STUDY OF IRR-TOPOLOGICAL GROUPS AND ISOCOMPACTNESS FOR LOCALLY SEMI-COMPACT SPACES

WANG ZHONGLI

UNIVERSITI SAINS MALAYSIA

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STUDY OF IRR-TOPOLOGICAL GROUPS AND ISOCOMPACTNESS FOR LOCALLY SEMI-COMPACT SPACES

by

WANG ZHONGLI

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$\mathbb{N}$	Positive integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real line
ω_1	First uncountable ordinal
$A \times B$	Cartesian product
$\prod_{i\in\mathbb{N}} \mathbb{X}_i$	Infinite cartesian products
X	In this work, X always represents the Topological space
A°	Interior of A
$\overline{A}$	Closure of A
$\overline{A^\circ}$	Closure of the interior of A
$\overline{A}^\circ$	Interior of the closure of A
$sop(A)$	Semi-interior of A
$sclA$	Semi-closure of A
$Fr(A)$	Boundary of A
$SO(X)$	Semi-open sets of X
G	In this work, G always represents the Abelian topological group
e	In this work, e always represents the identity element.

KAJIAN KUMPULAN IR-TOPOLOGI DAN ISO-KEPADATAN BAGI RUANG SEMI-PADAT SETEMPAT

ABSTRAK

Selama bertahun-tahun, set terbuka teritlak telah menjadi tumpuan utama penyelidkan ahli matematik. Banyak sifat topologi bagi ruang semi padat setempat belum diketahui, dan hubungan mereka dengan ruang topologi yang lain masih tidak jelas. Kajian ini memberi tumpuan kepada masalah ini dan terdiri daripada dua bahagian utama. Bahagian pertama ferumpu kepada pengitlakan dan aplikasi ruang semi padat setempat, manakala bahagian kedua khususnya menumpu kepada aplikasi ruang isopadat dalam kumpulan topologi padat setempat. Kajian ini telah menyeles aikan tiga permasalahan dalam bahagian pertama. Pertama, kami memperoleh beberapa sifat topologi baru bagi ruang semi padat setempat, dan memperkenalkan ruang semi-k, sebagai suatu pengitlakan kepada ruang semi padat setempat. Kedua, hubungan antara aksiom pemisahan dan ruang semi padat setempat, antara ruang semi terbilangkan padat dan ruang semi-k, dan ruang semi padat setempat dan ruang s-parapadat ditubuhkan. Ketiga, dan yang paling penting, kami memperoleh beberapa aplikasi ruang semi padat setempat dalam kumpulan s-topologi dan kumpulan Irr-topologi. Selain itu, kami membuktikan bahawa kumpulan Irr-topologi σ semi padat setempat ialah ruang semi padat. Dalam bahagian kedua, kami menunjukkan bahawa jika G ialah kumpulan topologi T_2 dengan sifat isoc, H ialah subkumpulan padat setempat, dan G/H adalah ruang isopadat, maka G adalah ruang isopadat. Kajian ini menyumbang kepada pemahaman tentang ruang semi padat setempat dan memperoleh aplikasi bagi ruang semi padat setempat dalam kumpulan s-topologi dan kumpulan Irr-topologi. Selain itu, kajian ini mengukuhkan hubungan antara ruang isopadat dan kumpulan topologi padat setempat. Salah satu ciri utama tesis ini ialah perbincangan sistematik tentang ruang semi padat setempat daripada aspek pemetaan.

STUDY OF IRR-TOPOLOGICAL GROUPS AND ISOCOMPACTNESS FOR LOCALLY SEMI-COMPACT SPACES

ABSTRACT

Over the years, generalized open sets have been a focal point of research for mathematicians. Many of the topological properties of locally semi-compact spaces remain unknown, and their connections with other topological spaces are still unclear. This work will focus on these problems and mainly consists of two parts. The first part focuses on the generalizations and applications of locally semi-compact spaces, while the second part primarily deals with the applications of isocompact spaces in locally compact topological groups. In the first part, we primarily accomplished three works. First, we obtain some new topological properties of locally semi-compact spaces, and introduce the new definition of semi-k spaces, which generalizes the concept of locally semi-compact. Second, the relations between separation axioms and locally semi-compact spaces, between semi-countably compact spaces and semi-k spaces, and between locally semi-compact spaces and s-paracompact spaces are established. Third, and most importantly, we obtain some applications of locally semi-compact spaces in s-topological groups and Irr-topological groups. Additionally, we prove when locally σ semi-compact Irr-topological groups are semi-compact spaces. In the second part, we show that if G is a T_2 topological group with the isoc property, H is a locally compact subgroup, and G/H is isocompact, then G is isocompact. This work enriches the study of locally semi-compact spaces and obtains the applications of locally semi-compact spaces in s-topological groups and Irr-topological groups. Additionally, it establishes connections between isocompact spaces and locally compact topological groups. One of the features of this work is the systematic discussion of locally semi-compact spaces from the perspective of mappings.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

This chapter describes the definitions used in this work and the relationships among them, comprising a total of eight sections. Additionally, this chapter introduces the research objectives and framework of this work. Sections one through four primarily serve for the definitions in chapters two and three of this work. The definitions in section five mainly contribute to chapters four and five. The definitions in section six primarily contribute to chapter six of this work.

In this chapter, we mainly review some definitions, and provide some examples to distinguish the differences in these definitions. For definitions not defined here, we refer the reader to [37].

1.1 Background

Topology began to emerge in the eighteenth century and has been one of the most intriguing fields of research for mathematicians over the years. Particularly after Poincaré published a series of articles related to topology, the field achieved many significant results. These results have found wide applications in many other branches of mathematics [37]. Topological spaces are the fundamental objects of study in topology, with open sets being the most basic elements of these spaces. Consequently, the study of open sets has become an area of interest for many topologists.

The research of generalized open sets has led to the development of the theory of topology, which enriches the framework of topological spaces and opens up new avenues for exploration and research. It was in 1963 that Levine introduced the theory of semi-open sets and semi-continuity within general topological spaces.

Definition 1.1.1. [44] Suppose A is a subset of X. If A is a subset of $A^{\circ -}$, then A is called a *semi-open* set. If $A^{-\circ}$ is subset of A, then A is called a *semi-closed* set.

