A NEW GENERALIZED TRIGONOMETRIC BERNSTEIN-LIKE BASIS FUNCTIONS AND ITS APPLICATIONS IN CURVE AND SURFACE CONSTRUCTIONS

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by

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In loving memory of Mrs. Nasira Javed,

— my aunt, my first teacher, and my eternal inspiration. For her unwavering love, boundless courage, and the dreams she planted in my soul. With love and gratitude.

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LIST OF ABBREVIATIONS

GT-Bernstein	Generalized Trigonometric-Bernstein
GT-Bézier	Generalized Trigonometric-Bézier
CAGD	Computer-Aided Geometric Design
CAD	Computer-Aided Design
CAM	Computer-Aided Manufacturing
CG	Computer Graphics
PSO	Particle Swarm Optimization

LIST OF SYMBOLS

ξ	xi (curve/surface parameter)
ν	nu (curve/surface parameter)
t	parameter
γ	design variable
b	curve control point
P	surface control point
R	control plane
C^0	point parametric continuity
C^1	tangent parametric continuity
C^2	curvature parametric continuity
C^3	rate of change of curvature parametric continuity
G^0	point geometric continuity
G^1	tangent geometric continuity
G^2	curvature geometric continuity
G^3	rate of change of curvature geometric continuity
$\mathcal{A}(\vec{S})$	surface area
$\mathfrak{D}$	Dirichlet functional
D	surface domain
η	scaling factor

ψ	scaling factor
$\kappa(\xi)$	curvature
$\kappa'(\xi)$	derivative of curvature
$\times$	cross product
$\cdot$	dot product
$\|$	parallel

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FUNGSI ASAS TRIGONOMETRI BAK-BERNSTEIN TERITLAK BAHARU DAN APLIKASINYA DALAM PEMBINAAN LENGKUNG DAN PERMUKAAN

ABSTRAK

Kajian ini membentangkan kaedah baharu untuk menjana lengkung dan permukaan menggunakan fungsi asas bak GT-Bernstein dengan dua pemboleh ubah reka bentuk. Lengkung dan permukaan yang dicadangkan ini meningkatkan keupayaan pelarasan bentuk dengan ketara berbanding dengan bentuk tradisional. Penyelidikan ini meneroka aplikasi lengkung dan permukaan, membolehkan penghasilan permukaan boleh laras dengan kawalan setempat, seperti permukaan tersapu, permukaan berayun, permukaan berputar, permukaan tergaris, permukaan rangkup dan permukaan spina yang boleh berkembang. Analisis menyeluruh terhadap parameter yang berbeza dengan mengkaji cara ia mempengaruhi bentuk lengkung dan permukaan ini, menjurus kepada pengenalpastian parameter teroptimum bagi pengoptimuman reka bentuk. Contoh-contoh berangka mempamerkan kefleksibelan dan kawalan bentuk setempat telah dicapai melalui kaedah yang dicadangkan. Tambahan pula, kajian ini memajukan kaedah baharu untuk menghasilkan permukaan dengan sempadan yang ditetapkan dalam reka bentuk geometri berbantu komputer (RGBK), dengan tumpuan khusus untuk meminimumkan luas permukaan. Algoritma pengoptimuman swarm zarah (PSZ) digunakan untuk menyelesaikan masalah Dirichlet, secara efektif mencari permukaan minimum yang dibatasi oleh kontur yang diberikan. Penyelidikan selanjutnya meneroka aplikasi praktikal permukaan minimum dengan sempadan bebas menggunakan fungsian Dirichlet untuk permukaan-permukaan TB-Coon. Contoh-contoh, termasuk permukaan boleh laras, permukaan boleh dibangunkan, dan permukaan minimum dengan batas bebas, menunjukkan keberkesanan dan aplikasi meluas kaedah ini dalam reka bentuk industri dan kejuruteraan.

A NEW GENERALIZED TRIGONOMETRIC BERNSTEIN-LIKE BASIS FUNCTIONS AND ITS APPLICATIONS IN CURVE AND SURFACE CONSTRUCTIONS

ABSTRACT

This study presents a novel methodology for constructing curves and surfaces using the GT-Bernstein-like basis function with two design variables. The proposed curves and surfaces substantially enhance shape-adjustment capabilities compared to traditional forms. The research explores the application of these curves and surfaces, enabling the creation of shape-adjustable surfaces with local control, such as swept surfaces, swung surfaces, rotation surfaces, ruled surfaces, enveloping surfaces, and spine curves of developable surfaces. A thorough analysis of different parameters shows the influence of the shape of these curves and surfaces, leading to the identification of optimized parameters for shape optimization design. Numerical examples showcase the flexibility and local shape control achieved through the proposed method. Additionally, the study advances a new method for generating surfaces with prescribed boundaries in computer-aided geometric design (CAGD), with a specific focus on minimizing surface area. The particle swarm optimization Algorithm (PSO) is employed to solve the Dirichlet problem, effectively finding minimal surfaces bounded by given contours. The study further explores the practical applications of minimal surfaces with free boundaries using the Dirichlet functional for TB-Coon surfaces. Examples, including adjustable surfaces, developable surfaces, and minimal surfaces by determining the optimized parameters with free boundaries, demonstrate the method's effectiveness and broad applicability in industrial design and engineering.

