

**MODELING OF HIGH PERFORMANCE
SURFACE MOUNT MOLDED PQFP PACKAGES**

by

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requirements for the degree
of Master of Science**

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ABSTRACT

Surface mount plastic packaging technology has evolved in order to meet the stringent standards demanded by high density and low cost integrated circuits devices. The internal heatsink Plastic Quad Flat Pack (PQFP) has been developed to replace the traditional single layer PQFP in order to enhance the package's electrical and thermal performance. Higher external package warpage is observed on heatsink package compared to the traditional package and thus this might affect the reliability and performance of the package.

This study aims to set some guidelines regarding the standard design rules for heatsink sizes and die sizes for 28x28 mm PQFP heatsink package. This will lead to improvement in the process assembly procedure and hence reduction in leadframe cost. The software MECHANICA developed by Rasna corporation which utilizes the *p*-version finite element method is used in this study to analyze the impact of heatsink sizes and die sizes on the package warpage and thermal performance of the package.

Modeling of different heatsink and die sizes are carried out to simulate the external package warpage and thermal resistance of the heatsink package. Simulation results indicate that heatsink size of 850 squared mils and die size of 340 squared mils are the optimum sizes for thermal performance improvement and package warpage reduction.

PEMODELAN UNTUK PERMUKAAN PENCANTUMAN ACUAN BENTUK
PQFP YANG BERPENCAPAIAN TINGGI

ABSTRAK

Teknologi pembentukan permukaan pencantuman plastik telah berkembang untuk memenuhi pempiawaian ketat yang bersesuaian dengan litar integrasi yang kepadatan tinggi dan kos rendah. Pakej "heatsink" dalaman bagi Plastic Quad Flat Pack (PQFP) telah dicipta untuk menggantikan pakej tradisi satu lapisan bagi memperkuatkan lagi prestasi litar integrasi dari segi elektrik dan pengaliran haba. Pakej heatsink telah menimbulkan pembengkokan bentuk yang lebih tinggi berbanding dengan pakej tradisi dan ini mungkin membangkitkan masalah-masalah mengenai prestasi dan keutuhan pakej pada keseluruhannya.

Kajian ini bertujuan bagi merumuskan panduan untuk pempiawaian rekaan untuk saiz heatsink dan saiz die bagi pakej PQFP heatsink berukuran 28x28 mm. Pempiawaian ini akan melicinkan lagi proses pembinaan pakej dan seterusnya mengurangkan kos pakej. Program MECHANICA yang dicipta oleh Rasna corporation di mana kaedah unsur terhingga rekaan-p

digunakan di dalam kajian ini bagi mengkaji keberkesanan saiz heatsink dan saiz die dalam mengurangkan pembengkokan pakej dan rintangan haba.

Pemodelan untuk mengkaji keberkesanan saiz heatsink dan die telah dilakukan untuk mensimulasikan pembengkokan pakej dan rintangan haba. Keputusan simulasi menunjukkan bahawa saiz heatsink berukuran 850 mils persegi dan saiz die 340 mils persegi adalah saiz optima bagi rintangan haba dan pembengkokan pakej yang lebih rendah.

CHAPTER ONE

INTRODUCTION

1.1 Introduction to Integrated Circuit

An Integrated Circuit (IC) is a small and complex circuit made of a single chip of silicon. Each IC is designed to perform and work to a specific function as required from as simple as a switch to a high speed microprocessor. Generally , there are two types of IC packages, namely the insert type package (Figure 1.1) and the surface mount package (Figure 1.2).