If U is open in X, U is a subset of A, and A is a subset of U^{-}, then A is semi-open. In addition, each open set is semi-open. However, each semi-open set may not be open.

Example 1.1.2. [44] Suppose $X = \{a, b, c\}$. Let

$$\mathscr{A} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}.$$

Then $(X, \mathscr{A})$ is a topological space, and the closed sets in X are

$$\mathscr{B} = \{\varnothing, X, \{b, c\}, \{a, c\}, \{c\}\}.$$

Since $\{b\} \subset \{b, c\} \subset \overline{\{b\}} = \{b, c\}$ and $\{a\} \subset \{a, c\} \subset \overline{\{a\}} = \{a, c\}$, it follows that $\{b, c\}$ and $\{a, c\}$ are semi-open sets. However, $\{b, c\}$ and $\{a, c\}$ are not open sets. Since $\{b, c\} \cap \{a, c\} = \{c\}$, it follows that the intersection of $\{b, c\}$ and $\{a, c\}$ is not semi-open.

In 1963, the author [44] said if $\mathscr{D}$ and $\mathscr{C}$ are two topologies of X, and $SO(X, \mathscr{D})$ is a subset of $SO(X, \mathscr{C})$, then $\mathscr{D}$ is a subset of $\mathscr{C}$. Example 1.1.3 shows that $\mathscr{D}$ is not a subset of $\mathscr{C}$.

Example 1.1.3. [56] Suppose $X = \{1, 2, 3\}$. Let

$$\mathscr{D} = \{X, \varnothing, \{1\}, \{1, 2\}\}$$

and

$$\mathscr{C} = \{X, \varnothing, \{1\}, \{3\}, \{1, 3\}\}.$$

Hence, $(X, \mathscr{D})$ and $(X, \mathscr{C})$ are topological spaces. Thus,

$$SO(X, \mathscr{D}) = \{X, \varnothing, \{1\}, \{1, 2\}, \{1, 3\}\}$$

and

$$SO(X, \mathscr{C}) = \{X, \varnothing, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$

Hence, $SO(X, \mathscr{D}) \subset SO(X, \mathscr{C})$ and $\mathscr{D} \not\subset \mathscr{C}$.

Since the semi-open sets play a significant role in general topology, they are now the research topics of many topologists worldwide. Research on semi-open sets mainly focuses on four topics. The first topic is the generalization of semi-open sets, such as α-open sets [56], regular-open sets [10], and pre-open sets [18]. These generalized open sets are distinct from semi-open sets. The second topic is the semi-separation axioms. Semi-separation axioms can be used to define more restricted classes of semi-topological spaces, such as semi-T_0 spaces [22], semi-T_1 spaces [22], semi-T_2 spaces [22], s-regular spaces [49], s-normal spaces [49], and semi-normal spaces [58]. The third topic is the covering properties generated by semi-open sets, such as semi-compact spaces [30], s-paracompact spaces [2], locally semi-compact spaces [41], semi-continuous mapping [44], irresolute mapping [21], pre-semi-open mapping [21], semi-open mapping, semi-homeomorphism mapping [53]. The fourth topic is the applications of generalized open sets in topological groups. Topologists in the 20th century worked on the properties of topological groups by relaxing the continuity requirements to extend their definition. They replaced open sets with semi-open sets and continuous mappings with irresolute, thus constructing some topological groups such as s-topological groups [13], and Irr-topological groups [68].

Recent research has focused on three questions. The first question is: How can we define a class of separation axioms using the aforementioned sets in topological spaces? The second question is: Certain topological spaces are preserved under what kinds of mappings? The third question is: When a locally compact space can be represented as a retract of a locally compact topological group?

1.2 Generalized open sets

Since the introduction of semi-open sets, many mathematicians have introduced and investigated generalized open sets. In 1965, Njåstad [56] not only introduced the definition of α-open sets but also utilized them to construct related topological structures.

Definition 1.2.1. [56] Suppose A is a subset in X. If A is subset of $A^{\circ-\circ}$, then A is called an *α-open* set.

According to the definition of α-open, each α-open set is semi-open. However, Example 1.2.2 shows that semi-open may not be α-open.

Example 1.2.2. [56] Suppose $X = \{1,2,3,4\}$. Let

$$\mathscr{A} = \{\varnothing, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}.$$

Hence, $\mathscr{A}$ is a topology of X. Let $A = \{1,3,4\}$. Thus, $\{1\} \subset A \subset \overline{\{1\}} = A$, and A is a semi-open set. However, A is not a α-open set.

Also, each open set is α-open. However, Example 1.2.3 shows that α-open may not be open.

Example 1.2.3. Suppose $X = \{1,2,3,4,5\}$. Let

$$\mathscr{A} = \{\varnothing, X, \{3\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}\}.$$

Hence, $\mathscr{A}$ is a topology of X. Let $A = \{2,3,4,5\}$. Thus,

$$A^{\circ-\circ} = \{3,4,5\}^{-\circ} = \{1,2,3,4,5\}^{\circ} = X.$$

Hence, $A \subset A^{\circ-\circ}$. Therefore, A is α-open and not open.

According to the definitions of α-open and semi-open, we know that each open set is α-open. Also, each α-open set is semi-open.

Definition 1.2.4. [10] Suppose A is a subset of X. If A is equal to $A^{-\circ}$, then A is called a *regular-open* set.

According to the definition of regular-open, each regular-open set is open. However, we known that open may not be regular-open. Also, Example 1.2.5 shows that the union of regular-open sets may not be regular-open.

Example 1.2.5. Let $\mathscr{A}$ be the usual topology of $\mathbb{R}$. Let $A = (0, 1/2)$ and $B = (1/2, 1)$. Thus, A and B are regular-open, but $A \cup B = (0, 1/2) \cup (1/2, 1)$ is not regular-open.

At the same time, some topologists, aiming to further study α-open sets, have extended the concept to construct some generalized open sets and related generalized continuous functions [18].

Definition 1.2.6. [18] Suppose A is a subset of X. If A is a subset of $\overline{A}^{\circ}$, then A is called a *pre-open* set.

