) is demonstrated through the fuzzy fractional heat equation of the form:

$$D_t^G \widetilde{V}(s,t) = \widetilde{A}(s)\frac{\partial^2 \widetilde{V}(s,t)}{\partial s^2} + \widetilde{B}(s),$$

where G represents either the generalized CK or ABC derivative. This work is also conducted in three main phases. **The first phase involves the defuzzification of the fuzzy fractional heat equation using a generalized fractional derivative, which transforms the fuzzy model into a crisp form suitable for further analysis. In the second phase, the FFOHAM is constructed and applied to the heat equation, allowing for the development of an effective iterative solution method that accommodates the complexities of the fractional operator. The third and final phase entail a detailed convergence analysis of the resulting FFOHAM series solution to ensure its accuracy, stability, and applicability in modeling heat transfer**

phenomena under fuzzy and fractional dynamics.

1.9 Thesis Outline

This thesis is organized into seven chapters:

- **Chapter 2** provides the foundational concepts of fractional calculus and fuzzy set theory. It also introduces the OHAM method as an approximate analytical technique.
- **Chapter 3** presents a review of existing literature related to approximate solutions of various types of FFDEs.
- **Chapter 4** introduces a modified OHAM framework and applies it to solve fuzzy fractional IVPs using the generalized CK derivative. A comparative analysis of the proposed scheme is presented.
- **Chapter 5** extends the method by applying the generalized ABC derivative with ML kernels to ordinary FFDEs. This chapter also discusses convergence and compares approximate results with known exact solutions.
- **Chapter 6** focuses on solving fuzzy fractional partial differential equations, particularly the heat equation, using OHAM with both CK and ABC fractional operators.
- **Chapter 7** summarizes the main findings, discusses the significance of the results, and outlines potential future research directions.

CHAPTER 2

BASIC CONCEPTS AND BACKGROUND

2.1 Introduction

Fuzzy fractional differential equations (FFDEs) are used to simulate real-world processes by accounting for uncertainty in the behavior of dynamical systems, leading to more realistic and adaptable models. Their primary purpose is to provide a mathematical framework for describing specific real-world phenomena where system behavior may involve ambiguity or fuzzy parameters.

To establish a suitable foundation for this thesis, this chapter presents the fundamental concepts of fuzzy set theory, the principles of fractional calculus, and an overview of the OHAM. Additionally, a brief review of key theoretical aspects of OHAM is provided.

2.2 Fuzzy Concept

Fuzzy set theory, introduced by Zadeh (Zadeh, 1965), is an extension of classical (crisp) set theory. In classical set theory, membership is binary: an element either belongs to a set or it does not. In contrast, fuzzy set theory allows gradual membership, where each element is assigned a degree of belonging through a membership function that maps to the interval $[0, 1]$.

Fuzzy sets generalize classical sets. In particular, in a positive universe, the membership function can reduce to an indicator function mapping all elements to 0 or 1, consistent with the classical concept. A crisp set is typically defined as a collection of items or objects $s \in S$, whether countable or uncountable. In fuzzy set theory, additional concepts are introduced, such as the support of a fuzzy set, convex fuzzy sets, ζ-levels, triangular fuzzy numbers, fuzzy functions, the extension principle, and fuzzy differentiation. These concepts are crucial for solving FFDEs in a fuzzy environment.

2.2.1 Fuzzy Set

A fuzzy set generalizes a crisp set by allowing the membership function to take any value within the interval $[0, 1]$.

Definition 2.2.1 Fuzzy Set (Gomez-Salazar et al., 2012)
Let s represent a collection of objects denoted by S. A fuzzy set $\widetilde{A}$ in S is defined as a set of ordered pairs:

$$\widetilde{A} = \{(s, \delta_{\widetilde{A}}(s)) : s \in S\},$$

where $\delta_{\widetilde{A}} : S \to [0, 1]$ is the membership function of the fuzzy set $\widetilde{A}$.

This definition establishes a link between the elements s and the interval $[0, 1]$, where the membership function $\delta_{\widetilde{A}}(s)$ quantifies the degree of membership. If an element s fully belongs to the set, its membership degree is 1; otherwise, it falls between 0 and 1.