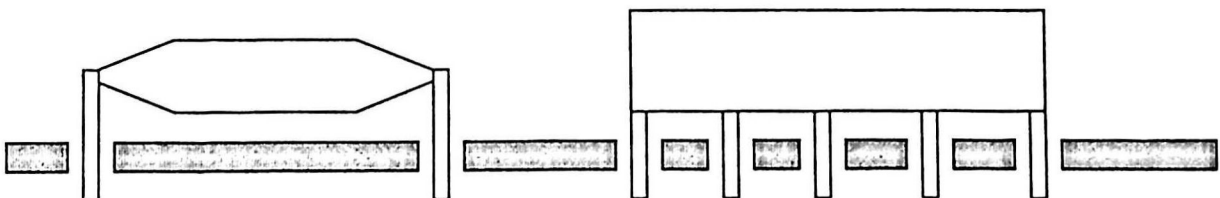


Figure 1.1 Example of insert type packages

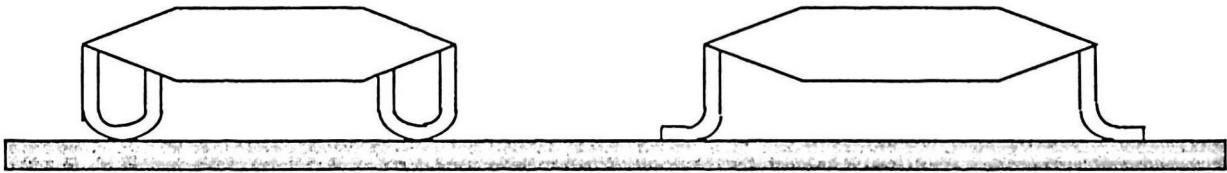


Figure 1.2 Example of surface mount packages

Surface mount technology (SMT) evolved from earlier technologies and has progressed rapidly to meet the strong desire and need for further miniaturization of electronic products with increased functional capability and reduced costs (Harper, 1991). The demand for smaller and smaller products with more and more functions at lower costs and increased reliability have resulted in SMT rapidly becoming the technology of choice for many of the new electronics products.

The common materials for IC encapsulation and packaging are ceramics and plastics. Selection of these materials is based on mechanical, thermal, and chemical properties as well as reliability, cost and applications. In this study, the focus will be on the modeling of high performance 208-lead surface mount molded Plastic Quad Flat Packages (PQFP) with particular reference to their thermal performance and package warpage.

1.2 Integrated Circuit Manufacturing

Assembly of a plastic IC package starts from an incoming wafer. A wafer is a piece of thin layer, typically 20 mils (a mil is one thousandth of an inch) of silicon which contains hundreds of dies (Kerridge, 1983). A die contains hundreds and thousands of transistors to form a complete electrical circuitry where various functions and controls are performed. The wafer is then mounted on adhesive tape and is ready for wafer saw. Next, the wafer will be sawn by high precision diamond saw to the designated die size. Then these dies will be attached to copper leadframe by silver epoxy. The interconnections between leadframe and die bond pads are performed by gold wire thermosonic bonder. The bonded leadframe will then be taken for plastic injection molding for encapsulation at high temperature. After encapsulation, the entire package is cooled from molding temperature to room temperature. The molded IC will then be sent for post mold cure, solder plating, trimming, forming and singulation. The completed ICs are then computer tested, marked, packed and are ready for shipment to customer. Figure 1.3 shows the typical processes involved in producing an IC (Hibberd, 1976).