According to the definition of the pre-open, each open set is pre-open. However, the pre-open sets need not be open. Also, we note that pre-open sets need not be semi-open, and semi-open sets need not be pre-open.

Example 1.2.7. Suppose $X = \{x_1, x_2, x_3, x_4\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{x_1, x_4\}, \{x_3\}, \{x_1, x_3, x_4\}\}.$$

Then $(X, \mathscr{A})$ is a topological space. Thus,

$$SO(X) = \{X, \varnothing, \{x_1, x_4\}, \{x_3\}, \{x_1, x_3, x_4\}, \{x_2, x_3\}, \{x_1, x_4, x_2\}, \}.$$

Let $A = \{x_1, x_2, x_3\}$. Thus, $A \subset (\overline{A})^{\circ} = X$ and A is pre-open. However, A is not semi-open.

Since $\{x_2, x_3\} \in SO(X)$ and $\overline{\{x_2, x_3\}}^{\circ} = \{x_3\}$, it follows that $\{x_2, x_3\}$ is semi-open. According to the definition of the pre-open, $\{x_2, x_3\}$ is not a pre-open.

It is clearly that each α-open is pre-open. However, the pre-open sets need not be α-open.

Example 1.2.8. Suppose $X = \{1, 2, 3, 4, 5\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{1\}, \{3, 4\}, \{1, 3, 4\}\}.$$

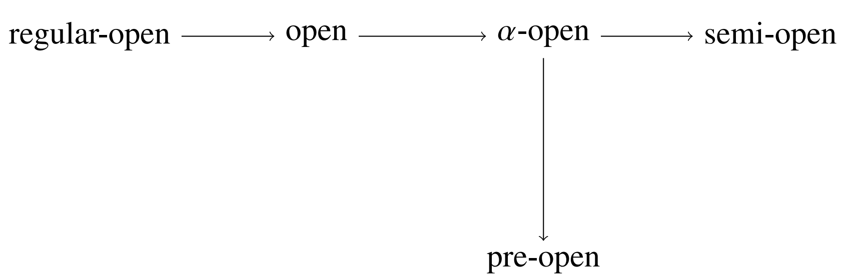
Then $(X, \mathscr{A})$ is a topological space. Since $\{1,4\}^{-\circ} = X^{\circ} = X$, it follows that $\{1,4\} \subset \{1,4\}^{-\circ}$. Thus, $\{1,4\}$ is pre-open. Hence,

$$\{1,4\}^{\circ-\circ} = \{1\}^{-\circ} = \{1,2,5\}^{\circ} = \{1\}.$$

Thus, $\{1,4\}$ is not α-open.

Remark 1.2.9. Suppose A is a pre-open subset of X. Then there has an open set B in X such that A is a subset of B, and B is a subset of $\overline{A}$.

Based on the previous descriptions of these definitions, we can use Figure 1.1 to illustrate their relationships. Generally, the relationships among these definitions are not reversible.



(In this work, the symbol $\rightarrow$ means ‘implies’.)

Figure 1.1: Relations of the general open sets

1.3 Semi-separation axioms

On the other hand, when open sets are replaced by semi-open sets, the separation axioms yield some results, known as the semi-separation axioms. Consequently, significant progress has been made in the study of semi-separation axioms through semi-open sets.

Definition 1.3.1. [22] Suppose $a \in X$, $b \in X$ and $a \neq b$. If at least one of a and b has a semi-open set that does not contain the other, then X is called a *semi-*T_0 space.

According to the definition of semi-T_0, each discrete space is semi-T_0 in the topological space. Here is a topological space that is semi-T_0.

Example 1.3.2. Suppose $X = \{1, 2\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{1\}\}.$$

Then $(X, \mathscr{A})$ is a topological space. Thus, $A = \{1\}$ is semi-open. Hence, $1 \in A$ and $2 \notin A$. Thus, X is semi-T_0.

Definition 1.3.3. [22] For a space X to be *semi-*T_1, it must have the property that each pair of distinct points has semi-open sets such that neither set contains the other point.

According to the definition of semi-T_1, each T_1 space is semi-T_1, and each semi-T_1 space is semi-T_0.

Now, we will show that the set of real space $\mathbb{R}$ with the usual topology is semi-T_1. Suppose a and b are distinct points in $\mathbb{R}$. Assume that $a < b$ and $b - a = r$. Let

$$A = [a - r/3, a + r/3]$$

and

$$B = [b - r/3, b + r/3].$$

Then A and B are semi-open such that $a \in A$ and $b \in B$. Because $a \notin B$ and $b \notin A$. Thus, $\mathbb{R}$ is semi-T_1.

Here is a topological space that is semi-T_0 and not semi-T_1.

Example 1.3.4. Suppose $X = \{1, 2, 3\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{1\}, \{1, 2\}, \{1, 3\}\}.$$

Thus, $\mathscr{A}$ is a topology of X.

Hence, there exists a semi-open set $A = \{1\}$ such that $1 \in A$, $2 \notin A$ and $3 \notin A$. Also, there exists a semi-open set $B = \{1, 2\}$ such that $2 \in B$ and $3 \notin B$. Thus, X is semi-T_0. Since $1 \neq 3$, and the only semi-open sets containing 3 are X and $\{1, 3\}$, it follows that X is not semi-T_1.

Definition 1.3.5. [22] A space X is said to be *semi-T_2* if, for any distinct points x, $y \in X$, there exist disjoint semi-open sets A and B such that $x \in A$, $y \in B$.

According to the definition of semi-T_2, each T_2 space is semi-T_2, and each semi-T_2 space is semi-T_1. Now, we give an example that shows that it is a semi-T_1 space, but not semi-T_2 space.

Example 1.3.6. Suppose $X = \{1, 2, 3\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{1\}, \{2, 3\}\}.$$

Then $(X, \mathscr{A})$ is a topological space. Clearly, X is semi-T_1. Hence, the semi-open sets containing 2 are X, $\{1, 2\}$ and $\{2, 3\}$. Thus, the semi-open sets containing 3 are X, $\{1, 3\}$ and $\{2, 3\}$. Therefore, X is semi-T_1 and not semi-T_2.