CHAPTER 1

INTRODUCTION

In early human history, objects were crafted without the use of mathematics, except perhaps unintentionally. As human civilization advanced, design techniques also evolved, resulting in more optimal shapes in the contemporary period (Burry & Burry, 2010). For instance, ships were built without the use of drafting techniques involving conic sections before the Renaissance. The invention of airplanes in the early 20th century brought about new design techniques that surpassed conventional drafting methods (Raymer, 2012). With the need for mass production, novel design techniques emerged, leading to the utilization of solid materials such as wood or steel to shape three-dimensional forms in the late 1950s. Nevertheless, the absence of sufficient software posed a bottleneck in production, giving rise to the emergence of a new field named computer-aided geometric design (CAGD), which addresses the design aspect of objects (Groover & Zimmers, 1983; Farin & Hansford, 2000).

Shape modeling often involves drawing lines and curves, which can be easy to do on paper but difficult to translate into a mathematical function that a computer can understand (Mortenson, 1997). As a result, there has been a demand for simpler and more efficient methods for designing and editing shapes in the computer age (Su & Liu, 1989).

To meet this demand, parametric surfaces have emerged as a valuable concept and offer the benefit of independence from an arbitrarily fixed coordinate system. This allows for greater flexibility and ease of use in shape modeling and editing (Farin, 2014). By using parametric surfaces, users can create complex shapes with less effort and in less time compared to traditional methods that rely on fixed coordinate systems (Sederberg, 1983).

CAGD experienced notable advancements when it pioneered the theory of Bézier curves and surfaces, which were subsequently integrated with B-spline techniques (Piegl & Tiller, 2012). These breakthroughs were the result of the independent work by De Casteljau and Pierre Bézier. Despite de Casteljau's earlier contributions, his work went unpublished, leading to the current association of the comprehensive understanding of polynomial type curves and surfaces in Bernstein form (Gordon & Riesenfeld, 1974).

Classical Bézier curves are widely used in CAGD due to their good properties, but they often need to be modified to meet design requirements. Creating more effective techniques for designing and modifying Bézier curves is a crucial research area in computer-aided design (CAD)/computer-aided manufacturing (CAM) and computer graphics (CG) technology. It is important to note that traditional Bézier curves have limitations when it comes to accurately representing conic shapes, including circles and ellipses. To overcome these limitations, trigonometric splines with design variables have garnered attention, particularly in modeling curve design (Han, 2002, 2004; Yan & Liang, 2011; Bashir et al., 2012).

1.1 Problem Statement

The following problem statements outline key challenges and limitations in CAD for surface modeling and design in engineering applications. Addressing these challenges is essential for improving the accuracy, smoothness, and flexibility of surface modeling and design, enabling designers to create complex and sophisticated surfaces that meet specific design requirements.

- Traditional Bernstein basis functions fall short as they are lack of design variables. As a result, an alternative method is required to enable the creation of Bézier-like curves with adjustable shapes.
- The traditional Bézier curve is limited in its ability to represent conic shapes, such as circles and ellipses. As a result, there is a need to develop new methods that can accurately represent such shapes.
- The lack of design variables in traditional Bernstein basis functions makes it impossible to construct shape-adjustable surfaces. Therefore, it is necessary to explore alternative approaches that incorporate design variables, in order to create surfaces that can be easily adjusted and developed to meet specific design requirements.
- The identification of the Bézier surface with the smallest area that fits within a given set of boundaries, known as the plateau-Bézier problem (Monterde, 2003), remains a significant challenge in CAD. Current methods are limited in their ability to accurately identify the minimal surface, highlighting the need for new approaches that can address this problem.

1.2 Research Objectives

The primary aim of this research is to explore the potential of a novel Generalized Trigonometric-Bernstein (GT-Bernstein)-like basis with two design variables within the domain of CAGD. This investigation focuses on utilizing Generalized Trigonometric-Bézier (GT-Bézier)-like curves and surfaces as a practical application. The study aims to achieve the following specific objectives:

1. To develop a novel nth-degree GT-Berstein-like basis function with two design variables for modeling complex curves and surfaces.
2. To derive the parametric (C^3) and geometric (G^3) continuity conditions for adjacent GT-Bézier-like curves and surfaces.
3. To improve nth-degree GT-Bézier-like surface smoothness, adjust shape flexibility, explore developable surfaces, and optimize surface construction.
4. To investigate the Plateau GT-Bézier problem for surface area minimization and develop a minimal area Trigonometric Bézier Coon (TB-Coon) surfaces with fixed boundaries.