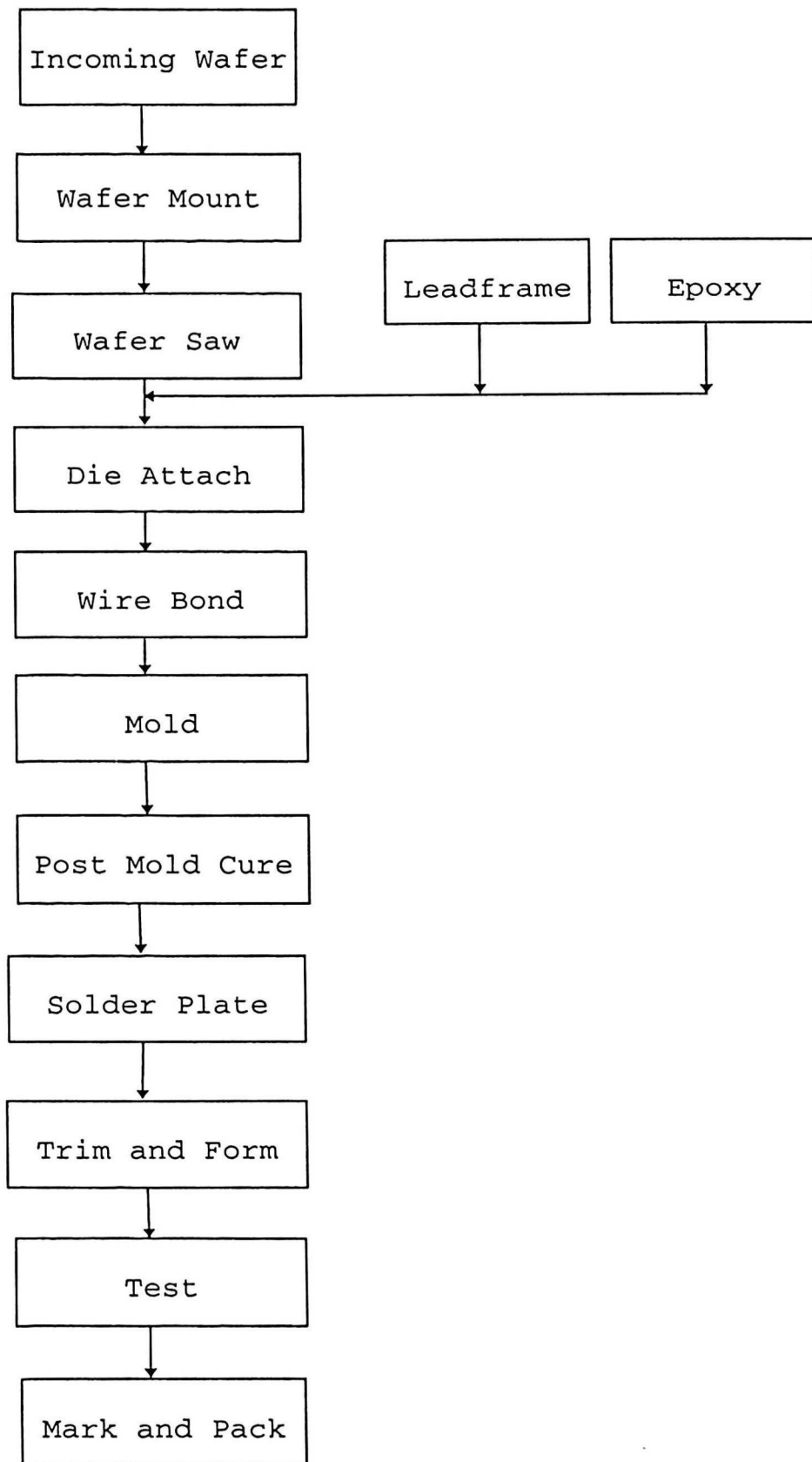


Figure 1.3 Integrated circuit process flow

1.3 Integrated Circuit Packaging

The issue of thermal and mechanical concerns in the packaging of microelectronics devices is of the utmost importance. As these devices decrease in size while increase in power dissipation, thermal management becomes a primary issue. Furthermore, the newer devices require a vast increase in I/Os. These increase in performance demands translate into a more demanding packaging and production environment (Harper, 1991).

Conventional plastic packages are not capable of handling high pin count and high density devices where the capacity for fast heat dissipation is crucial. Hence internal heatsink PQFP package was introduced in order to meet the electrical and thermal performance requirements of today's high performance devices (Aghazadeh and Mallik, 1990). Nevertheless conventional plastic assembly methods can still be applied to accommodate internal heatsink packages without the need for developing new processes and equipments.

The structure of internal heatsink leadframe is different from that of the conventional single layer copper leadframe. An additional layer of copper heatsink is attached to internal lead finger by adhesive polimide tape. Figure 1.4 and Figure 1.5 are the cross sectional views of

conventional and heatsink packages respectively. It is noted that the die attach pad of conventional package is replaced by the copper heatsink in heatsink package. Both packages use silver epoxy to attach the die onto the die attach pad.