Definition 1.3.7. [58] Suppose A and B are semi-closed in X, and the intersection of A and B is empty. If there are semi-open sets U and V such that A is a subset of U, B is a subset of V and the intersection of U and V is empty, then X is called a *semi-normal* space.

Definition 1.3.8. [48] A space X is said to be *s-regular* if, for any closed set A, $x \notin A$, there exist disjoint semi-open sets B and C such that $A \subset B$, $x \in C$.

Definition 1.3.9. [49] A space X is said to be *s-normal* if, for any disjoint closed sets A and B, there exist disjoint semi-open sets C and D such that $A \subset C$, $B \subset D$.

Example 1.3.10 shows that s-normal may not be s-regular.

Example 1.3.10. Suppose $X = \{a, b, c, d\}$. Let

$$\mathscr{A} = \{X, \varnothing, \{b\}, \{c\}, \{a, b\}, \{c, b\}, \{b, c, d\}, \{a, b, c\}\}.$$

Then $(X, \mathscr{A})$ is a topological space. Hence, X is s-normal, and $\{a, c, d\}$ is closed and $b \notin \{a, c, d\}$. However, there are no semi-open sets A and B such that $b \in A$, $\{a, c, d\} \subset B$ and the intersection of A and B is empty. Therefore, X is not s-regular.

1.4 Covering properties and mappings

Covering properties plays an important role in general topology, and many topological properties of locally compact spaces have been obtained by topologists [34, 36, 46, 50, 52, 59, 62], but research on locally semi-compact spaces is not as extensive.

Definition 1.4.1. [28] A subset A of space X is said to be a dense subset of X if any of the closures of A is equal to X.

Definition 1.4.2. [28] A topological space is called $separable$ if it contains a countable dense subset.

Clearly, $\mathbb{R}$ is a separable space, and the rational numbers $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

Definition 1.4.3. [28] Suppose $\mathscr{A}$ is an open cover of X. If $\mathscr{A}$ has a finite subcover of X, then X is called a *compact* space.

Definition 1.4.4. [45] A topological space is said to be σ compact if it is the union of countably many compact subspaces.

Definition 1.4.5. [30] A topological space X is said to be *semi-compact* if every cover of X by semi-open sets has a finite subcover.

Since each open set is semi-open, it follows that each semi-compact space is compact. However, the compact spaces need not be semi-compact.

Example 1.4.6. Let $X = \mathbb{R}$ have the standard topology $\mathscr{A}$. Suppose $Y = [-a, a]$ is a subset of X, and $\mathscr{B} = \mathscr{A}|_Y$. Thus, Y is compact. Let

$$\mathscr{C} = [-a, 0] \cup \{(a/n + 1, a/n) : n \in \mathbb{N}\}.$$

Thus, $\mathscr{C}$ is a semi-open cover of Y. However, $\mathscr{C}$ does not has finite subcover. Thus, Y is not semi-compact.

Definition 1.4.7. [9] Suppose $\mathscr{B}$ is a family of X, and x is a point in X. If there is an open set U such that x in U, and $\{B \in \mathscr{B} : B \cap U \neq \varnothing\}$ is a finite set. Then X is called *locally finite*.

Definition 1.4.8. [9] Suppose $\mathscr{A}$ is an open cover of X. If $\mathscr{A}$ has a locally finite open refinement, then X is called a *paracompact* space.

Clearly, each compact space is a paracompact space, and the converse need not be true.

Definition 1.4.9. [1] Suppose $\mathscr{A}$ is a family of X, and a is a point in X. If there is a semi-open set U such that x in U, and $\{B \in \mathscr{B} : B \cap U \neq \varnothing\}$ is a finite set. Then X is called *s-locally finite*.

Clearly, if a collection is locally finite, it is s-locally finite, but Example 1.4.10 shows that s-locally finite need not be locally finite.

Example 1.4.10. [1] Suppose $\mathbb{R}$ has the usual topology $\mathscr{A}$. Let

$$\mathscr{B} = \{\{1/n\} : n = 1, 2, \cdots\}.$$

Suppose A is an open neighborhood of 0. Thus, $\{B \in \mathscr{B} : B \cap A \neq \varnothing\}$ is an infinite set. Hence, $\mathscr{B}$ is not locally finite. Let $C = (-2, 0]$. Then $0 \in C$ and C is semi-open. For each $B \in \mathscr{B}$, $B \cap C = \varnothing$. Suppose x is a point in $\mathbb{R}$, and $x \neq 0$. Then there exists a semi-open neighborhood D such that D intersects at most finitely many members of $\mathscr{B}$. Hence, $\mathscr{B}$ is s-locally finite.

Definition 1.4.11. [2] A space X is said to be *s-paracompact* if every open cover of X has a locally finite semi-open refinement.

According to the definition of s-paracompact, each paracompact space is s-paracompact. But the converse need not be true. A topological space is locally compact if every point has a compact open neighborhood [28]. Locally compact spaces are one of the most widely used topological spaces and have many good applications, such as locally compact topologies on abelian groups and the action of the group of isometries on a locally compact metric space [20, 42, 43, 46, 50, 52, 55, 59, 62, 69]. Following the concept of locally compact spaces, topologists have constructed locally semi-compact spaces.

Definition 1.4.12. [41] A space X is said to be *locally semi-compact* if, for any point a of X, there is an open semi-compact neighborhood A such that $a \in A$.

According to the definition of locally semi-compact, each semi-compact space is locally semi-compact. However, Example 1.4.13 shows that locally semi-compact spaces need not be semi-compact.

Example 1.4.13. Let X be an infinite discrete topological space. Thus, X is not semi-compact. Since the singletons can serve as semi-compact neighborhoods, it follows that X is locally semi-compact.

Clearly, locally semi-compact spaces arose as a natural generalization of local compactness and semi-compactness. However, the research on the locally semi-compact spaces could be more extensive. In this work, we continue to study the properties of locally semi-compact spaces.

Another line of research on locally compact frames stems from the k-spaces [34, 38, 39]. In the same way, we introduce the concept of semi-k spaces by locally semi-compact spaces. The study of semi-k spaces is important in general topology as it provides insights into the structure and properties of topological spaces that lie between k-spaces and general topological spaces.