1.3 Significance of the Study

The research presented in this study has significant implications in the field of CAGD for surface modeling and designing in engineering applications. The following points highlight the significance of this study based on the identified problem statements and research objectives:

- The proposed method for developing flexible GT-Bézier-like curves with shape adjustability has significant implications for CAGD applications, as it enables greater flexibility and control in modeling complex shapes and surfaces.
- By developing new methods that can accurately represent conic shapes using GT-Bézier-like curves, this study will contribute to the advancement of surface modeling and design techniques in engineering applications.
- The exploration of alternative approaches to constructing shape-adjustable surfaces, such as ruled surfaces, rotation surfaces, swept surfaces, developable surfaces, and swung GT-Bézier-like surfaces.
- The identification of the GT-Bézier-like surface with the smallest area that fits within a given set of boundaries, known as the plateau-GT-Bézier problem, is a significant challenge in CAGD. The proposed approach to address this problem will enhance the accuracy and smoothness of the resulting surfaces, leading to improved surface modeling and design outcomes.
- The proposed method for constructing Minimal Area TB-Coon surfaces has significant implications for surface modeling and design in engineering applications, as it enables accurate modeling of complex shapes by fixing boundaries with minimal surface area.

1.4 Thesis Outline

This thesis outline provides a logical flow of chapters, starting with an introduction of the topic, followed by a literature review, and then progressing through the specific contributions of each chapter. It concludes with a summary of the research findings and their significance.

In Chapter 1, the topic of geometric design and its significance in various industries are introduced. The limitations of traditional methods in terms of shape adjustability and design flexibility are also discussed, highlighting the need for a new approach. Additionally, this chapter presents the problem statement, research objectives, and the significance of the study.

Chapter 2 consists of a comprehensive literature review on geometric design, with a specific focus on Bézier curves and surfaces. The limitations and challenges associated with traditional methods will be discussed, and the existing body of literature will be explored. This chapter will set the foundation for the proposed new approach by identifying the gaps in current research.

Chapter 3 presents a novel method for constructing GT-Bézier-like curves using the GT-Bernstein-like basis function with two design variables. C^3 and G^3 continuity specifications for constructing composite curves and conics will be derived, showcasing the advantages of the proposed method over standard Bézier curves. The potential of the proposed method in shape modification of engineering designs will also be discussed.

In Chapter 4, the use of GT-Bézier-like curves with design variables in constructing shape-adjustable surfaces will be explained. The theory of constructing various GT-Bézier-like surfaces and their continuity conditions will be explored. Additionally, the impact of design variables on different types of surfaces will be examined, highlighting the increased design flexibility and efficiency offered by GT-Bézier-like surfaces with design variables.

Chapter 5 focuses on the construction of GT-Bézier-like developable surfaces with local control for enveloping and spine developable surfaces. The approach for constructing these surfaces will be presented, and the theoretical background and conditions for evaluating their properties will be discussed. The application of the Particle Swarm Optimization (PSO) method for obtaining optimal developable surfaces will be demonstrated, and continuity conditions between adjacent GT-Bézier-like surfaces for complex shape handling will be derived.

In Chapter 6, the Plateau-GT-Bézier problem and the Dirichlet functional approach for surface generation with prescribed boundaries will be introduced. An improved method that utilizes GT-Bernstein-like basis functions and particle swarm optimization will be proposed. Additionally, harmonic GT-Bézier-like surfaces and their relationship with minimal surfaces will be investigated. Lastly, the Dirichlet functional will be applied for constructing minimal TB-Coon surfaces.

Finally, in Chapter 7, the conclusion of the thesis will be provided. The key findings and contributions of the research will be summarized, discussing the implications of the proposed methodology in geometric design.

CHAPTER 2

LITERATURE REVIEW AND BACKGROUND

2.1 Literature Review

For many years, the field of computer graphics and geometric modeling has struggled to find suitable methods for representing free-form curves and surfaces that are both mathematically accurate and computationally efficient (Preparata & Shamos, 2012). In recent decades, however, significant progress has been made in this area with the development of novel mathematical representations designed to make complex shapes more compatible with computer systems (Hughes & Foley, 2014).

While there are many different approaches to representing curves and surfaces, one of the most widely used methods is parametric representation. Parametric representations of curves and surfaces have become popular due to their simplicity and efficiency, as they can be easily bounded in the parameter range and programmed. This makes them an attractive option for designers and engineers working in a range of industries, from product design to aerospace engineering (Farin, 2014).

However, as the demand for more complex and aesthetically pleasing shapes has increased, the limitations of parametric representations have become more apparent. To address this, Bézier curves and surfaces was introduced, making a significant impact on geometric modeling (Hansford, 2002)

2.1.1 Bézier Curves and Surfaces

Bézier curves and surfaces, unlike their typical parametric counterparts, are specified by a set of control points that control the shape of the curve or surface (Marsh, 2005). Although Bézier curves and surfaces find widespread usage among designers and engineers engaged in complex projects due to their flexibility and versatility, the global control property associated with them can impose certain limitations on their applicability in fields beyond design, such as engineering and automotive design (Piegl & Tiller, 2012). This is because changes made to the curve at one point can affect the entire shape, making localized modifications difficult to achieve.

To address this issue, researchers have developed alternative methods such as the Rational Bézier model (Farin, 1983), which is a robust tool for constructing free-form curves and surfaces. This model allows for the adjustment of curves through weight factors without changing their control points, but it also has several drawbacks such as producing curves of a very high degree due to repeated differentiation (Rogers, 2001).