Definition 2.2.2 Support of a Fuzzy Set (Zimmermann, 2010)
The support of a fuzzy set $\widetilde{A}$ within the universal set S is defined as:

$$\text{Supp}(\widetilde{A}) = \{s \in S : \delta_{\widetilde{A}}(s) > 0\}.$$

It consists of all elements in S that have nonzero membership values in $\widetilde{A}$.

Definition 2.2.3 Convex Fuzzy Set (Chakraverty et al., 2016)
Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $E(\mathbb{R}^n) = \widetilde{E}$ represent the set of all nonempty fuzzy sets in $\mathbb{R}^n$. A fuzzy set with membership function $\delta_{\widetilde{E}} : \mathbb{R}^n \to [0, 1]$ is called *convex* if, for all $s, t \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, the following holds:

$$\delta(\lambda s + (1 - \lambda)t) \geq \min\{\delta(s), \delta(t)\}.$$

Additionally, a fuzzy set is called a *cone* if, for all $s \in \mathbb{R}^n$ and $\lambda > 0$:

$$\delta(\lambda s) = \delta(s).$$

2.2.2 The ζ-Level Sets

Many properties that hold for classical sets can also be demonstrated for fuzzy sets through the concept of ζ-level sets.

Definition 2.2.4 **ζ-cut** (Gomes and Barros, 2015)
Let $\widetilde{A} : S \to [0, 1]$ be a fuzzy set. The ζ-level set (or ζ-cut) of $\widetilde{A}$ is defined as:

$$[\widetilde{A}]_\zeta = \{s \in S \mid \delta_{\widetilde{A}}(s) \geq \zeta\}, \quad \zeta \in [0, 1].$$

The strong ζ-level set (Gomes and Barros, 2015) is similarly defined as:

$$[\widetilde{A}]^+_\zeta = \{s \in S \mid \delta_{\widetilde{A}}(s) > \zeta\}, \quad \zeta \in [0, 1].$$

Strong ζ-level sets are important because they preserve more structural properties of fuzzy sets compared to classical ζ-cuts. It can be easily verified that for any $\zeta, \eta \in [0, 1]$, the following properties hold:

1. $(\widetilde{S} \cup \widetilde{T})_\zeta = \widetilde{S}_\zeta \cup \widetilde{T}_\zeta$,
2. $(\widetilde{S} \cap \widetilde{T})_\zeta = \widetilde{S}_\zeta \cap \widetilde{T}_\zeta$,
3. If $\widetilde{S} \subseteq \widetilde{T}$, then $\widetilde{S}_\zeta \subseteq \widetilde{T}_\zeta$,
4. If $\zeta \leq \eta$, then $\widetilde{S}_\eta \subseteq \widetilde{S}_\zeta$,
5. $\widetilde{S} = \widetilde{T}$ if and only if $\widetilde{S}_\zeta = \widetilde{T}_\zeta$ for all $\zeta \in [0, 1]$,
6. $\widetilde{S}_\zeta \cap \widetilde{S}_\eta = \widetilde{S}_\eta$ and $\widetilde{S}_\zeta \cup \widetilde{S}_\eta = \widetilde{S}_\zeta$ if $\zeta \leq \eta$.

For a fuzzy set $\widetilde{A}$, the collection $\{\widetilde{A}_\zeta\}_{\zeta \in [0,1]}$ represents a family of nested crisp subsets of S, and the fuzzy set $\widetilde{A}$ can be reconstructed as:

$$\widetilde{A} = \bigcup_{\zeta \in [0,1]} \zeta \widetilde{A}_\zeta.$$

2.2.3 Fuzzy Numbers

Fuzzy numbers are generalizations of real numbers that allow for vagueness in value. The degree of membership expresses how strongly an element belongs to a set, representing uncertainty. Typically, a fuzzy interval is defined by two endpoints α_1 and α_3, with a peak at α_2, and denoted as $[\alpha_1, \alpha_2, \alpha_3]$.