Definition 1.4.14. [29] A space X is said to be a k space if X is T_2 and each $A \subset X$ is closed, providing $B \cap A$ is closed in X for any compact set B.

Since various types of mappings serve as tools for comparing and classifying abstract geometric objects, such as topological spaces, topological groups, metric spaces, and function spaces, they play a crucial role in the study of these objects. To better introduce the connections between topological spaces, we will now introduce some mappings.

Definition 1.4.15. [44] Let f be a mapping from X to Y. Suppose V is an open set in Y. If $f^{-1}(V)$ is semi-open in X, then f is called *semi-continuous*.

Since f and g are continuous, it follows that fg is continuous. However, Example 1.4.16 shows that even if f and g are semi-continuous, fg may not be semi-continuous.

Example 1.4.16. Let $X = X_1 = X_2 = [a, b]$ with the standard topology. Let $f : X \to X_1$ be the following mapping.

$$f(x) = \begin{cases} x, & \text{if } a \le x \le \frac{b-a}{2}, \\ 0, & \text{if } \frac{b-a}{2} < x \le b. \end{cases}$$

Let $g : X \to X_2$ be the following mapping.

$$g(x) = \begin{cases} 0, & \text{if } a \le x < \frac{b-a}{2}, \\ \text{b}, & \text{if } \frac{b-a}{2} \le x \le b. \end{cases}$$

Then f and g are both semi-continuous, but fg is not semi-continuous.

Definition 1.4.17. [21] A mapping $f : X \to Y$ is said to be *irresolute* if $f^{-1}(V)$ is semi-open in X for each semi-open set V in Y.

According to the definition of irresolute, each irresolute mapping is semi-continuous. However, semi-continuous need not be irresolute.

Definition 1.4.18. [21] A mapping $f : X \to Y$ is said to be *pre-semi-open* if $f(V)$ is semi-open in Y for each semi-open set V in X.

Definition 1.4.19. [15] A mapping $f : X \to Y$ is said to be *semi-open* if $f(V)$ is open in Y for each semi-open set V in X.

Definition 1.4.20. [53] A mapping $f : X \to Y$ is said to be *semi-homeomorphism* if f is bijective, irresolute and pre-semi-open.

Example 1.4.21 shows that the restriction of semi-homeomorphism need not be semi-homeomorphism.

Example 1.4.21. Suppose $X = \{a, b, c, d\}$. Let

$$\mathscr{A} = \{X, \varnothing, a\},$$

$$\mathscr{B} = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

Then $(X, \mathscr{A})$ and $(X, \mathscr{B})$ are topological spaces. Let $f : (X, \mathscr{A}) \to (X, \mathscr{B})$ be a identity mapping. Then f is a semi-homeomorphism mapping. Let $A = \{b, c\}$. Thus,

$$\mathscr{A} |_A = \{\varnothing, \{b, c\}\}$$

and

$$\mathscr{B} |_{f(A)} = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}.$$

Hence, $f |_A : (A, \mathscr{A} |_A) \to (f(A), \mathscr{B} |_{f(A)})$ is not semi-homeomorphism.

Definition 1.4.22. [45] A mapping $f : X \to Y$ is said to be *continuous* if $f^{-1}(V)$ is open in X for each open set V in Y.

According to the definition of continuous, each continuous mapping is semi-continuous, and Example 1.4.23 shows that the converse need not be true.

Example 1.4.23. Suppose $X = Y = [0, 1]$ with the standard topology. Let $f : X \to Y$ be the following mapping.

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2, \\ 0, & \text{if } 0 < x \le 1. \end{cases}$$

Thus, $f : X \to Y$ is a semi-continuous mapping, and not continuous mapping.

Definition 1.4.24. [72] A closed continuous onto mapping $f : X \to Y$ is said to be *perfect* if $f^{-1}(y)$ is compact in X for each y in Y.

Definition 1.4.25. [32] A mapping $f : X \to Y$ is said to be *almost open* if, for each $y \in Y$, there is $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y in Y whenever U is a neighborhood of x.

According to the definition of almost open, each open surjection is almost open. But the converse need not be true. According to the definitions of almost open mapping and open mapping, each continuous almost open surjection is open.

According to the definitions, we have the following relations about the mappings of generalized open sets in Figure 1.2, and none of these implications is reversible in general. Also, we will show the relations about the covering properties of generalized open sets in Figure 1.3.

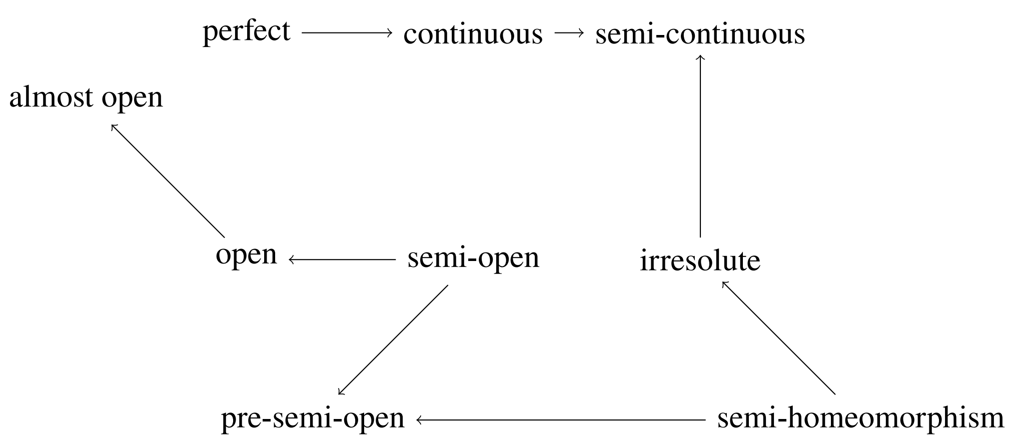


Figure 1.2: Relations of the mappings

1.5 Applications of generalized open sets in topological groups

A topological group represents a blend of different branches of mathematics and serves as an important tool for studying the properties of topological spaces. It has consistently been a subject of interest for topologists and has broad applications across various mathematical fields, helping us to understand many complex topological structures [5].