To address the limitations of Bézier and Rational Bézier curves, researchers have made advancements by incorporating shape factors into the curve formulation, leading to improve precision and efficiency in modeling (Han et al., 2009; Liu et al., 2012; Li, 2013; Li & Zhao, 2013; Bashir et al., 2013a, 2013b; Li, 2016; Sharma, 2016; Misro et al., 2017a, 2017b; Xie & Li, 2018; Misro et al., 2019). These approaches employ parameters that do not modify or disrupt the underlying mathematical functions governing the curve. Consequently, these parameters have the ability to influence the original basis while maintaining the original position of the control points (Maqsood et al., 2020). The adaptability of the shape design variable enables more control over the positioning and generation of the curves and surfaces (Han et al., 2009).

To build upon the advancements in curve modeling, researchers have continued to explore novel approaches to enhance the capabilities of Bézier curves. One such area of focus has been the development of trigonometric Bézier curves that incorporate shape design variables and normalization adjustments. These advancements aim to further improve the precision, flexibility, and control over the shape of curves. Li (2013) and Li and Zhao (2013) introduced new trigonometric-type cubic Bézier curves capable of describing conic functions and envisioning enlargement estimations. Additionally, Usman et al. (2020) presented a new class of trigonometric cubic Bézier curves and their applications in approximating conic curves and font designing. Xie and Li (2018) suggested the use of a single design variable to determine the ellipse and parabola frameworks. Liu et al. (2012) provided a description of the B-spline C^3 curves of degree three, modifying the design variables of the curves to resemble the control points more convincingly. Furthermore, Sharma (2016) proposed a novel kind of fourth-degree trigonometric (T-Bézier) curves and surfaces through the weighting process.

In continuation of the evolution of shape-alterable surfaces, Hu et al. (2018b); Hu and Wu (2019); Hu et al. (2020a) presented a new kind of T-Bézier curve with many design variables and discussed the implications for surface modeling. Furthermore, several instances of curves and surfaces with varying degrees of local and global form characteristics have been shown and studied. Yang and Zeng (2009) introduced a novel variation of Bézier curves that incorporates n shape parameters, allowing for more flexible curve designs. They also extended this concept to triangular Bézier surfaces, which involve $3n(n+1)/2$ design variables.

The research conducted by Misro et al. (2017a) resulted in the development of the trigonometric Bézier curve. In addition, Misro et al. (2017b) illustrated spiral Bézier curves using the advantage of the associated design variables. Han et al. (2009) and Li (2016) proposed third-degree trigonometric curves characterized by two design variables. The open and closed quasi C^2 T-Bézier of fifth-degree and degree two G^2 rational curves are shown in Bashir et al. (2013a, 2013b). BiBi et al. (2019) proposed a hybrid type generalized T-Bézier basis function with design variables to create symmetric curves and surfaces. Similarly, Maqsood et al. (2020) introduced a geometrically defined generalized blended T-Bézier curve of degree m incorporating two design variables.

Moreover, Liu and Xu (2008) proposed a technique for modeling free-form surfaces and provided illustrative instances to examine the impact of shape factors on these surfaces. The study conducted by Hu et al. (2017b) focused on examining the continuity conditions that exist between generalized Bézier-like surfaces that include numerous shape parameters. Additionally, the authors discussed various characteristics and uses of smooth continuity by presenting modeling instances. Lasser (2002) introduced a novel approach that addresses the conversion of a rectangular surface from a T-Bézier to a conventional type of Bézier representation. Additionally, the authors delve into the corner issue associated with such surfaces.

In addition, the use of developable surfaces is a key aspect of geometric design, especially when it comes to the manufacture of substances such as automobile components and clothing (Frey & Bindschadler, 1993; Tang & Wang, 2005; Chu et al., 2008; Nguyen & Ko, 2014). Unlike other surfaces that need to be stretched or broken to fit a particular shape, developable surfaces can be expanded onto a plane without losing their structural integrity (Liu et al., 2006; Tang et al., 2016).

2.1.2 Bézier Developable Surfaces

Researchers have conducted numerous experiments in the field of developable surfaces, with a focus on their design and accurate description. These investigations primarily revolve around the construction and precise characterization of these surfaces (Weiss & Furtner, 1988). Quasi-developable surfaces have also been explored (Chen & Tang, 2013), along with fitting techniques (Peternell, 2004), geometrical restrictions (Zhao & Wang, 2008; Li et al., 2011), and smooth splicing methods (Chu & Chen, 2004). Numerical representation has facilitated the use of developable surfaces in architecture and engineering, which can be categorized into representing geometric points, lines, and planes (Aumann, 1991; Lang & Röschel, 1992; Bodduluri & Ravani, 1993a; Pottmann & Farin, 1995; Zhao & Wang, 2008).

For Bézier developable surfaces, Aumann (1991) established essential conditions for constructing a Bézier developable surface with two boundary curves constrained to parallel planes and derived a fundamental threshold. Lang and Röschel (1992) presented the general characterization, including weights and control nets, of the developable rational Bézier surface. However, the use of rational Bézier curves and surfaces introduces non-linearity in the defining equations.