Definition 2.2.5 Triangular Fuzzy Number (Dubois and Prade, 1982)

A triangular fuzzy number is defined by three real numbers $\alpha_1 < \alpha_2 < \alpha_3$. Its base spans $[\alpha_1, \alpha_3]$ and reaches a peak membership of 1 at $s = \alpha_2$, with membership function:

$$\delta(s; \alpha_1, \alpha_2, \alpha_3) = \begin{cases} 0, & s < \alpha_1, \\ \frac{s-\alpha_1}{\alpha_2-\alpha_1}, & \alpha_1 \le s \le \alpha_2, \\ \frac{\alpha_3-s}{\alpha_3-\alpha_2}, & \alpha_2 \le s \le \alpha_3, \\ 0, & s > \alpha_3. \end{cases}$$

The ζ-level set of a triangular fuzzy number is:

$$[\delta(s)]_\zeta = [\alpha_1 + \zeta(\alpha_2 - \alpha_1), \alpha_3 - \zeta(\alpha_3 - \alpha_2)], \quad \zeta \in [0, 1].$$

Definition 2.2.6 (Mansouri and Ahmady, 2012a)

Let $\widetilde{E}$ denote the set of all upper semi-continuous, normal, convex fuzzy numbers with bounded ζ-level sets. A fuzzy number $\delta(s)$ satisfies:

1. **Normality:** There exists $s_0 \in \mathbb{R}$ such that $\delta(s_0) = 1$,

2. **Convexity:** For all $s, t \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\delta(\lambda s + (1 - \lambda)t) \ge \min\{\delta(s), \delta(t)\},$$

3. **Upper semi-continuity:** For any $s_0 \in \mathbb{R}$, $\delta(s_0) \ge \lim_{s \to s_0^\pm} \delta(s)$,

4. **Bounded support:** $\overline{\{s \in \mathbb{R} \mid \delta(s) \ge \zeta\}}$ is a compact subset of $\mathbb{R}$.

In parametric form, a fuzzy number is represented as an ordered pair:

$$[\delta]_\zeta = [\underline{\delta}(\zeta), \overline{\delta}(\zeta)], \quad \zeta \in [0, 1],$$

where:

1. $\underline{\delta}(\zeta)$ is a bounded, left-continuous, non-decreasing function,
2. $\overline{\delta}(\zeta)$ is a bounded, left-continuous, non-increasing function,
3. $\underline{\delta}(\zeta) \leq \overline{\delta}(\zeta)$.

2.2.4 Fuzzy Function

Definition 2.2.7 Fuzzy Function (Guo and Shang, 2013)
A mapping $\widetilde{g} : T \to \widetilde{E}$, where T is an interval and $\widetilde{E}$ is the set of upper semi-continuous normal convex fuzzy numbers, is called a fuzzy function or fuzzy process. The ζ-level set is:

$$[\widetilde{g}(s)]_\zeta = [\underline{g}(s; \zeta), \overline{g}(s; \zeta)], \quad s \in T, \quad \zeta \in [0, 1].$$

Thus, a fuzzy function maps each crisp value to a fuzzy set, and mathematically, the fuzzifying function and the fuzzy relation are equivalent.

2.2.5 The Extension Principle

The extension principle plays a central role in fuzzification, allowing classical mathematical concepts to be extended to fuzzy contexts.

Definition 2.2.8 Extension Principle (Gerla and Scarpati, 1998)
Given a function $g : S \to T$, where $S = S_1 \times S_2 \times \ldots \times S_n$, and $\widetilde{A} = \widetilde{A}_1 \times \widetilde{A}_2 \times \ldots \times \widetilde{A}_n$ is an n-fuzzy subset in S, the image fuzzy set $\widetilde{B} = g(\widetilde{A})$ in T is defined as:

$$\widetilde{B} = \{(t, \delta_{\widetilde{B}}(t)) : t = g(s_1, s_2, \ldots, s_n),\ s_1, s_2, \ldots, s_n \in S\},$$

where

$$\delta_{\widetilde{B}}(t) = \begin{cases} \sup\limits_{(s_1,\ldots,s_n)\in g^{-1}(t)} \min\{\delta_{\widetilde{A}_1}(s_1),\ldots,\delta_{\widetilde{A}_n}(s_n)\}, & \textit{if } g^{-1}(t) \neq \emptyset, \\ 0, & \textit{otherwise}. \end{cases}$$