Definition 1.5.1. [11] Suppose G is a group with a topology, and $G \times G$ is given the product topology. If the multiplication mapping $G \times G \to G$, $(x, y) \to xy$ and the inverse mapping $G \to G$, $x \to x^{-1}$ are both continuous, then G is called a *topological group*.

Definition 1.5.2. [66] Suppose G is a topological group. If G is T_2 and locally compact, then G is called a *locally compact group*.

Definition 1.5.3. [5] Suppose f is a mapping from group G to group F. Let a and b be points in G. If $f(ab)$ is the same as $f(a)f(b)$, then f is called a *homomorphism*.

In [12], Bohn and Lee defined semi-topological groups and obtained some properties. In [13], the author constructed a series of generalized topological groups based on generalized open sets. These generalized topological groups differ from existing topological groups and represent a generalization of them. The emergence of these topological groups provides some research directions for the study of topological groups. On the other hand, they provided some examples to illustrate the differences and connections among these topological groups.

Definition 1.5.4. [13] Suppose G is a group with a topology, x and y are points in G, and W is a neighborhood of xy^{-1}. If there are semi-open neighborhoods U of x and V of y such that $UV^{-1} \subset W$, then G is called an *s-topological group*.

Remark 1.5.5. For any subgroups A, B, and C of an s-topological group G, $AB \cap C \neq \emptyset$ if and only if $A \cap CB^{-1} \neq \emptyset$.

It was noticed in [13] that each topological group is s-topological group. If G is a topological group, and $a \in G$, then the mappings $x \to ax$ and $x \to xa$ are homeomorphisms, and the inverse mapping is a homeomorphism [5].

In 2015, the authors proposed Irr-topological groups in [68], analyzed their features, and established their distinctions from topological groups. In [51], the authors proved the relation between s-topological groups and Irr-topological groups.

Definition 1.5.6. [68] A topologized group G is said to be an *Irr-topological group* if both the multiplication mapping $f : G \times G \to G$ and the inverse mapping $g : G \to G$ are irresolute.

It was noticed in [68] that each s-topological group is Irr-topological group. However, the relationships among Irr-topological groups and other topological spaces have not been obtained.

Locally compact groups are significant, and many topologists have explored the relationships among various topological groups and locally compact spaces [24, 61, 71, 74]. Using this method, we will utilize locally semi-compact spaces to study the properties of s-topological groups. At the same time, we will explore the connections between Irr-topological groups and other topological spaces.

One of the main areas of research on topological groups is the study of the properties of their quotient groups. The quotient spaces of some topological groups have yet to be proposed either. One of the primary purposes of this work is to establish relationships between Irr-topological groups and locally semi-compact spaces and define a quotient topology on the Irr-topological group.

In this work, we use locally semi-compact spaces to study generalized topological groups, and obtain some relationships between these topological groups and other topological spaces.

1.6 The relations of locally compact subgroups and isocompact spaces

Locally compact groups play an important role in many areas of topology [14, 47, 57, 60, 73]. For example, locally compact groups have the stronger property of being normal, and every locally compact group which is first countable is metrisable as a topological group [5], then many results about locally compact subgroups have been obtained [8, 16, 17, 54].

In this work, we use locally compact subgroups to study isocompact space introduced by Bacon.

Definition 1.6.1. [7] A space X is called an *isocompact* space if every closed countably compact set in X is a compact set.

According to the definition of isocompact, each paracompact spaces is isocompact, and there is a locally compact space that is not isocompact, for example ω_1 [45]. A large number of topological spaces imply isocompactness, for example, pure spaces [4], and neat spaces [64].

The relationships between isocompact spaces and Tychonoff spaces also have been established. In [27], the authors proved that a nice isocompact Tychonoff space is not cl-isocompact. However, a Tychonoff space X is cl-isocompact if and only if X is an isocompact space with property CC [19].

Definition 1.6.2. [45] A T_1 space X is said to be $Tychonoff$ if and only if for each $x \in X$ and each closed set $A \subset X$ so that $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for each $a \in A$.

In [64], the author showed that an ω_1-compact ω_1-neat T_1-space is α-realcompact and k-neatness is an inverse invariant of maps under some conditions. Due to the importance of isocompact spaces, many authors used the mapping to study isocompact spaces. For example, Shiraki [67] pointed out that if a topological space has a point countable pseudo-base, then it is an isocompact space. In [75], the authors proved that each weakly $\delta\theta$ refinable spaces is isocompact. The relationships among the isocompact spaces, ω-Lindelöf spaces, and paracompact spaces have been established in [6].

Definition 1.6.3. [70] Suppose $\mathscr{A}_n$ is an open cover of X for each $n \in \mathbb{N}$, and let x be a point in X. If $x_n \in (\cup\{A_n \in \mathscr{A}_n : x \in \mathscr{A}_n\})$, then $\{x_n : n \in \mathbb{N}\}$ has an accumulation point. Thus, X is called a $w\Delta$ space.

Definition 1.6.4. [23] Suppose G is a topological group, and V is an open neighborhood of e. If there is a countable set A in G, and $VA = AV = G$, then G is called ω $narrow$.

1.7 Problems of statement

In this work, we continue to study locally semi-compact spaces, locally compact spaces, and isocompact spaces. Firstly, many properties of locally semi-compact spaces have already been established, but their applications in semi-topological spaces have not yet been explored, nor have the connections between them and topological groups been established. Naturally, we propose the following questions.

Question 1.7.1. *What is the connections between semi-countably compact spaces and semi-k spaces?*

Question 1.7.2. *Suppose X is a locally semi-compact space. Under what conditions is X an s-paracompact space?*

To establish the connections between locally semi-compact spaces and topological groups, we pose the following questions.

Question 1.7.3. *Suppose G is an s-topological group, H is a semi-compact subgroup. Under what conditions on G/H is G a locally semi-compact space?*

Question 1.7.4. *Suppose G is a locally σ semi-compact space. Is G a semi-compact space?*

Secondly, topologists have obtained some topological properties of isocompact spaces. However, the relationship between isocompact spaces and topological groups has not yet been studied. Since H and G/H are locally compact, it follows that topological group G is also locally compact. Naturally, we propose the following question.