An alternative approach to modeling developable surfaces is projective geometry, as explored by Pottmann and Farin (1995). By representing developable surfaces as projective space curves, this approach overcomes the non-linearity of characterizing equations. However, in the context of developable surfaces, Laguerre geometry emerges as a more suitable model. Laguerre geometry studies oriented circles, lines, spheres, and hyperplanes, utilizing bijective transformations that preserve oriented contact (Peternell & Pottmann, 1998; Bobenko et al., 2019).

In Laguerre geometry, a developable surface, which retains tangent planes along its rulings, can be transformed into a curve (Skopenkov et al., 2020). Furthermore, Laguerre geometry provides practical solutions to geometric architecture issues (Pottmann et al., 2009). However, computational challenges arise from the lack of unambiguous specifications about developable surfaces and the non-linearity of characteristic equations, which limits their applicability.

The concept of dual Bézier and B-spline interpretations for creating developable surfaces was introduced by Bodduluri and Ravani (1993a, 1994). These researchers focused on the efficient utilization of cubic Bézier and B-spline basis functions to construct developable surfaces. Building upon this work, Pottmann and Farin (1995) expanded the understanding and application of these surfaces in geometry by introducing the concept of rational developable Bézier and B-spline surfaces.

Notably, Zhou et al. (2006) made a significant contribution to the field of developable surfaces by constructing quartic developable Bézier surfaces using QT-Bernstein basis functions. Their geometric approach allowed for the creation of these surfaces with enhanced control and precision. In recent years, Hu et al. have presented a number of simple methodologies for developable surfaces, encompassing H-Bézier, Bézier-like developable, generalized quartic H-Bézier, and generalized C-Bézier surfaces (Hu et al., 2017c; Hu et al., 2018a; Hu & Wu, 2019; Hu et al., 2020b). These methodologies combine multiple form parameters and effectively handle G^2 continuity requirements, making them suitable for applications in geometric modeling.

Furthermore, in the pursuit of achieving developable surfaces of exceptional accuracy and precise degrees of developability, researchers have explored various bio-inspired optimization methods. These methods have shown significant promise in the field of surface modeling and design. Among them, Particle Swarm Optimization (PSO) has been notably effective. For instance, Cao et al. (2022) utilized PSO to generate quasi-developable Q-Bézier strips, optimizing shape parameters to fine-tune the developability of the surfaces. Additionally, BiBi et al. (2023) applied the PSO algorithm for the shape optimization of GHT-Bézier developable surfaces, further validating the versatility and efficiency of PSO in handling complex geometrical configurations.

The Sparrow Search Algorithm (SSA) has recently been adapted for the optimization of developable surfaces. Miao et al. (2021) utilized SSA for optimizing developable Bézier-like surfaces with multiple shape parameters. Their work demonstrates SSA's capability to effectively search and optimize large parameter spaces, which is crucial for achieving the desired geometric properties and developability of surfaces. Moreover, the Non-dominated Sorting Genetic Algorithm (NSGA-II) has also been employed to address multi-objective optimization challenges. Lu et al. (2023) explored this method for the shape optimization of developable Bézier-like surfaces, achieving a balanced trade-off between multiple conflicting objectives such as surface smoothness and developability.

Other bio-inspired algorithms offer complementary advantages in surface optimization. The Cuckoo Search algorithm, as applied by Hu et al. (2020a), has proven effective in shaping developable surfaces, showcasing the versatility of nature-inspired techniques in engineering applications. Further refinements in the field are evident through the work of Nadimi-Shahraki et al. (2021) and Hou et al. (2022), who enhanced the Grey Wolf Optimizer (I-GWO) to address issues such as population diversity and premature convergence. These improvements are vital for ensuring robust exploration and exploitation capabilities, essential for complex optimization tasks.

2.1.3 Minimal Surfaces

Minimal surfaces have captivated researchers in the field of geometric design due to their unique properties and diverse applications. Extensive research has been conducted on minimal surfaces, as evidenced by several published works (Nitsche, 1989; Johnstone & Sloan, 1995; Monterde, 2004; Kapfer et al., 2011).

These surfaces are not only fascinating from a mathematical perspective but also occur naturally, making them a subject of study for many years (Tråsdahl & Rønquist, 2011). Minimal surfaces have found practical use in surface design, architecture, material science, shipbuilding, biology, and related fields (Giusti & Williams, 1984; Colding & Minicozzi, 2011; Pan & Xu, 2011; Yoo, 2011). Moreover, there is ongoing speculation regarding the application of nanotechnology to explore the potential of minimal surfaces in areas such as molecular engineering and materials science (Wang, 2007).

One specific topic of interest within the realm of minimal surfaces is the Plateau problem, which has been extensively studied. The Plateau-Bézier problem is a modification of the Plateau problem in differential geometry, focusing on finding the minimal surface area that covers a given set of Bézier curves (Monterde, 2003; Hao et al., 2012).

While the Dirichlet functional is not a reliable lower bound for the area functional in the case of Bézier surfaces, Monterde (2003) demonstrated that Dirichlet extremals can still provide a reasonable approximation of the area functional's extremals. Thus, although the minimizer of the area functional for a Bézier surface may not precisely minimize the area, it should closely approach the minimum value.