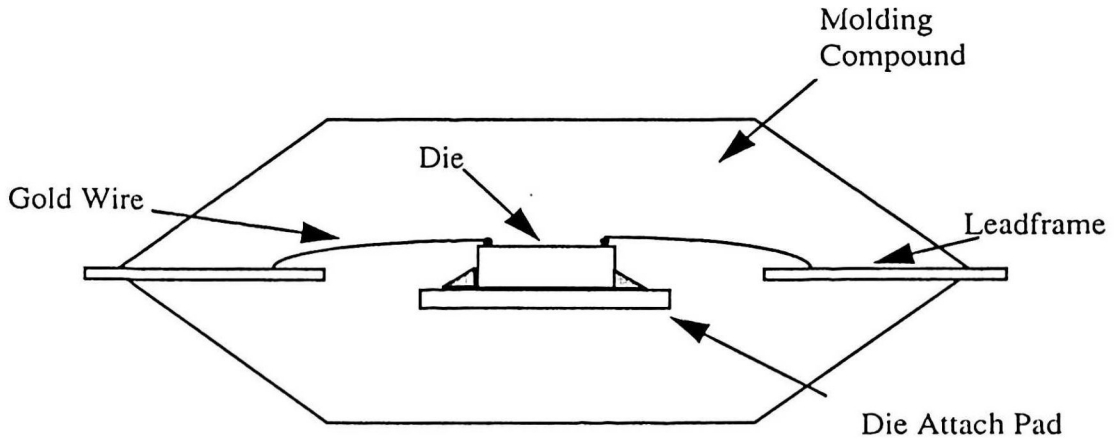


Figure 1.4 Conventional package

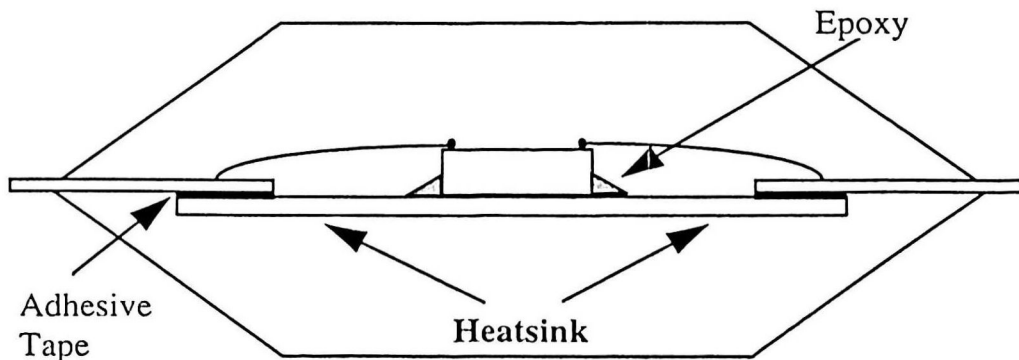


Figure 1.5 Internal heatsink package

1.3.1 Package Warpage

Typical IC packages are comprised of a mixture of materials ranging from brittle materials like silicon die and epoxy molding compound to ductile materials like copper leadframe and heatsink metal. Thermal stress will be induced during assembly of IC packages due to the mismatch of the different coefficients of thermal expansion of dissimilar materials. As a consequence, external package warpage will result from these thermal stresses when the package is being cooled down from mold temperature to room temperature (Kiang et. al., 1991). The package warpage may affect the downstream IC assembly processes following mold encapsulation. The common failures resulting from package warpage include package crack, die crack and delamination between interfaces of different materials.

Conventional type of package has less package warpage compared to heatsink package and hence in this respect the impact to failure is far less. The increased warpage in internal heatsink package is possibly due to larger surface area of copper heatsink attached between epoxy molding compound, that results in higher thermal stress. Further the structure of heatsink package also results in the imbalance between the volumes of epoxy molding compound at the top and bottom portion of the molded package. This imbalance has been identified to be one of the factors that contributes to package warpage (Kiang et. al., 1991).

1.3.2 Thermal Performance

High performance and high speed devices normally have higher power dissipation levels. Therefore improving the thermal characteristics and performance of PQFP packages is necessary in order to meet the overall reliability and performance requirements of these devices. The advantage of heatsink package lies in the improvement in the package thermal performance. The copper heatsink layer in heatsink package improves heat dissipation, by redistributing the heat flow paths and increasing the heat transfer area. Thus, heat generated by the devices will dissipate faster, resulting in better thermal performance and improved reliability.

The thermal resistance (θ_{ja}) for a internal heatsink PQFP package is 20% to 30% lower than that of conventional single layer PQFP package. The thermal resistance of heatsink packages can achieve θ_{ja} of below 25 °C/W as compared with θ_{ja} of around 30 °C/W in a typical conventional package (Tanaka and Takeuchi, 1992). A lower thermal resistance implies better thermal performance. Hence the reduction of thermal resistance in an internal heatsink package has further extended the application limits of conventional plastic packaging to accommodate devices that generate more heat.

1.4 Objective and Scope of The Study

PQFP packages are categorized by the number of lead count which is determined by the number of external I/O required. A particular lead count is further categorized by its die size which depends on the various applications and functionalities. These various characteristics have resulted in the use of different die sizes and heatsink sizes for a particular type of heatsink package. Different die sizes and heatsink sizes need different process parameters during mold encapsulation, hence require different design standards.