Question 1.7.5. *Suppose H and G/H are isocompact spaces. Is G an isocompact space? If not, under what conditions G is an isocompact space.*

On the other hand, seeking characterizations of isocompact spaces is still an open problem. This work continues to study the problem by topological groups, and obtain some properties of isocompact spaces.

1.8 Objectives of the study

Based on the problems stated, we will continue our research on locally semi-compact spaces and locally compact spaces. In this work, we have four objectives.

1. To obtain some applications and properties of locally semi-compact spaces.

2. To introduce some concepts, such as SOP properties, semi-k spaces, semi-nets, and isoc properties.

3. To establish relationships among locally semi-compact spaces, semi-Lindelöf spaces, s-paracompact spaces, semi-countably compact spaces, paracompact spaces, semi-k spaces and k spaces.

4. To obtain the relationships between locally semi-compact spaces and s-topological groups, locally semi-compact spaces and Irr-topological groups, isocompact spaces and locally compact groups.

The relationships among these topological spaces and topological groups are illustrated in Figure 1.3, and none of these implications is reversible in general.

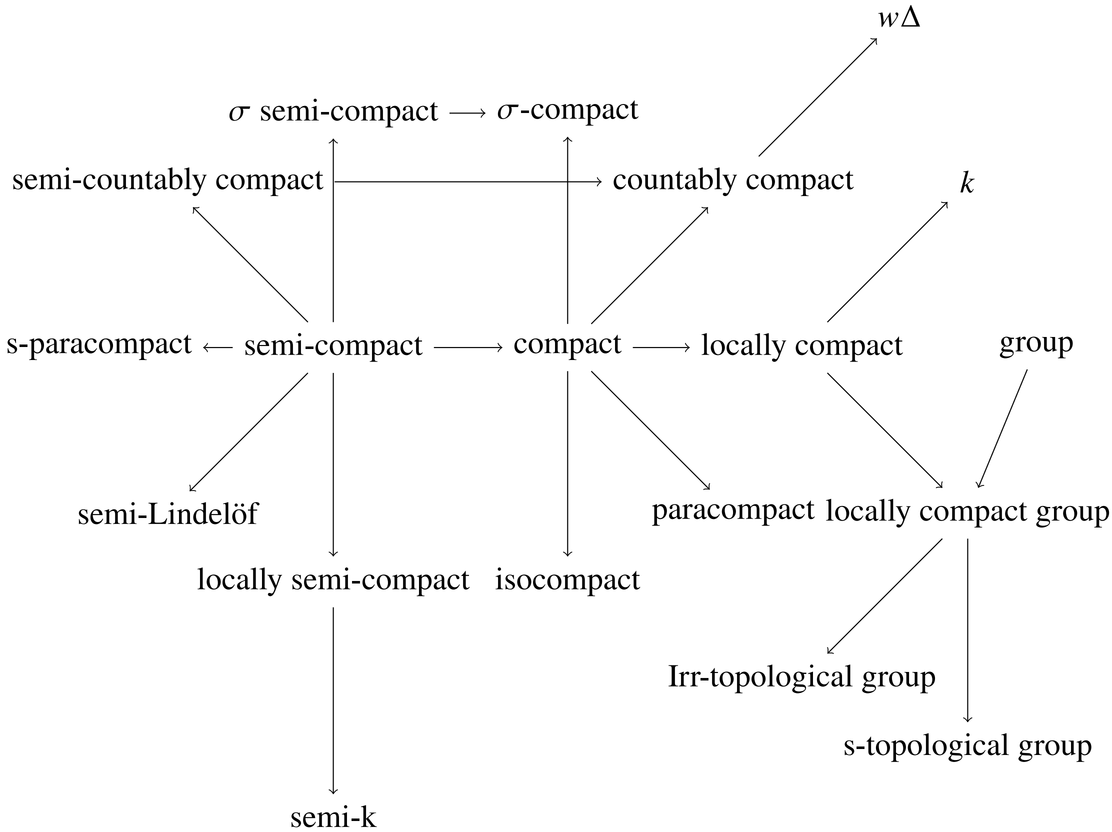


Figure 1.3: Relations of the topological spaces

1.9 Thesis Outline

This work mainly consists of two parts. The first part focuses on the generalizations and applications of locally semi-compact spaces, while the second part primarily deals with the applications of isocompact spaces in locally compact groups. This document is organized as follows.

In chapter 2, we continue the study of the properties of locally semi-compact spaces, and some properties have been obtained. In this chapter, we show that locally semi-compact space is inversely preserved under pre-semi-open continuous bijection. At the same time, the relations among separation axioms and locally semi-compact spaces are also established. Meanwhile, we have established the connection between s-regular spaces and locally semi-compact spaces. To better study locally semi-compact spaces, we introduce a definition of the SOP topological property. According to the SOP property, we obtain some topological properties of locally semi-compact spaces.

Chapter 3 begins by generalizing locally semi-compact spaces to semi-k spaces. At the same time, we have obtained some topological properties of semi-k spaces, such as the topology sum of semi-k spaces being semi-k spaces, regular-open subspaces of semi-k spaces being semi-k spaces, and semi-k spaces being preserved under semi-homeomorphism mappings. Also, we establish the connections between locally semi-compact spaces and semi-k spaces by irresolute and pre-semi-open mappings and obtain the relations between semi-countably compact spaces and semi-k spaces. In order to establish the connections between semi-countably compact spaces and semi-k spaces, we introduced the definition of semi-nets in this chapter and obtained some topological properties of semi-nets. We also obtained the connections between s-paracompact spaces and locally semi-compact spaces.

In chapter 4, we obtained some applications of locally semi-compact spaces, and established the relations between locally semi-compact spaces and s-topological groups. There are mainly two parts, and the first one is about the properties of s-topological groups. We show that if a locally semi-compact s-topological group is separable, then it is a semi-Lindelöf space. The second part deals with the quotient groups of s-topological groups, where we establish a relationship between quotient groups and locally semi-compact spaces.

In chapter 5, we obtain some properties of the locally semi-compact Irr-topological groups and establishes the relationships between Irr-topological groups and semi-compact spaces. Also, we show that if a locally σ semi-compact Irr-topological group has the countable semi-star property, and each semi-open set is pre-open, then the group is semi-compact. At the same time, we define a quotient topology on the Irr-topological group and obtain some properties of the quotient topology group.