For $n = 1$, this simplifies to:

$$\widetilde{B} = \{(t, \delta_{\widetilde{B}}(t)) : t = g(s),\ s \in S\},$$

with

$$\delta_{\widetilde{B}}(t) = \begin{cases} \sup\limits_{s\in g^{-1}(t)} \delta_{\widetilde{A}}(s), & \textit{if } g^{-1}(t) \neq \emptyset, \\ 0, & \textit{otherwise}. \end{cases}$$

2.2.6 Fuzzy Differentiation

The concept of the fuzzy derivative was first introduced by Chang (1972) (Chang and Zadeh, 1972). Later, Dubois and Prade (1982) (Dubois and Prade, 1982) expanded this concept using the extension principle, presenting two definitions: one based on Hukuhara differentiability within a convex cone, and another defined in Banach spaces.

Definition 2.2.9 Hausdorff Distance (Lakshmikantham and Mohapatra, 2004)

The Hausdorff distance $D[s,t]$ between two fuzzy numbers is defined as:

$$D[s,t] = \sup_{0\leq\zeta\leq1} \max\{|\underline{s}(\zeta) - \underline{t}(\zeta)|, |\overline{s}(\zeta) - \overline{t}(\zeta)|\}.$$

Definition 2.2.10 Hukuhara Difference (Lakshmikantham and Mohapatra, 2004)

Let $s,t \in \widetilde{E}$. If there exists $r \in \widetilde{E}$ such that $s = t + r$, then r is called the Hukuhara difference of s and t, denoted by $s \ominus t$.

Definition 2.2.11 H-differentiability (Chalco-Cano et al., 2008)

Let $g : J \longrightarrow \widetilde{E}$ and $s_0 \in J$. The fuzzy function g is said to be Hukuhara differentiable

(H-differentiable) at s_0 if there exists $g'(s_0) \in \widetilde{E}$ such that, for small $h > 0$, the differences $g(s_0 + h) \ominus g(s_0)$ and $g(s_0) \ominus g(s_0 - h)$ exist and:

$$g'(s_0) = \lim_{h \to 0} \frac{g(s_0 + h) \ominus g(s_0)}{h} = \lim_{h \to 0} \frac{g(s_0) \ominus g(s_0 - h)}{h}.$$

The ζ-level set $[\widetilde{g}'(s_0)]_\zeta$ represents the Hukuhara derivative (H-derivative) of $[\widetilde{g}(s)]_\zeta$ at s_0.

With respect to the metric D, if a fuzzy function $g : J \to \widetilde{E}$ is H-differentiable at s_0, then all its ζ-level sets $[\widetilde{g}(s)]_\zeta$ are H-differentiable at s_0 for all $\zeta \in [0, 1]$, satisfying the above limit conditions.

Theorem 2.2.12 (Stefanini et al., 2006)

Let $g : [s_0 + t, S] \to \widetilde{E}$ be H-differentiable. Then, for all $\zeta \in [0, 1]$,

$$[\widetilde{g}'(s)]_\zeta = [\underline{g}'(s; \zeta), \overline{g}'(s; \zeta)],$$

where $\underline{g}'(s; \zeta)$ and $\overline{g}'(s; \zeta)$ are differentiable functions, and

$$[\widetilde{g}'(s)]_\zeta = [(\underline{g}(s; \zeta))', (\overline{g}(s; \zeta))'].$$

Theorem 2.2.13 (Mansouri and Ahmady, 2012b)

Let $g : [s_0 + t, S] \to \widetilde{E}$ be H-differentiable. Then, for all $\zeta \in [0, 1]$, the n-th order fuzzy derivative is given by:

$$[\widetilde{g}^{(n)}(s)]_\zeta = [\underline{g}^{(n)}(s; \zeta), \overline{g}^{(n)}(s; \zeta)].$$

2.2.7 Fuzzy Integration

Definition 2.2.14 (Wu, 1998)

