HALF-DYON SOLUTIONS IN SU(2) YANG-MILLS-HIGGS THEORY AND WEINBERG-SALAM MODEL

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HALF-DYON SOLUTIONS IN SU(2) YANG-MILLS-HIGGS THEORY AND WEINBERG-SALAM MODEL

by

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ABSTRAK

Monokutub magnet—zarah hipotetikal dengan kutub magnet tunggal—mencabar teori elektromagnetik dengan mencadangkan simetri elektrik-magnetisme. Mula-mula diintegrasikan ke dalam mekanik kuantum oleh Dirac dan kemudian disokong oleh 't Hooft dan Polyakov dalam teori penyatuan agung, ia berkembang dengan monokutub elektrolemah Cho dan Maison yang berpotensi dihasilkan dalam pemelanggaran moden kerana jisimnya dalam julat TeV. Kajian kami mengembangkan penyelidikan terdahulu dengan mengkaji cas topologi separuh integer untuk $n \geq 1$ dengan parameter cas elektrik η, memperkenalkan jenis baru separuh-dyon yang membawa cas magnet negatif. Kami mengkaji penyelesaian separuh-dyon simetri paksi teritlak dalam teori SU(2) Yang-Mills-Higgs (YMH) dengan dua jenis berbeza: Jenis I (cas magnet positif sepanjang paksi-z negatif) dan Jenis II (cas magnet negatif sepanjang paksi-z positif). Kami menganalisis sifat-sifat konfigurasi ini—jumlah tenaga, cas elektrik, dan momen dwikutub magnet—sambil mengubah nombor lilitan-ϕ n, parameter cas elektrik η, dan pemalar gandingan-diri Higgs β. Kami membandingkan konfigurasi separuh-monokutub dual kami ($n = 2$) dengan monokutub 't Hooft-Polyakov. Penyelidikan kami meluas kepada penyelesaian separuh-dyon dalam teori SU(2)×U(1) Weinberg-Salam, membina separuh-dyon simetri paksi kedua-dua jenis serupa dengan yang terdapat dalam teori SU(2) YMH. Konfigurasi ini memiliki separuh cas magnet dari monokutub Cho-Maison. Kami menganalisisnya pada pemalar gandingan-diri Higgs fizikal $\beta = 0.7782$, memeriksa kedua-dua cas neutral "magnet" dan "elektrik" bersama dengan sifat-sifat lain. Tenaga yang berkaitan dengan konfigurasi separuh-dyon ini adalah tak terhingga disebabkan ketakseragaman pada lokasi separuh-dyon. Kami membandingkan monokutub Cho-Maison dengan separuh-monokutub dengan $n = 2$, $\eta = 0$. Kewujudan penyelesaian separuh-dyon Jenis I dan Jenis II memberikan pandangan tentang kemungkinan penghasilan pasangan monokutub Cho-Maison, meningkatkan pemahaman kita tentang struktur topologi dalam teori medan tolok dan memberikan perspektif baru mengenai monokutub magnet dalam Model Standard, membuka jalan untuk penyelidikan mengenai interaksi pasangan monokutub dan implikasi kosmologi.

HALF-DYON SOLUTIONS IN SU(2) YANG-MILLS-HIGGS THEORY AND WEINBERG-SALAM MODEL

ABSTRACT

Magnetic monopoles—hypothetical particles with single magnetic poles—challenge electromagnetic theories by suggesting electricity-magnetism symmetry. First integrated into quantum mechanics by Dirac and later supported by 't Hooft and Polyakov in grand unified theories, they evolved with Cho and Maison's electroweak monopole potentially producible in modern colliders due to its TeV-range mass. Our research expands previous work by examining half-integer topological charge for $n \geq 1$ with electric charge parameter η, introducing a new type of half-dyon carrying negative magnetic charge. We study generalized axially symmetric half-dyon solutions in SU(2) Yang-Mills-Higgs (YMH) theory with two distinct types: Type I (positive magnetic charge along negative z-axis) and Type II (negative magnetic charge along positive z-axis). We analyze these configurations' properties—total energy, electric charge, and magnetic dipole moment—while varying the ϕ-winding number n, electric charge parameter η, and Higgs self-coupling constant β. We compare our dual half-monopole ($n = 2$) configurations with the 't Hooft-Polyakov monopole. Our investigation extends to half-dyon solutions in SU(2)×U(1) Weinberg-Salam theory, constructing axially symmetric half-dyons of both types similar to those in SU(2) YMH theory. These configurations possess half the magnetic charge of the Cho-Maison monopole. We analyze them at the physical Higgs self-coupling constant $\beta = 0.7782$, examining both "magnetic" and "electric" neutral charges along with other properties. The energy associated with these half-dyon configurations is infinite due to singularity at the half-dyon location. We compare the Cho-Maison monopole to the half-monopole with $n = 2$,

$\eta = 0$. The existence of both Type I and Type II half-dyon solutions provides insight into possible pair production of Cho-Maison monopoles, enhancing our understanding of topological structures in gauge field theories and providing new perspectives on magnetic monopoles within the Standard Model, paving the way for investigations into monopole pair interactions and their cosmological implications.