Various approaches have been employed to tackle the Plateau-Bézier problem for tensor product Bézier surfaces. Previous methods have utilized the Dirichlet energy as a substitute for the area functional (Dziuk, 1990; Osserman, 2013). Another approach involves a multiresolution analysis using B-splines to achieve a parametric surface with minimal area (Hao et al., 2013). By solving a set of linear equations with a predominantly sparse structure, the proposed method obtains the solution surface.

Additionally, Polthier (2002) explored the Dirichlet energy associated with triangular Bézier surfaces. Moreover, Hao et al. (2012) investigated the smallest quasi-type Bézier surfaces in non-polynomial space using the Dirichlet technique and the harmonic approach, focusing on boundaries defined by catenaries and circular arcs.

In the field of constructing parametric polynomial minimal surfaces, several methods have been proposed for achieving surfaces of lower degrees. Xu et al. provide various methods in their works (Xu & Wang, 2008, 2010). They explore the construction of parametric polynomial minimal surfaces using quintic and parametric approaches.

Another area of research focuses on the development of approximate minimal surfaces constrained by geodesics. Li et al. (2022) conducted a study in this direction and proposed an approach for generating an approximate minimal surface constrained by geodesics. This research provides insights into incorporating geodesic constraints in the creation of minimal surfaces.

Additionally, Kassabov (2014) made a significant contribution by discovering a theory for constructing canonical fundamental factor minimal surface equations. This discovery opens up new possibilities for incorporating minimal surfaces into parametric functions.

Furthermore, Ahmad and Masud (2014) developed a variational method to improve the shape of a surface by minimizing the area bounded by a certain number of curves. Their method initiates a variational improvement process to decrease the area while maintaining the desired curve constraints.

2.2 Background

In this section, the fundamental concepts, preliminary definitions, and key insights involved in CAGD will be reviewed. References to some of the most significant findings pertaining to our specific topic are also included. The materials covered in this section are predominantly found in conventional literature such as applied geometry for computer graphics (Marsh, 2005), curves and surfaces for computer graphics (Salomon, 2007), a practical guide to splines (De Boor, 1978), and the differential geometry of curves and surfaces (Toponogov, 2006).

2.2.1 Bernstein Polynomials

The $(m+1)$ Bernstein Polynomials $\{B_{j,m}(\nu)\}_{j=0}^{m}$ of degree $m \in \mathbb{N}$ are defined as:

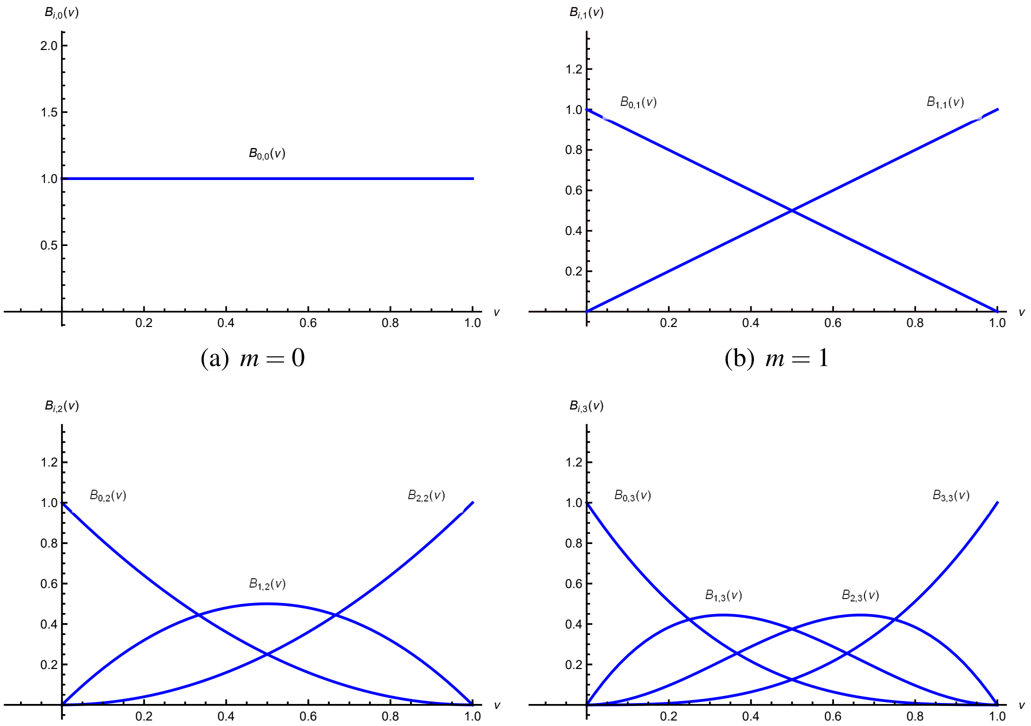
$$B_{j,m}(\nu) = \begin{cases} \frac{m!}{(m-j)!j!}\nu^{j}(1-\nu)^{m-j}, & \text{if } 0 \leq j \leq m, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\nu \in [0,1]$. The degree m of Bernstein polynomials serve as the foundation for power polynomials of degree m.