Let $g : J \to \widetilde{E}$ be a bounded fuzzy-valued function defined on a closed interval $J = [t_0, T] \subseteq \mathbb{R}$. Assume the ζ-level sets are given by $[\widetilde{h}(t)]_\zeta = [\underline{h}(t, \zeta), \overline{h}(t, \zeta)]$, where $\underline{h}(t,\zeta)$ and $\overline{h}(t,\zeta)$ are Riemann integrable on J for each $\zeta \in [0,1]$. Define:

$$\widetilde{A}(t,\zeta) = \left[\int_{t_0}^{T} \underline{h}(t,\zeta)\,dt, \int_{t_0}^{T} \overline{h}(t,\zeta)\,dt\right].$$

Then, $[\widetilde{h}(t)]_\zeta$ is said to be fuzzy Riemann integrable on J, and

$$\delta_{\int_{t_0}^{T}\widetilde{h}(t,\zeta)dt}(t) = \sup_{\zeta\in[0,1]} \alpha 1_{\widetilde{A}(t,\zeta)},$$

where $1_{\widetilde{A}(t,\zeta)}$ is the indicator function over the set $\widetilde{A}(t,\zeta)$.

2.2.8 Illustrative Examples of Fuzzy Concepts

Example: Triangular Membership Function

Let a triangular fuzzy number be $\tilde{A} = (1,3,5)$. Its membership function is given by:

$$\delta_{\tilde{A}}(s) = \begin{cases} 0, & s \le 1 \\ \frac{s-1}{2}, & 1 < s \le 3 \\ \frac{5-s}{2}, & 3 < s < 5 \\ 0, & s \ge 5 \end{cases}$$

This function increases linearly from 0 to 1 between 1 and 3, then decreases from 1 to 0 between 3 and 5.

ζ-Cuts and Level Sets

For a fuzzy number $\tilde{A} = (1,3,5)$, the α-cut $\tilde{A}_\zeta$ is a crisp interval:

$$\tilde{A}_\zeta = [1+2\zeta,\ 5-2\zeta], \quad \zeta \in [0,1]$$

Example values:

- $\tilde{A}_0 = [1,5]$ (support)
- $\tilde{A}_{0.5} = [2,4]$

- $\tilde{A}_1 = [3, 3]$ (core)

Fuzzy-Valued Functions

Let $\tilde{v}(s) = (s, s+1, s+2)$. Then for any fixed s, $\tilde{v}(s)$ is a triangular fuzzy number. This represents a fuzzy-valued function depending on time.

For instance, at $s = 2$:

$$\tilde{v}(2) = (2, 3, 4)$$

As time increases, both the core and the support of the fuzzy value shift accordingly.

Fuzzy Differential Equation: Example

Consider the fuzzy initial value problem (FIVP):

$$\frac{d\tilde{v}(s)}{ds} = \tilde{v}(s), \quad \tilde{v}(0) = (1, 2, 3)$$

Using the ζ-cut method, we write:

$$\tilde{v}_\zeta(0) = [1 + \zeta, 3 - \zeta]$$

Then we solve the following crisp initial value problems for each $\zeta \in [0, 1]$:

$$\begin{cases} \frac{ds_L^\zeta}{ds} = v_L^\zeta, & y_L^\zeta(0) = 1 + \zeta \\ \frac{dy_R^\zeta}{dt} = y_R^\zeta, & y_R^\zeta(0) = 3 - \zeta \end{cases} \quad \Rightarrow \tilde{v}^\zeta(s) = \left[(1 + \zeta)e^t,\ (3 - \zeta)e^t\right]$$

Example at $s = 1$:

- $\zeta = 0$: $\tilde{v}_0(1) = [e^1,\ 3e^1] \approx [2.72,\ 8.15]$
- $\zeta = 1$: $\tilde{v}_1(1) = [2e^1,\ 2e^1] \approx [5.44,\ 5.44]$

This demonstrates that uncertainty is greatest at $\zeta = 0$ and collapses to a crisp solution at $\zeta = 1$.