CHAPTER 1

INTRODUCTION

1.1 Introduction to Magnetic Monopole

1.1.1 Theoretical Studies

A magnetic monopole is a hypothetical elementary particle in particle physics that is an isolated magnet with only one magnetic pole (a north pole without a south pole, or vice versa). The concept of magnetic monopoles has intrigued scientists for centuries, dating back to the early observations of magnetism where it was thought that magnetic poles could exist independently, akin to the positive and negative charges in electricity. The existence of such a particle would imply a symmetry between electricity and magnetism, significantly impacting our understanding of fundamental forces and field theories.

The idea of a magnetic monopole within the framework of Maxwell's equations in classical electrodynamics presents an intriguing theoretical anomaly. Maxwell's equations, which form the cornerstone of classical electrodynamics, describe the behavior of electric and magnetic fields and their interaction with matter. These equations are symmetric in many ways but notably asymmetrical in their treatment of electric and magnetic sources. Specifically, while electric charges are treated as sources or sinks of electric field lines, the standard formulation of Maxwell's equations posits that there are no analogous magnetic charges–hence, no magnetic monopoles, as indicated by Gauss's law for magnetism which states that the net magnetic flux through any closed surface is zero.

A compelling argument to the existence magnetic monopole was initially introduced by Dirac (1931), who demonstrated that quantum mechanics permits the existence of certain quantized magnetic charges. Moreover, the presence of these magnetic charges could elucidate the observed quantization of electric charge, meaning all electric charges must be integer multiples of a fundamental unit. This quantization of electric charge is indeed observed in nature, and no alternative explanation for this profound phenomenon had been proposed at the time.

Years later, another persuasive argument came to light. In 1974, 't Hooft and Polyakov showed that monopoles are a natural outcome of general concepts regarding the unification of fundamental forces. A popular belief among many particle physicists is that the strong and electroweak forces, characterized by three distinct gauge coupling constants, actually merge at very small distances into a single gauge interaction governed by one gauge coupling constant. 't Hooft and Polyakov demonstrated that any such grand unified theory in particle physics inherently includes magnetic monopoles.

While Dirac's work laid the groundwork by aligning the concept of magnetic monopoles with the principles of quantum electrodynamics, the work of 't Hooft and Polyakov went a step further. They did not just suggest the compatibility of monopoles with grand unified theories (GUTs); they asserted their inevitability within such frameworks. Moreover, they established that the characteristics of these monopoles are not just speculative but are instead concrete, calculable predictions within any given unified theory. This progression from theoretical possibility to a necessary element of grand unification theories marks a significant evolution in our understanding of magnetic monopoles.

Magnetic monopoles were not traditionally anticipated within the framework of the Standard Model, primarily because with the spontaneous symmetry breaking the quotient space SU(2) × U(1) / $U(1)_{em}$ has a trivial second homotopy. However, there has been ongoing debate regarding the potential existence of monopoles within the Standard Model. Cho and Maison (1997) argued that there is an alternative topological framework that supports the presence of monopoles in the Standard Model. They introduced the electroweak monopole, which generalized the Dirac monopole. This electroweak monopole is a hybrid of Dirac and 't Hooft-Polyakov monopoles, possessing a magnetic charge that is twice that of the Dirac monopole and its energy is singular. More recently, regularization technique was suggested by Kimm et al. (2015) aimed at estimating the total energy of such monopoles. The calculated mass of these electroweak monopoles falls within the TeV range, making it theoretically detectable by today's particle colliders.

In addition to GUTs, magnetic monopoles find a place within the field of supersymmetry and string theory (Duff et al., 1995; Seiberg & Witten, 1994), which represent more speculative yet ambitious attempts to unify the fundamental forces of nature, including gravity. Supersymmetry, which proposes a partner particle for every particle in the Standard Model, offers scenarios where magnetic monopoles could arise through new types of symmetry-breaking. String theory, with its premise that point-like particles are replaced by one-dimensional strings, provides a framework where monopoles can emerge in various guises, depending on the specific vibrational modes of the strings. The study of magnetic monopoles in these contexts not only deepens our understanding of the theoretical underpinnings of the universe but also bridges the gap between high-energy particle physics and cosmology, hinting at the profound interconnectedness of the microscopic and cosmic scales.

In the standard cosmological model, the predicted abundance of magnetic monopoles far exceeds the limits set by current observations. This discrepancy highlights a critical challenge: it suggests that either our understanding of the evolution of universe or some fundamental aspects of particle physics at very short distances may need revision. This issue has spurred significant advances in theoretical cosmology (Albrecht & Steinhardt, 1982; Guth, 1981; Linde, 1982a, 1982b). The discovery of a magnetic monopole would be a landmark event, rich with intriguing consequences. For instance, the existence of monopoles would confirm that the early universe reached extremely high temperatures. Additionally, it would impose strict limitations on cosmological modeling, as any viable model would need to account for the observed quantities of monopoles.