The main objective of this study is to help define the standard design rules on die sizes and heatsink sizes for internal heatsink PQFP packages which will lead to standardization of heatsink packages regardless of lead count. This standardization will reduce the process engineering effort for the characterization and optimization of various types of heatsink design. Furthermore, leadframe cost could be further reduced if standardized heatsink designs could be more efficiently used on high volume packages. In this thesis, heatsink size and die size are considered as the main factors that control package warpage and thermal resistance.

The finite element package MECHANICA will be used in this study to analyze the influence of heatsink size and die

size on package warpage and thermal resistance. This analysis, hopefully, will help to pave the way towards the standardization of internal heatsink designs to meet the stringent demands of high performance PQFP Packages.

The remaining thesis will be organized as follows. Chapter Two will give a brief introduction to the finite element methods relevant to this study, leading to the exposition of the proprietary package called MECHANICA which is used extensively in this study. Some test modeling using MECHANICA and a different software will be reported in Chapter Three to illustrate some salient features of importance. Modeling of package warpage will be discussed in Chapter Four while Chapter Five will focus on the thermal performance of internal heatsink packages as predicted by MECHANICA and measured in the laboratory. Both modeling as well as laboratory testing, measurement and confirmation are performed at AMD Sunnyvale, California. Based upon the results contained in the previous chapters, some analysis of optimal design rules for the internal heatsink packages will be presented in Chapter Six, which is followed by some concluding remarks in Chapter Seven.

CHAPTER TWO

FINITE ELEMENT METHOD

2.1 Introduction to Finite Element

Scientists, engineers and applied mathematicians are often faced with practical problems whose solution by conventional analytical method is either too difficult or even impossible. A numerical method can be used to obtain an approximate solution when an analytical solution cannot be developed. There are several numerical methods available and finite element method is one of the choices. The finite element method (FEM) is a numerical approximation procedure for solving physical problems governed by a differential equation or an energy principle. It uses an integral formulation to generate a system of algebraic equations and utilizes continuous piecewise smooth functions for approximating the unknown quantities. The finite element method is thus a particular class of discretization

procedure by which the original governing equations having infinite degrees of freedom are transformed into approximation equations with finite degrees of freedom (Martin and Carey, 1973).

The basic steps of the finite element process can be listed as follows.

- (i) The structure is first divided into distinct non overlapping region known as elements over which the main variables are interpolated.
- (ii) The discretized elements are connected at a discrete number of points, known as nodal points or simply nodes, along their periphery.
- (iii) For each element the element "stiffness matrix" and "applied load" vector are calculated (Seegerlind, 1984).
- (iv) The element stiffness matrix and load vector of each element are assembled to give respectively, the global stiffness matrix and global load vector for the complete structure.
- (v) The resulting system of simultaneous equations is then solved for the unknown nodal variables; which for structural problems are the displacement components, for example.
- (vi) Finally subsidiary quantities such as stress components are evaluated for each element.

Some detail of the procedures and concepts mentioned above will be illustrate by the examples in the following sections.

2.2 One-Dimensional Linear Element

One-dimensional element is the simplest and easiest to begin and hence will be used to obtain an approximate solution to the one-dimensional heat equation

$$D \frac{d^2\phi}{dx^2} + Q = 0 \tag{2.1}$$

with the given boundary conditions

$$\phi(0) = \phi_0 \quad \text{and} \quad \phi(H) = \phi_1 \tag{2.2}$$

In one dimension, a linear element is a line segment with a length L and two nodes, one at each end (Figure 2.1). The nodes are denoted by i & j and the nodal values by Φ_i and Φ_j respectively.