The Bernstein polynomials possess a variety of advantageous characteristics. A more in-depth description of these characteristics can be found in Farin (2014). Figure 2.1 offers a visual representation of the Bernstein polynomial basis.

There are a few distinguishing characteristics of Bernstein basis polynomials:

- $B_{j,m}(\nu) \geq 0$ for $\nu \in [0,1]$ and $j = 0, \ldots, m$ (Non-negativity)
- $\sum_{j=0}^{m} B_{j,m}(\nu) = 1$ (Partition of unity)
- $B_{j,m}(\nu) = B_{m-j,m}(1-\nu)$ (Symmetry)



(a) $m = 0$ (b) $m = 1$

(c) $m = 2$ (d) $m = 3$

Figure 2.1: Bernstein polynomial basis

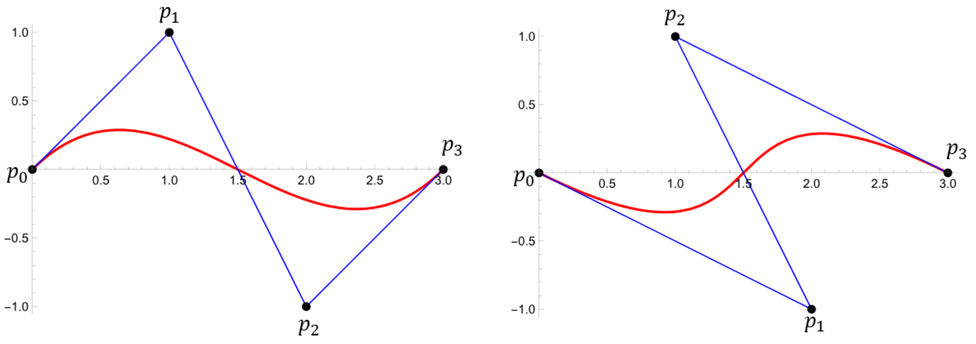
- $B_{j,m}(v) = (1-v)B_{j,m-1}(v) + vB_{j-1,m-1}(v)$, and $B_{0,0}(v) = 1$, $B_{j,m}(v) = 0$ for $j < 0$ or $j > m$ (Recursion)

2.2.2 Bézier Curves

The field of computer graphics has made great progress over the last two decades. This includes the development of dedicated 3D hardware, computer-generated liveliness, and faster PCs. Using computer graphics, real-life modeling can be shown in a computer model. A computer model makes the work much easier, faster, and more effective. It also makes it more successful (Eastman, 1975). In the beginning, the computer model is shown as a wire-frame model. In the end, the model is changed and rendered. It can also be used to make the computer model look like the original model by making it move. Computer graphics are used in a wide range of fields, including engineering, scientific visualization, image processing, design, and entertainment (Ganovelli et al., 2014).

One of the research development topics is concerned with the presentation of smooth curves and surfaces that are suited for modeling fascinating landscapes, faces, and other topologies, among other things. The usage of parametric curves, such as the Bézier, becomes apparent in this situation (De Boor, 1978). Bézier curves are a particular type of curve that are often used to create smooth lines and shapes in various applications. They are defined by a set of control points and incorporate mathematical principles to determine the shape and trajectory of the curve. A broad number of diverse applications have been developed for them, and establishing themselves as a standard tool in CAD, computer images, animation, and other closely related industries (Marsh, 2005).

A Bézier curve, denoted as $\mathbf{C}(v)$, with a degree of m, is determined by a series of control points spanning p_0 to p_m, where p_0 represents the starting point and p_m represents the endpoint. These control points determine the design and trajectory of the Bézier curve. While it is not a strict requirement for the intermediate control points to lie directly on the curve, they can still have an impact on the curve design. The influence exerted by these intermediate points depends on the degree of weighting assigned to them, whether it be a stronger or weaker influence on the curve's overall shape (Salomon, 2007). A designer may quickly and intuitively change the design of a curve by adjusting the intermediate points. The Bézier curve may be expressed in a



(a) (b)

Figure 2.2: Cubic Bézier curves and their associated control shapes

more generic way as follows:

$$\mathbf{C}(v) = \sum_{j=0}^{m} p_i B_{j,m}(v). \tag{2.2}$$

where $v \in [0,1]$ and $B_{j,m}(v)$ is a basis of Bézier curve. In addition to the Bernstein basis, the coefficient p_i is liable for determining the shape of the curves. A control polygon is created by connecting consecutive points with lines.

Before studying the characteristics of Bézier curves, let us look at a cubic Bézier curve in Figure 2.2(a) to see what these curves look like. The curve can be described by four control points, $p_0, \cdots, p_3$, with p_0 representing the starting point and p_3 representing the endpoint of the curve. The curve does not go through p_1 or p_2. Additionally, it should be noted that a curve of degree m has $(m+1)$ control points.