The exploration of magnetic monopoles continues to be an active area of research, not just in high energy physics, but also in condensed matter physics, where phenomena analogous to magnetic monopoles have been observed in specific materials (Castelnovo et al., 2008; Fang et al., 2003; Qi et al., 2009). These quasiparticles, known as flux tubes, exhibit properties reminiscent of theoretical magnetic monopoles, providing physicists with a laboratory-scale playground to explore monopole-like behavior. While these systems do not constitute the discovery of true magnetic monopoles, they offer valuable insights into the dynamics of magnetic fields and contribute to the broader understanding of magnetic phenomena. As research progresses, the pursuit of magnetic monopoles, whether in high-energy particle experiments or condensed matter systems, remains a tantalizing goal that could unlock new realms of physics and revolutionize our understanding of the fundamental forces that govern the universe.

1.2 Magnetic Monopoles Searches

Theoretically, magnetic monopoles could be produced in particle accelerator experiments, provided the collision energies surpass a threshold of $2Mc^2$. A Dirac magnetic monopole, considered a fundamental point-like particle, could possess any given spin or mass. However, the monopoles predicted by Grand Unified Theories (GUTs) (Polyakov, 1974; 't Hooft, 1974) are substantially different. These are not point-like but rather extended objects with a massive scale, typically around 10^{17} GeV, produced cosmologically through the Kibble mechanism (Kibble, 1976). Given their enormous mass, it is highly unlikely that such monopoles could be produced with the energies achievable by current or foreseeable particle accelerators. However, recent theoretical advancements in extensions of the Standard Model suggest the possibility of electroweak monopoles (Cho & Maison, 1997; Ellis et al., 2016; Kimm et al., 2015), which might exist at a lower mass scale in the TeV range.

Searches for magnetic monopoles have taken place at particle accelerators, through cosmic ray studies, and by looking for monopoles bound in matters. To date, no definitive detection of such charged exotic particles has been reported. While the constraints on the production cross sections at accelerators or on the flux limit of cosmic monopoles are made assuming that the monopole has purely magnetic charge, most of the experiments are also sensitive to dyon (particle that possesses both magnetic and electric charges).

1.2.1 Searches at Colliders

Various searches have been performed across different types of collider experiments, including those involving hadron-hadron (Aad et al., 2020, 2023; Acharya et al., 2019), electron-positron (Kinoshita et al., 1989; Musset et al., 1983), and lepton-hadron collisions (Aktas & others (H1), 2005). These searches can be categorized into direct and indirect methods. Direct searches aim to find physical evidence of the passage of monopoles through medium, such as traces similar to those left by charged particles. On the other hand, indirect searches assume that virtual monopole processes might affect the production rates of certain final states (De Rujula, 1995; Ginzburg & Schiller, 1999).

Investigations into monopoles generated at the peak energies achievable in hadron-hadron collisions, were conducted at the Large Hadron Collider (LHC) by the ATLAS (Aad et al., 2020, 2023) and MoEDAL (Acharya et al., 2019) experiments. These studies aimed to detect monopoles in proton-proton (pp) collisions. The ATLAS experiment sought out particles with high ionization capabilities that would leave distinct energy trails, while MoEDAL employed the induction technique to identify stopped monopoles. The findings from these experiments were able to extend the charge-dependent mass limits up to approximately 4 TeV (Workman et al., 2022). Specifically, MoEDAL explored the possibility of monopole-pair production through photon fusion, a method alongside the Drell-Yan mechanism, which is a standard approach for studying hadron-hadron collisions (Baines et al., 2018).

The absence of experimental evidence for monopoles might be explained by Dirac proposal (Dirac, 1948; Dirac, 1931), suggesting that monopoles do not exist freely but rather form a bound state known as monopolium (Epele et al., 2008; Hill, 1983), held together by intense magnetic forces. The monopolium, being a neutral entity, poses significant challenges for direct detection in collider experiments. Nevertheless, its potential decay into two photons could provide a distinct signature detectable by the ATLAS and CMS detectors.

1.2.2 Searches for Cosmic Monopoles

It is hypothesized that magnetic monopoles of cosmic origins, were created shortly after the Big Bang. The magnetic monopoles possibly formed due to topological defects that emerged as the Universe expanded and cooled down. The existing galactic magnetic field, with a strength of $\simeq 3$ μG, is thought to accelerate these monopoles, extracting energy from the magnetic field in the process. For the galactic magnetic field to be sustained, the energy it loses must not surpass the energy it gains. This requirement suggests there is a maximum allowable flux, known as the Parker bound (Turner et al., 1982)

$$\Phi \lesssim 10^{-15}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{sr}^{-1}. \tag{1.1}$$

The MACRO experiment, short for Monopole Astrophysics and Cosmic Ray Observatory, conducted the most sensitive hunt for cosmic supermassive magnetic monopoles. Positioned in the Gran Sasso underground facility in Italy, this large detector operated in various configurations from 1989 until December 2000. It was specifically designed to detect GUT magnetic monopoles moving at velocity $v/c \geq 4\times10^{-5}$, where v is the velocity of the monopole and c is the speed of light. However, its underground placement meant it was ineffective at detecting lower-energy monopoles. Throughout its operational period, MACRO was able to establish an upper limit for the monopole flux $\sim 10^{-16}$ cm^{-2} s^{-1} sr^{-1} (Ambrosio et al., 2002) that fell significantly below the Parker bound for nearly the entire velocity range of GUT monopoles.