In chapter 6, we give a necessary and sufficient condition for a ω narrow to be Lindelöf by locally compact subgroup. On the other hand, we obtain the relations between isocompact spaces and locally compact subgroups.

CHAPTER 2

PROPERTIES OF LOCALLY SEMI-COMPACT SPACES

2.1 Introduction

This chapter is primarily devoted to the study of locally semi-compact spaces and consists of two parts. In the first part, we show that locally semi-compact spaces are preserved by almost open irresolute surjection, and inversely preserved by pre-semi-open continuous bijection. At the same time, we show that if A and B are locally semi-compact α-open in X, then $A \cup B$ is locally semi-compact.

In the second part, we introduce the definition of the SOP property, and establish the relations between locally semi-compact spaces and s-regular spaces. At the same time, Theorem 2.3.18 shows that if X and Y are locally semi-compact spaces with the SOP property, then $X \times Y$ is locally semi-compact.

2.2 Properties of locally semi-compact spaces

In 1984, the author [41] shows open irresolute surjection can preserve locally semi-compact spaces. Now, we will show that almost open irresolute surjection can preserve locally semi-compact spaces also, and it is a generalization of Theorem 2.2.1.

Theorem 2.2.1. *[41] If $f : X \to Y$ is an open irresolute surjection and X is locally semi-compact, then Y is locally semi-compact.*

Since f is an open surjection, it follows that f is an almost open mapping [45]. However, the almost open mapping may not be open mapping [32]. Theorem 2.2.2 is a generalization of Theorem 2.2.1 to locally semi-compact spaces.

Theorem 2.2.2. *If $f : X \to Y$ is an almost open irresolute surjection and X is locally semi-compact, then Y is locally semi-compact.*

Proof. Suppose y in Y. Since f is an almost open mapping and X is locally semi-compact, it follows that there is x in $f^{-1}(y)$ and a semi-compact neighborhood V containing x such that $f(V)$ is a neighborhood of y. Suppose $\{W_\alpha\}_{\alpha \in I}$ is a semi-open cover of $f(V)$. Since f is an irresolute surjection, it follows that $\{f^{-1}(W_\alpha)\}_{\alpha \in I}$ is a semi-open cover of V. Since V is semi-compact, it follows that there exists a finite semi-open subcover

$$\{f^{-1}(W_1), f^{-1}(W_2), \cdots, f^{-1}(W_r)\}.$$

Hence, V is a subset of $\cup_{i=1}^{r} f^{-1}(W_i)$, and $f(V)$ is a subset of $\cup_{i=1}^{r} W_i$. Therefore, $f(V)$ is a semi-compact neighborhood of y, and Y is locally semi-compact. □

Example 2.2.3 shows that Y need not be locally semi-compact when $f : X \to Y$ is an irresolute surjection and X is a locally semi-compact space. Hence, the hypothesis that f is almost open is essential in Theorem 2.2.2.

Example 2.2.3. Suppose $\mathbb{R}$ is the Euclidean topology space. Let $X = \{-1\} \cup (0, 1]$ be a subspace of $\mathbb{R}$. Then X is a locally semi-compact space. Let

$$Y = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, 0)\}.$$

Then Y is a subspace of Euclidean topology space $\mathbb{R}^2$.

Suppose V is an open neighborhood of $(0, 0)$. Then there exists an open ball E_r centered at $(0, 0)$, where r is the radius of E_r, such that $E_r \cap Y \subset V$. Thus,

$$\{(x, r/3) : x \in \mathbb{R}\} \cap E_r \cap Y$$

is an infinite set of V and does not have semi-cluster point in Y. Then V is not semi-compact. Thus, Y is not locally semi-compact. Let $f : X \to Y$ be a following mapping.

$$f(x) = \begin{cases} (0, 0), & if\ x = -1, \\ (x, \sin(1/x)), & if\ 0 < x \leq 1. \end{cases}$$

Then $f : X \to Y$ is an irresolute surjection, and X is a locally semi-compact space, but the locally semi-compact is not preserved under f.

Theorem 2.2.4. *[41] Each locally semi-compact space is a locally compact space.*

Lemma 2.2.5. *If $f : X \to Y$ is a semi-open continuous bijection and Y is locally compact, then X is locally semi-compact.*

Proof. Suppose $x \in X$ and $y = f(x)$. Then there exists a compact neighborhood U such that $y \in U$ and $x \in f^{-1}(U)$. Then $f^{-1}(U)$ is a neighborhood of x. Suppose $\{V_\alpha : \alpha \in I\}$ is a semi-open cover of $f^{-1}(U)$. Since f is a semi-open surjection, it follows that $\{f(V_\alpha) : \alpha \in I\}$ is an open cover of U. Thus, there is a finite subcover

$$\{f(V_1), f(V_2), \ldots, f(V_r)\}$$

of U. Then there is a finite subcover

$$\{V_1, V_2, \ldots, V_r\}$$

of $f^{-1}(U)$. Therefore, $f^{-1}(U)$ is a compact neighborhood of x, and X is locally semi-compact. □

Corollary 2.2.6. *If $f : X \to Y$ is a semi-open continuous bijection and Y is locally compact, then X is locally compact.*

Proof. According to Lemma 2.2.5, X is locally semi-compact. According to Theorem 2.2.4, X is locally compact. □

Lemma 2.2.7. *If $f : X \to Y$ is a pre-semi-open continuous bijection and Y is locally semi-compact, then X is locally semi-compact.*

Proof. Suppose x in X and $y = f(x)$. Since Y is a locally semi-compact space, it follows that there exists an open semi-compact neighborhood A such that $y \in A$. Since f is a continuous bijection, it follows that $f^{-1}(A)$ is an neighborhood of x. Suppose $\{B_\alpha : \alpha \in I\}$ is a semi-open cover of $f^{-1}(A)$. Since f is a pre-semi-open mapping, it follows that $\{f(B_\alpha) : \alpha \in I\}$ is a semi-open cover of A. Since A is semi-compact, it follows that there exists a finite subcover $\{f(B_1), f(B_2), \ldots, f(B_r)\}$ of A. Thus, there