2.3 Fractional Derivative

Fractional differential calculus, a branch of mathematical analysis, investigates the properties and applications of integrals and derivatives of arbitrary (non-integer) order. Compared to classical integer-order derivatives, fractional derivatives offer a more precise and flexible framework for modeling various phenomena and material properties, making them significantly more suitable for describing complex real-world systems.

2.3.1 General Fractional Integral and Derivative

Fractional calculus (FC) extends traditional calculus and has a history spanning over 300 years. Over time, it has evolved significantly, with applications now found in fields such as applied mathematics, physics, engineering, and economics.

The Riemann–Liouville fractional integral of a function $f(s)$ generalizes the classical Cauchy formula. If a continuous function vanishes at the origin, then:

$$\int_{s_0}^{s}\int_{s_0}^{s_1}\cdots\int_{s_0}^{s_{n-1}} f(s)\,ds_n\ldots ds_2 ds_1 = \frac{1}{(n-1)!}\int_{s_0}^{s}(s-t)^{n-1}f(t)\,dt. \tag{2.1}$$

By replacing the positive integer n with an arbitrary positive real number α, using the Gamma function, the integral of arbitrary order is defined as (Samei et al., 2021):

$$^{RL}I^{\alpha}f(s) = \frac{1}{\Gamma(\alpha)}\int_{0}^{s}(s-t)^{\alpha-1}f(t)\,dt, \tag{2.2}$$

where Γ denotes the Gamma function. When $\alpha = 0$, this reduces to the identity operator.

Numerous attempts have been made to define fractional derivatives, most utilizing an integral representation. One of the most notable definitions is the Riemann–Liouville fractional derivative, given by (Chen et al., 2007 Chen et al., 2023):

$${}_{a}^{RL}D^{\alpha}f(s) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\dfrac{d^m}{ds^m}\displaystyle\int_{a}^{s}(s-t)^{m-\alpha-1}f(t)\,dt, & m-1<\alpha<m, \\ \dfrac{d^m}{ds^m}f(s), & \alpha = m, \end{cases} \tag{2.3}$$

where m is the smallest integer greater than α.

The Caputo fractional derivative is defined similarly as:

$$
{}_{a}^{C}D^{\alpha}f(s) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{a}^{s} (s-t)^{m-\alpha-1} f^{(m)}(t)\,dt, & m-1 < \alpha < m, \\ \dfrac{d^m}{ds^m} f(s), & \alpha = m. \end{cases} \tag{2.4}
$$

An important property distinguishing these two fractional derivatives is their behavior when applied to constant functions. If $f(s) = C$ (where C is a constant), then:

$$
{}_{a}^{RL}D^{\alpha}C = \frac{C(s-a)^{-\alpha}}{\Gamma(m-\alpha)}, \quad \text{and} \quad {}_{a}^{C}D^{\alpha}C = 0.
$$

This shows that, unlike the Riemann–Liouville derivative, the Caputo fractional derivative yields zero when applied to a constant function, which aligns more naturally with classical calculus intuition.

Another important distinction is related to initial conditions. The Riemann–Liouville derivative requires initial conditions in the form of fractional-order limits, which often lack clear physical interpretation. As a result, solutions based on the Riemann–Liouville derivative are less frequently used in applications.

To overcome these limitations, the Caputo fractional derivative is preferred, as it allows initial conditions to be expressed in terms of integer-order derivatives, thus providing a clearer physical meaning and making it more suitable for practical modeling and engineering applications.

2.3.2 Caputo–Katugampola Fractional Derivative

The CK fractional derivative, characterized by two parameters, generalizes both the Caputo and Caputo–Hadamard fractional derivatives. In this section, we present some basic definitions and key properties.

Definition 2.3.1 (Almeida et al., 2016)

The left and right generalized fractional integrals of a function f, known as the Katugampola fractional integrals, are respectively defined as:

$$I_{a+}^{\alpha,\rho} f(s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^s t^{\rho-1}(s^\rho - t^\rho)^{\alpha-1} f(t)\, dt,$$

$$I_{b-}^{\alpha,\rho} f(s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_s^b t^{\rho-1}(t^\rho - s^\rho)^{\alpha-1} f(t)\, dt,$$

where $0 < a < b < \infty$, $f : [a, b] \to \mathbb{R}$ is an integrable function, and $\alpha > 0$, $\rho > 0$ are fixed real numbers.