1.2.3 Monopoles Bound in Matter

Efforts to detect monopoles have included searching within various bulk materials that are believed to have captured cosmic ray monopoles over extensive periods, potentially spanning millions of years. In recent decades, the induction technique has predominantly been used to identify monopoles trapped within substances like lunar rocks (Eberhard et al., 1971; Ross et al., 1973), meteorites (Jeon & Longo, 1995; Kovalik & Kirschvink, 1986), seawater (Kovalik & Kirschvink, 1986), iron ores (Ebisu & Watanabe, 1987), and ferromanganese modules (Kovalik & Kirschvink, 1986). This involves passing these material samples through superconducting coils. Using this approach, a rigorous upper limit has been established for the monopoles per nucleon ratio at $\sim 10^{-29}$ (Jeon & Longo, 1995; Kovalik & Kirschvink, 1986).

In a different experimental approach, researchers utilized several ancient (4.6×10^8 yr) mica samples. These samples underwent a chemical etching process and were then examined under a polarizing microscope using transmitted light. The goal was to identify etch pits that could have been created by elastic collisions between a monopole and a nucleus. The lack of any detected monopole tracks within the mica samples has led to the establishment of an upper limit of 10^{-17} to 10^{-16} $\mathrm{cm^{-2}\,s^{-1}\,sr^{-1}}$ on the flux of GUT monopoles having velocity in the range $3 \times 10^{-4} \lesssim v/c \lesssim 1.5 \times 10^{-3}$ (Price et al., 1984).

While experimental searches have thus far yielded no definitive evidence of magnetic monopoles, theoretical work continues to evolve. Of particular interest is the development of models that predict monopoles at potentially observable energy scales. In this context, the electroweak monopole introduced by Cho and Maison (1997) presents an especially promising avenue for investigation, as its predicted mass falls within the TeV range potentially accessible by current particle colliders.

1.3 Problem Statement

The discovery by Cho and Maison (1997) that gauge theory of electroweak interactions could allow magnetic monopole solutions opened new avenues for theoretical exploration. Concurrently, investigations into monopole solutions in SU(2) Yang-Mills-Higgs theory have yielded important insights into the nature of these elusive particles. This dual theoretical approach provides complementary perspectives on monopole structures with half-integer topological charges. Currently, there is limited exploration of monopole solutions with half-integer topological magnetic charges in both SU(2) Yang-Mills-Higgs theory and the Weinberg-Salam model. The axially symmetric half-dyon exhibits a distinctive teardrop shape in both frameworks. Prior studies have focused on the case of $n = 1$ for half-monopoles in these models, with limited attention to the role of electric charge or the existence of negative magnetic charge configurations. We expand this research by first exploring generalized half-integer dyon solutions in SU(2) Yang-Mills-Higgs theory, considering $n \geq 1$ and incorporating an electric charge parameter η. Building upon these results, we extend our investigation to the Weinberg-Salam model, constructing both Type I solutions (possessing positive magnetic charge along the negative z-axis) and Type II solutions (possessing negative magnetic charge along the positive z-axis). The existence of these complementary solutions provides a complete picture of potential pair production of Cho-Maison monopoles.

1.4 Objectives

- To construct generalized half-integer dyon solutions in SU(2) Yang-Mills-Higgs theory with ϕ-winding number n greater than or equal to one
- To construct generalized half-integer dyon solutions in SU(2) $\times$ U(1) Weinberg-Salam theory with ϕ-winding number n greater than or equal to one
- To investigate physical properties of the half-dyon solutions constructed, including total energy, Higgs modulus, magnetic charge, electric charge, magnetic dipole moment, and the relationships between these properties under varying parameters

1.5 Organization of Thesis

This dissertation is structured into five interconnected chapters that progressively develop the research narrative. Chapter 1 establishes foundational concepts of magnetic monopoles, provides a historical review of search efforts, defines the problem statement, and articulates the research objectives. In Chapter 2, a comprehensive literature review examines various magnetic monopole theories, including Dirac monopoles, Wu-Yang monopoles, 't Hooft-Polyakov monopoles, and Cho-Maison monopoles, thereby establishing the theoretical framework underpinning this research. The methodological approach is detailed in Chapter 3, outlining the numerical techniques and mathematical constructions employed to develop half-dyon solutions in both SU(2) Yang-Mills-Higgs theory and Weinberg-Salam theory. Chapter 4 presents results of numerical simulations, where properties of generalized half-dyon solutions across both theoretical frameworks are analyzed, with particular attention to energy characteristics, magnetic charge distribution, and electric properties. The dissertation concludes with Chapter 5, offering a synthesis of key findings, discussion of their implications for magnetic monopole theory, and suggestions for future research directions.