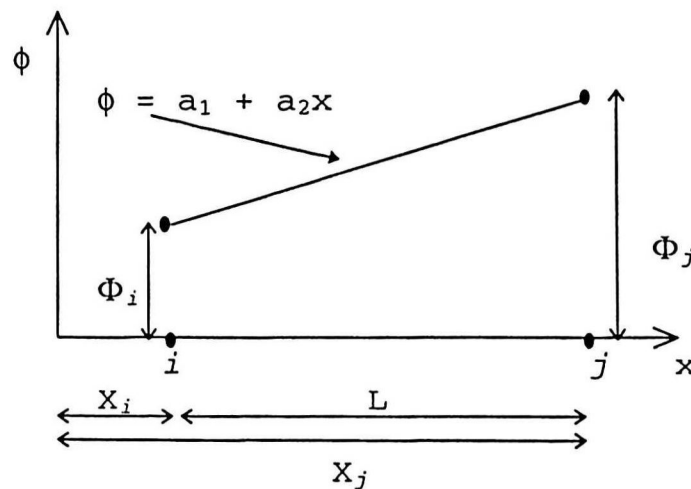


Figure 2.1 The one-dimensional linear element

The equation for ϕ is

$$\phi = a_1 + a_2 x . \quad (2.3)$$

Referring to Figure 2.1, it is easy to show that

$$a_1 = \frac{\Phi_i X_j - \Phi_j X_i}{X_j - X_i}$$
$$a_2 = \frac{\Phi_j - \Phi_i}{X_j - X_i} , \quad (2.4)$$

and hence ϕ may be written as

$$\phi = \left(\frac{X_j - x}{L} \right) \Phi_i + \left(\frac{x - X_i}{L} \right) \Phi_j . \quad (2.5)$$

Equation (2.5) is now in a standard finite element form from which the concept of shape functions may be derived. It is noted that nodal values Φ_i and Φ_j are multiplied by linear functions of x of the same order as in (2.3), which are called shape functions or interpolation function (Seegerlind, 1984). The shape functions derived in (2.5) are denoted by N_i and N_j with

$$N_i = \left(\frac{X_j - x}{L} \right)$$

and

$$N_j = \left(\frac{x - X_i}{L} \right) \quad . \quad (2.6)$$

In term of the shape functions, equations (2.5) can be rewritten as

$$\phi = N_i \Phi_i + N_j \Phi_j \quad (2.7)$$

or

$$\phi = [N] \{ \Phi \} \quad , \quad (2.8)$$

where $[N] = [N_i \ N_j]$ is a row vector of shape functions and

$$\{ \Phi \} = \left\{ \begin{array}{c} \Phi_i \\ \Phi_j \end{array} \right\}$$

is a column vector containing the element nodal values.

A few comments about the shape function are in order.

- (i) Each shape function has a value of one at its associated node and zero at the other node.
- (ii) The two shape functions sum to one.
- (iii) The shape functions are always polynomials of the same order as the original interpolation equation.
- (iv) The derivatives of the shape function with respect to x sum to zero.

Some of these important concepts and properties of shape functions are crucial in the development of other elements in higher dimensions, the details of which are deemed inappropriate for inclusion in this thesis in order to maintain consistency of purpose.

2.2.1 The Galerkin's Method

The following discussion demonstrates the derivations of the finite element equations using Galerkin's formulation whereby a system of linear equations is generated by evaluating the weighted residual integral with appropriate choices of the weighting function $W(x)$.

$$- \int_0^1 W(x) \left(D \frac{d^2\phi}{dx^2} + Q \right) dx = 0. \quad (2.9)$$

Here $W(x)$ is a set of weighting functions to be chosen, one for each node where ϕ is unknown, as explained below. Basically Galerkin's formulation of the weighted residual method requires that weighting functions be constructed using the shape functions N_i and N_j . Consider the node s . Let r, s and t be a sequence of adjacent nodes of s (Figure 2.2). The weighting function W_s for node s is chosen as

$$W_s(x) = \begin{cases} N_s^{(e)} & , \quad X_r \leq x \leq X_s \\ N_s^{(e+1)} & , \quad X_s \leq x \leq X_t \end{cases} \quad (2.10)$$

and is shown in Figure 2.2. Hence W_s has the value of 1 at node s and 0 elsewhere.