The Katugampola fractional integral operator $I^{\alpha,\rho}$ satisfies the following properties for $\alpha \geq 0$, $\rho > 0$, and $c \in \mathbb{R}$:

1. $I^{\alpha,\rho} I^{\beta,\rho} f(s) = I^{\alpha+\beta,\rho} f(s) = I^{\beta+\alpha,\rho} f(s) = I^{\beta,\rho} I^{\alpha,\rho} f(s)$,
2. $I^{\alpha,\rho}(cf(s)) = cI^{\alpha,\rho}(f(s))$,
3. $I^{\alpha,\rho}(s^n) = \frac{\rho^{-\alpha}\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)} s^{n+\alpha\rho}$.

The first two properties are discussed in (Tran et al., 2020).

We now prove property (3):

Proof 2.3.2 By the definition above (setting $a = 0$), we have:

$$I^{\alpha,\rho}(s^n) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^s t^{\rho-1}(s^\rho - t^\rho)^{\alpha-1} t^n\, dt.$$

Using the substitution $u = \frac{t^\rho}{s^\rho}$ and the Beta function $\mathcal{B}(\cdot,\cdot)$, we obtain:

$$\begin{aligned}
I^{\alpha,\rho}(s^n) &= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} s^{\alpha\rho+n} \int_0^1 (1-u)^{\alpha-1} u^{\frac{n}{\rho}}\, du, \\
&= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} s^{\alpha\rho+n} \mathcal{B}\left(\alpha, \frac{n}{\rho}+1\right), \\
&= \frac{\rho^{-\alpha}}{\Gamma(\alpha)} s^{\alpha\rho+n} \frac{\Gamma(\alpha)\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)}, \\
&= \frac{\rho^{-\alpha}\Gamma\left(\frac{n}{\rho}+1\right)}{\Gamma\left(\alpha+\frac{n}{\rho}+1\right)} s^{n+\alpha\rho}.
\end{aligned}$$

Definition 2.3.3 (Almeida et al., 2016)

Let $0 < a < b < \infty$, $f : [a,b] \to \mathbb{R}$ be integrable, and let $m-1 < \alpha \le m$ with $m \in \mathbb{N}$ and $\rho > 0$. The left and right CK fractional derivatives of order α are respectively defined as:

$$^{C}D^{\alpha,\rho}_{a+} f(s) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^s \frac{t^{(\rho-1)(1-m)}}{(s^\rho - t^\rho)^{\alpha-m+1}} f^{(m)}(t)\, dt,$$

$$^{C}D^{\alpha,\rho}_{b-} f(s) = \frac{(-1)^m \rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_s^b \frac{t^{(\rho-1)(1-m)}}{(t^\rho - s^\rho)^{\alpha-m+1}} f^{(m)}(t)\, dt.$$

The CK fractional derivative satisfies the following properties for $c \in \mathbb{R}$, $a < s \le b$, with $f \in C^j[a,b]$ (Almeida et al., 2016; Odibat and Baleanu, 2020):

1. $^{C}D^{\alpha,\rho}(c) = 0$,
2. $^{C}D^{\alpha,\rho}_{a+} I^{\alpha,\rho}_{a+} f(s) = f(s)$,
3. $I^{\alpha,\rho}_{a+}\, {}^{C}D^{\alpha,\rho}_{a+} f(s) = f(s) - \sum_{j=0}^{m-1} \frac{1}{\rho^j j!}(s^\rho - a^\rho)^j \left[\left(s^{1-\rho}\frac{d}{ds}\right)^j f(s)\right]_{s=a}$,
4. $^{C}D^{\alpha,\rho}(s^n) = \frac{\Gamma(n+1)\Gamma\left(\frac{n}{\rho}-m+1\right)}{\Gamma(n-m+1)\Gamma\left(\frac{n}{\rho}-\alpha+1\right)} \rho^{\alpha-m} s^{n-\alpha\rho}$.

We now prove property (4):