CHAPTER 2

LITERATURE REVIEW

2.1 Theory of Dirac Monopole

2.1.1 Introduction

Maxwell's equations completely describe how electric and magnetic fields interact and behave in classical electrodynamics. In vacuum, where there is no electric or magnetic sources, they read

$$\nabla \cdot \mathbf{E} = 0, \tag{2.1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0, \tag{2.4}$$

where vector fields **E** and **B** are electric and magnetic fields, respectively. In the absence of sources, Maxwell's equations exhibit invariance under the following duality transformation,

$$\mathbf{E} \to \mathbf{B}, \quad \mathbf{B} \to -\mathbf{E}, \tag{2.5}$$

which interchanges electricity with magnetism but maintaining the underlying physics. However, this elegant symmetry breaks down in the presence of sources, as the equations traditionally only incorporate electric charges ρ_e and currents $\mathbf{j}_e$. The equations (2.1) and (2.2) have sources while equations (2.3) and (2.4) do not. This discrepancy suggests at the theoretical potential for the existence of objects with magnetic charge, which, through their own unique charge densities and currents, could restore the symmetry. By introducing magnetic monopoles, Maxwell's equations can be generalized to include magnetic charge densities ρ_m and currents $\mathbf{j}_m$:

$$\nabla \cdot \mathbf{E} = \rho_e, \tag{2.6}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{j}_e, \tag{2.7}$$

$$\nabla \cdot \mathbf{B} = \rho_m, \tag{2.8}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_m. \tag{2.9}$$

Then the modified Maxwell's equations (2.6)–(2.9) maintain their invariance under the transformation (2.5) if the source terms are transformed in a similar way:

$$\rho_e \longrightarrow \rho_m, \;\; \mathbf{j}_e \longrightarrow \mathbf{j}_m, \tag{2.10}$$

$$\rho_m \longrightarrow -\rho_e, \;\; \mathbf{j}_m \longrightarrow -\mathbf{j}_e. \tag{2.11}$$

Thus, with the introduction of new magnetic source terms, the symmetry of Maxwell's equations as in the vacuum case is again achieved.

In quantum physics, the possible existence of magnetic monopoles presents a more complex and fascinating challenge. This is because in quantum physics, electromagnetic interactions have to be expressed through scalar ϕ and vector $\mathbf{A}$ potentials (Aharonov & Bohm, 1959), rather than through the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields. These potentials serve as position-dependent fields within the theory, with their variations result in electric and magnetic fields according to the relations

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \;\; \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.12}$$

While in classical electrodynamics these potentials serve as useful tools for calculations, with the scalar potential essentially representing the familiar electric potential, their application is not mandatory, and calculations can also be effectively done using electric and magnetic fields.

The introduction of scalar and vector potentials through equation (2.12) destroys the duality symmetry between **E** and **B**. Nevertheless, these potentials are not direct physical observables but represent the same physical phenomena as described by the original Maxwell's equations (2.1)–(2.4). Interestingly, there exists an infinite number of different potentials that lead to identical electric and magnetic fields, because the fields are unaffected if one transforms ϕ and **A** accordingly,

$$\phi \to \phi - \frac{\partial \lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla \lambda, \tag{2.13}$$

where λ is any arbitrary function. This process of shifting between equivalent potential configurations via equation (2.13) is termed a gauge transformation, and since physical quantities remain unchanged under such transformations, the theory is said to have a gauge symmetry. Maxwell's equations exhibit a gauge symmetry known mathematically as U(1). This concept of gauge symmetry, albeit in more complex forms, is central to present particle physics theories, guiding our understanding of elementary particle interactions.

However, the formalism surrounding these potentials Eq. (2.12) seem to preclude the existence of magnetic charges. This stems from a basic result in vector analysis, which asserts that the divergence of the curl of vector field is always zero, so that one obtains

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{2.14}$$

Consequently, this apparently implies that magnetic monopoles cannot be described using the vector potential. This limitation becomes significant in quantum mechanics, when characterizing an electrically charged particle, the complex phase of the wavefunction is influenced by the vector potential **A**.

2.1.2 Dirac Monopole

Dirac (1931) revisited the concept of magnetic monopoles within the framework of quantum mechanics. He demonstrated that the quantum mechanics governing an electrically charged particle could be coherently extended to include a point magnetic charge, given that the magnetic charge follows a specific condition. Dirac conceptualized a magnetic monopole as a semi-infinitely long, extremely thin solenoid. The end of the solenoid could be perceived as a magnetic charge, and it would only be meaningful to classify it as a magnetic monopole if it were impossible for any experimental approach to detect this infinitesimally thin solenoid. For instance, an attempt to detect the solenoid through an electron interference experiment would only result in no observable effect if the electron wavefunction has trivial phase change when moving around a closed path that encloses the solenoid.