In Figure 2.2(b), the shift in the curve is evident when the control points p_1 and p_2 are swapped in their positions. One of the primary reasons Bézier curves are widely used in design is their ability to facilitate easy modification of a curve's curvature by adjusting the position of one or more control points. This flexibility empowers users to swiftly and intuitively alter the shape and curvature of the curve, providing a powerful tool for design and editing processes.

2.2.2(a) Properties of Bézier curves

To understand Bézier curves, it is necessary to understand their characteristics. This will provide the necessary framework for the discussion that will occur later on in this work. The characteristics of Bézier curves have been thoroughly summarised in Farin (2014), and the most significant of them are stated below.

- **Endpoint Interpolation:** $\mathbf{C}(0) = p_0$ and $\mathbf{C}(1) = p_m$.
- **Endpoints Tangent:** $\mathbf{C}^{'}(0) = m(p_1 - p_0)$ and $\mathbf{C}^{'}(1) = m(p_m - p_{m-1})$.
- **Convex Hull:** Every Bézier curve is within its control points' convex hull. The Bézier curve's convex hull is an acronym for the control points' convex hull.

$$\forall\, v \in [0,1],\ \ \mathbf{C}(v) \in \mathrm{CH}\{p_0, \ldots, p_m\}.$$

- **Symmetry:** The Bézier symmetric curves by the pair of control points $p_0, \ldots, p_m$ and $p_m, \ldots, p_0$ can be defined as:

$$\mathbf{C}(v; p_0, \cdots, p_m) = \mathbf{C}(1 - v; p_m, \cdots, p_0).$$

2.2.2(b) Derivatives of Bézier curves

Many curve-related tasks, such as finding tangents and normals, need the calculation of first and second derivatives. Because of the usage of Bernstein basis functions, calculating the derivatives at a given point on a Bézier curve is a rather simple process. As a result, Bézier curve derivatives are derived from Bernstein basis function derivatives. For example, to calculate the derivative of the Bernstein polynomial of degree 3,

$$B_{0,3}^{'}(v) = -3(1-v)^2 = -3B_{0,2}(v),$$

$$B_{1,3}^{'}(v) = 3(1-v)^2 - 6v(1-v) = 3(1-4v+3v^2) = 3B_{0,2}(v) - 3B_{1,2}(v),$$

$$B_{2,3}^{'}(v) = 3v(2-3v) = 3B_{1,2}(v) - 3B_{2,2}(v),$$

$$B_{3,3}^{'}(v) = 3v^2 = 3B_{2,2}(v).$$

So,

$$\begin{aligned}\mathbf{C}^{'}(v) &= -3p_0B_{0,2}(v) + 3p_1(B_{0,2}(v) - B_{1,2}(v)) + 3p_2(B_{1,2}(v) - B_{2,2}(v)) + 3p_3B_{2,2}(v)\\ &= 3(p_1-p_0)B_{0,2}(v) + 3(p_2-p_1)B_{1,2}(v) + 3(p_3-p_2)B_{2,2}(v)\\ &= \sum_{j=0}^{2} p_i^{(1)} B_{j,2}(v)\\ &= \sum_{j=0}^{m-1} p_i^{(1)} B_{j,m-1}(v)\end{aligned}$$

where $p_j^{(1)} = m(p_{j+1} - p_j)$.

However, kth derivative generalizations of aforementioned formula for degree m is written as:

$$\mathbf{C}^{(k)}(v) = \sum_{j=0}^{m-k} p_j^{(k)} B_{j,m-k}(v), \tag{2.3}$$

where $p_j^{(k)} = m(m-1)\cdots(m-k+1)\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}p_{i+j}$.

2.2.3 Bézier Surfaces

It is possible to create a tensor product surface patch by moving a curve across space while letting deformations occur in that curve. One way to think about it is to imagine that each control point p_i may sweep a curve over space. Using Bernstein polynomials to describe the surface in consideration, a Bézier surface patch is constructed, which may be expressed by the following formula:

$$S(\xi,\nu) = \sum_{i=0}^{p}\sum_{j=0}^{q} P_{i,j} B_{i,p}(\xi) B_{j,q}(\nu), \qquad 0 \le \xi,\nu \le 1. \tag{2.4}$$

$P_{i,j}$ is $(p+1)\times(q+1)$ Bézier surface control points and $B_{i,p}(\xi)$ and $B_{j,q}(\nu)$ are the basis functions for parameters ξ and ν respectively, defined as follows:

$$B_{i,p}(\xi) = \frac{p!}{(p-i)!i!}\xi^i(1-\xi)^{p-i},$$

$$B_{j,q}(\nu) = \frac{q!}{(q-j)!j!}\nu^j(1-\nu)^{q-j}.$$

Since the basis function has p and q degree, it is appropriate to refer this surface as a Bézier surface of degree (p,q). However, a Bézier surface for $(p=3,q=3)$ degree is depicted in Figure 2.3.

2.2.3(a) Properties of Bézier Surfaces

- **Endpoint Interpolation:** The Bézier surface $S(\xi,\nu)$ passes through the corner points:

$$S(0,0) = P_{0,0},\ S(1,0) = P_{p,0},\ S(0,1) = P_{0,q},\ S(1,1) = P_{p,q}.$$