Suppose a point monopole with a magnetic charge g positioned at the origin, generating a static, Coulomb-like magnetic field

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3}. \tag{2.15}$$

Away from the origin, all the conventional vacuum Maxwell's equations hold true, but there is a delta-function source for the magnetic field

$$\nabla \cdot \mathbf{B} = 4\pi g \delta^3(\mathbf{x}), \tag{2.16}$$

and consequently the total magnetic flux passing through any closed surface S that encloses the origin is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = 4\pi g. \tag{2.17}$$

The presence of magnetic charges affects the divergenceless nature of **B** Eq. (2.14), rendering it impossible to represent **B** globally as a curl. Nonetheless, if magnetic charges were point-like or confined within a finite space, it might be feasible to express **B** as a curl in areas where magnetic charges were absent. Dirac's argument is fundamentally mathematical in nature, with quantum mechanics playing a secondary role. He insisted that there must be a vector potential for the magnetic field, despite it inevitably having a singularity at the origin. This singularity is inconsequential because the vector potential, not being a measurable physical quantity, is not required to be smooth or globally well defined.

In spherical coordinates (r, θ, φ), the Dirac potential is

$$\mathbf{A}^{\mathbf{I}} = g \frac{(1 - \cos\theta)}{r \sin\theta} \hat{\mathbf{e}}_\varphi, \tag{2.18}$$

where $\hat{\mathbf{e}}_\varphi = -\hat{\mathbf{e}}_x \sin\varphi + \hat{\mathbf{e}}_y \cos\varphi$, or written in covariant form

$$\mathbf{A} = \frac{g}{r} \frac{\mathbf{r} \times \mathbf{n}}{r - (\mathbf{r} \cdot \mathbf{n})}, \tag{2.19}$$

where the unit vector $\mathbf{n}$ points along the z-axis: $\mathbf{n} = (0, 0, 1)$. The potential Eq. (2.18) has a singularity along the negative z-axis ($\theta = \pi$), known as the Dirac string. Ignoring the singularity, the magnetic field Eq. (2.15) can be obtained by taking the curl of $\mathbf{A}^{\mathbf{I}}$. We can consider a second appropriate potential which gives the same $\mathbf{B}$, given as

$$\mathbf{A}^{\mathbf{II}} = -g\frac{(1+\cos\theta)}{r\sin\theta}\hat{\mathbf{e}}_{\varphi}. \tag{2.20}$$

The Dirac string of this potential is along the positive z-axis ($\theta = 0$). In fact, through a suitable yet singular gauge transformation, the Dirac string can be aligned with any chosen curve emanating from the origin to infinity. However, it is important to note that this string cannot be completely eliminated through gauge transformation.

The situation is more sophisticated in the quantum theory, where the vector potential can have effect on the phase of a wavefunction (Aharonov & Bohm, 1959). Specifically, suppose that an electrically charged particle with charge q traverses a closed path C that encircles a space with a non-zero magnetic flux. Upon completing the loop, wavefunction of the particle would acquire a gauge-invariant Aharonov-Bohm phase factor

$$U[C] = \exp\left(iq\int_C d\ell\cdot\mathbf{A}\right). \tag{2.21}$$

This makes the particle sensitive to the presence of magnetic field, even if the wavefunction is zero in regions where the $\mathbf{B}$ is non-zero.

Dirac (1931) highlighted the profound implications that even a single magnetic monopole might have due to this phenomenon. For instance, if we consider a magnetic monopole with its associated Dirac string aligned along the negative z-axis, the location of the spring, being arbitrary and dependent on the gauge choice, should theoretically be undetectable. However, the Aharonov-Bohm effect seems to offer a way to detect the string. Imagine moving a electrically charged particle of charge q without magnetic charge around a small closed loop C that surrounds the string. The wave function of the particle would then be multiplied by the phase factor in equation (2.21). Absence of the string would result in the integral around the loop equating to zero, giving $U = 1$, in the limit in which the loop was shrunk to a point. To ensure the string is unobservable, the same result must be obtained when C is contracted to an infinitesimal loop around the string.

We can then evaluate the line integral in spherical coordinates by setting the loop C to be at a constant value of θ and applying equation (2.18), we get

$$q \int_C d\ell \cdot \mathbf{A} = q \int r \sin\theta d\phi A_\phi = 2\pi q g(\cos\theta - 1). \tag{2.22}$$

As C tightens into an infinitesimal curve around the string, with $\theta \to \pi$, the line integral becomes $-4\pi q g$. Incorporating this into equation (2.21), for the Dirac string to be invisible we must demand that

$$e^{4\pi i q g} = 1, \tag{2.23}$$

or, equivalently, that

$$qg = \frac{n}{2}, \tag{2.24}$$

where n is an arbitrary integer. This requirement, known as the Dirac quantization condition, has extensive implications. Moreover, similar requirement would arise from considering loops around the Dirac strings of any other magnetic monopole, implying that the quantization condition must hold for all magnetic charges. The quantization condition remains valid even when q represents the electric charge of a particle whose charge is not minimal but is instead an integer multiple of the minimal charge unit. Therefore the existence of even a single monopole anywhere in the universe could provide a compelling explanation for the observed quantization of electric charge.

The Dirac monopole cannot be classified as a topological soliton due to its singular characteristics at the point where $r = 0$. It seemingly possesses an infinite mass, as the energy density in the magnetic field is proportional to $1/r^4$. This relationship leads to a linear divergence in total energy upon integration over three-dimensional space $\mathbb{R}$ as $r \to 0$. Additionally, formulating a quantum theory for the dynamics involving Dirac monopoles and electrically charged particles presents significant challenges (Zwanziger, 1982). A comprehensive quantum field theory that accounts for Dirac monopoles, including phenomena like the creation of monopole-antimonopole pairs, is lacking. While such issues might be theoretically addressable within frameworks that treat monopoles as topological solitons, practical calculations and predictions within these models remain difficult.

2.1.3 Dyons

So far our discussions have primarily involved particles characterized by either electric or magnetic charge, not both. Dirac (1948) was uncertain about the possibility of such dual-charged particles existing. Schwinger (1969) introduced the concept of dyons, which are hypothetical particles that have both electric and magnetic charges.

Imagining a dyon in the above scenario, it would possess its own Dirac string, which would intertwine with the monopole's string when the dyon moves around the path C. If we consider a stationary monopole bearing only magnetic charge, this interaction would not alter the outcome. Yet, there will be an additional phase factor when both interacting particles are dyons. We can work out the result directly from a symmetry argument as the quantization condition must remain consistent under the duality transformation Eq. (2.5). Moreover, it must agree with our earlier findings Eq. (2.24) when one of the particles has only magnetic charge. The only formulation that satisfies these requirements is (Schwinger, 1968; Zwanziger, 1968)

$$q_1 g_2 - q_2 g_1 = \frac{n}{2}. \tag{2.25}$$

2.2 Wu-Yang Monopole

It may seem troubling that the vector potential describing a Dirac monopole has a string singularity along which the magnetic field is infinite, despite arguments that the string is imperceptible. Wu and Yang (1975, 1976) offered a sophisticated solution to this dilemma by introducing a point-like magnetic monopole within the SU(2) Yang-Mills gauge theory, particularly without the Dirac strings. Their approach is particularly captivating because it delves into the core principles of field theory and unveils a novel perspective on how topology connects with physics. This elegant topological approach, expressed in terms of differential geometry, accurately reflects the underlying physical context.

To construct their stringless model, Wu and Yang took advantage of the inherent ambiguity in the direction of the Dirac string, which is essentially arbitrary due to gauge invariance. In this logic, the Dirac string singularity can be avoided by parametrizing the three-space enclosing the monople, excluding the origin, $\mathbb{R}^3/\{0\}$, with two slightly overlapping hemispheres, termed the north R^N and south R^S hemispheres. The overlap region $R^N \cap R^S$ is identified as the "equator". The electromagnetic potentials are then properly defined and regular across both hemispheres and their overlapping region, where they are connected through non-Abelian gauge transformations. There still exists a singularity at the origin, but this is physically meaningful which corresponds to a singular, point-like particle that acts as the magnetic field source. Hence, we can write the two potentials that do not contain singularity everywhere in their defined domains as

$$\begin{cases} \mathbf{A}^N = g\dfrac{1-\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi, & 0 \le \theta < \dfrac{\pi}{2}+\Delta, \\ \mathbf{A}^S = -g\dfrac{1+\cos\theta}{r\sin\theta}\hat{\mathbf{e}}_\varphi, & \dfrac{\pi}{2}-\Delta < \theta \le \pi, \end{cases} \tag{2.26}$$

where Δ is an infinitesimal value. On the intersecting region $R^N \cap R^S$, both $\mathbf{A}^N$ and $\mathbf{A}^S$ describe the same magnetic field, so they must differ by a gauge transformation,

$$\mathbf{A}^N - \mathbf{A}^S = \frac{2g}{r\sin\theta} = \partial_\mu\Lambda, \tag{2.27}$$

where $\Lambda = 2g\phi$. Through this gauge transformation, the wavefunction of an electrically charged particle with charge q would transform according to

$$\psi \longrightarrow \psi' = e^{iq\Lambda}\psi = e^{2iqg\phi}\psi. \tag{2.28}$$

Imposing the requirement that the wavefunction stays single-valued, with $\psi'(r,\theta,\phi+2\pi) = \psi'(r,\theta,\phi)$, yields Eq. (2.23), hence we regain the Dirac quantization condition Eq. (2.24). This provides an alternate derivation to the Dirac quantization condition without involving the string singularity.

It is clear that this quantization condition is applicable to any vector potential on the sphere, not solely to the specific form mentioned in Eq. (2.26). Generally, if the vector potentials $\mathbf{A}^N$ and $\mathbf{A}^S$ on the upper and lower hemispheres differ only by a gauge transformation $\Omega(\phi)$ at the equator, where $\Omega(\phi) = \exp(i2qg\phi)$, then $\Omega(\phi)$ can be interpreted as an object that measures the total magnetic flux Φ through the sphere. If

$\Omega(\phi = 0) = 1$, then $\Omega(\phi = 2\pi)$ satisfies

$$\begin{aligned}\Omega(\phi = 2\pi) &= \exp[iq \oint dx\,(A^N - A^S)] = \exp[ie(\Phi^N + \Phi^S)] \\ &= \exp[iq(4\pi g)],\end{aligned} \tag{2.29}$$

where the line integral is taken along the equator, and g is the magnetic charge across the sphere. Requirement on $\Omega(\phi)$ to be single-valued again implies Eq. (2.24).

The integer n, representing the magnetic charge of the monopole in Dirac units, acts as a winding number. This is the number of times $\Omega(\phi)$ covers the $U(1)_{\text{em}}$ gauge group as the variable ϕ changes from 0 to 2π. This finding lays a topological foundation for the Dirac quantization condition, asserting that magnetic charge must be quantized since the winding number must be an integer.

Suppose if we now let the radius r of the sphere to vary, $\Omega(r, \phi)$ and the winding number n remain continuous functions of r as long as $\mathbf{A}^N$ and $\mathbf{A}^S$ are non-singular. Since n must be an integer, it remains constant regardless of r. If n is non-zero, it implies that the magnetic charge g is localized within an arbitrarily small sphere, indicating that the monopole manifests as a point singularity.

The non-Abelian Wu-Yang monopole presents a subtle issue in that its magnetic charge is not easily identifiable. As a solution within pure Yang-Mills gauge theory that lacks a Higgs field, it does not have a topological charge that can be clearly defined as a magnetic charge. This is in contrast to the 't Hooft-Polyakov monopole (Polyakov, 1974; 't Hooft, 1974), which we will discuss later. Another complexity, identified by Wu and Yang themselves (Wu & Yang, 1975), involves the non-satisfaction of the Bianchi identity for the electromagnetic field strength at the origin, which would indicate that the corresponding gauge field does not behave as a proper gauge field there.

2.3 Theory of 't Hooft-Polyakov Monopole

2.3.1 Introduction

In 1974, 't Hooft and Polyakov independently uncovered the significant finding that non-abelian gauge theories can possess magnetic monopole solutions without singularities. This was a pivotal moment, highlighting that unlike the Dirac monopole, which fits within an Abelian framework, certain non-Abelian frameworks, such as the Georgi-Glashow model (Georgi & Glashow, 1972), naturally include solutions resembling monopoles.

Historically, the strongest argument for the existence of monopoles was their potential to explain the quantization of electric charge, a concept initially rooted in Dirac's theories. However, contemporary perspectives consider the electric charge operator as a generator of the U(1) group, and charge quantization is seen as evidence for unified models where the electromagnetic subgroup is part of a larger, semi-simple non-Abelian gauge group. Here, nontrivial commutation relations are formed between the electric charge generators and all other generators of the gauge theory. Hence, the quantization of electric charge is now viewed as a supporting argument for the unified theory approach. Interestingly, both traditional and modern interpretations of charge quantization ultimately address the same fundamental issue. It has been recognized that any unification model embedding an electromagnetic U(1) subgroup into a semi-simple gauge group, which is subsequently broken down spontaneously through the Higgs mechanism, necessarily contain monopole solutions (Preskill, 1984).

The Derrick theorem states that pure Yang-Mills equations (C. N. Yang & Mills, 1954) lack topological soliton solutions in three-dimensional space (Manton & Sutcliffe, 2004). Yet, when Yang-Mills fields are coupled to Higgs scalar fields, monopole solutions that are topologically stable and have finite energy become possible. These solutions, complex at their core but with long-range electromagnetic fields identical to a Dirac monopole, can be seen as Dirac monopoles embedded in the Yang-Mills-Higgs theory, with the singularity effectively resolved. Their stability is ensured by the topological nature of the magnetic charge, and remains invariant under any smooth deformation of the field.

The Yang-Mills-Higgs theory can be formulated using any compact Lie group G as the gauge group, with the Higgs field transforming under any finite-dimensional representation of G. In this context, we will consider the Higgs field transforms via the adjoint representation of G, meaning it is valued in the Lie algebra of G and transforms through conjugation. The following discussion will focus on the Georgi-Glashow model or Yang-Mills-Higgs (YMH) theory with the SU(2) gauge group.

2.3.2 Georgi-Glashow Model: An Introduction

The Lagrangian of the Georgi-Glashow model is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - V(\Phi). \tag{2.30}$$