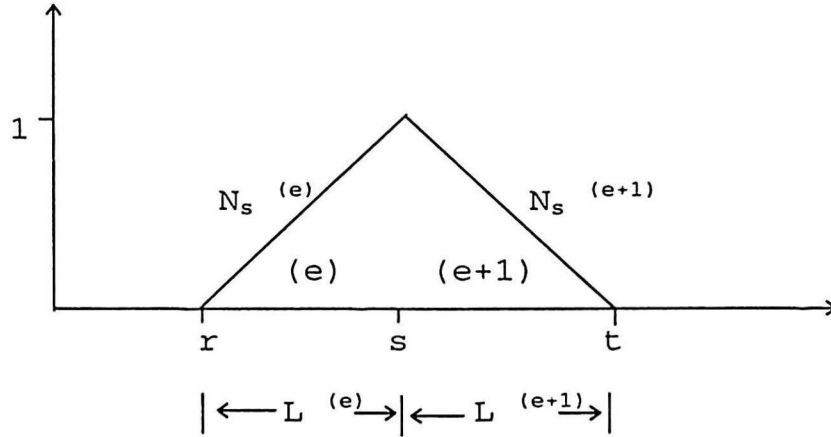


Figure 2.2 The weighting function W_s at node s

Then the residual equation R_s for the node s is

$$\begin{aligned} R_s &= - \int_{x_r}^{x_s} \left[N_s \left(D \frac{d^2\phi}{dx^2} + Q \right) \right]^{(e)} dx \\ &\quad - \int_{x_s}^{x_t} \left[N_s \left(D \frac{d^2\phi}{dx^2} + Q \right) \right]^{(e+1)} dx = 0 \\ &= R_s^{(e)} + R_s^{(e+1)} \end{aligned} \quad (2.12)$$

The integral splits into two parts because $W_s(x)$, in terms of the shape functions, is defined by two separate equations within the interval $X_r \leq x \leq X_t$. By reducing the second derivative of ϕ to first derivative via integration by parts, (2.11) becomes

$$\begin{aligned}
 R_s = & \left(D \frac{d\phi}{dx} \right)^{(e)} \Big|_{x = x_s} \\
 & + \int_{x_r}^{x_s} \left(D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e)} dx \\
 & + \left(D \frac{d\phi}{dx} \right)^{(e+1)} \Big|_{x = x_s} \\
 & + \int_{x_s}^{x_t} \left(D \frac{dN_s}{dx} \frac{d\phi}{dx} - N_s Q \right)^{(e+1)} dx = 0. \quad (2.12)
 \end{aligned}$$

These residual equations R_s for each node s are then assembled to form a global residual equation R for the complete system as explain below.

2.2.2 Element Matrices

Let $\{R\}$ be the column vector, each component of which represents a residual equation associated with a particular node. Further, analyzing from an element point

of view with an arbitrary element with nodes i and j , $R_i^{(e)}$ is equivalent to $R_s^{(e+1)}$, with $s=i$ and $t=j$ whereas $R_j^{(e)}$ is the same as $R_s^{(e)}$, with $r=i$ and $s=j$. Hence, (2.12) can be rewritten in two parts as

$$R_i^{(e)} = D \left. \frac{d\phi}{dx} \right|_{x=x_i} + \frac{D}{L} (\Phi_i - \Phi_j) - \frac{QL}{2} \quad (2.13)$$

and

$$R_j^{(e)} = D \left. \frac{d\phi}{dx} \right|_{x=x_j} + \frac{D}{L} (-\Phi_i + \Phi_j) - \frac{QL}{2} . \quad (2.14)$$

Combining (2.13) and (2.14) gives

$$\begin{Bmatrix} R_i^{(e)} \\ R_j^{(e)} \end{Bmatrix} = \begin{Bmatrix} I_i^{(e)} \\ I_j^{(e)} \end{Bmatrix} + \frac{D}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_i \\ \Phi_j \end{Bmatrix} - \frac{QL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad (2.15)$$

or

$$\{ R^{(e)} \} = \{ I^{(e)} \} + [K^{(e)}] \{ \Phi^{(e)} \} - \{ f^{(e)} \}. \quad (2.16)$$

Here $\{ R^{(e)} \}$ is the contribution of element (e) to the final system of equations. This contribution consists of an element stiffness matrix $[K^{(e)}]$, an element force vector $\{ f^{(e)} \}$, nodal values $\{ \Phi^{(e)} \}$ and interelement requirement $\{ I^{(e)} \}$. The vector $\{ R \}$ represents a system of equations that symbolically is

$$\{ R \} = [K] \{ \Phi \} - \{ F \} = \{ 0 \} \quad . \quad (2.17)$$

The direct stiffness method is used for incorporating the element matrices into the final system of equation. By solving the system of equation, after incorporating the given boundary conditions, quantities of interest can be calculated (Segerlind, 1984). This completes the description of the one-dimensional linear finite elements.

2.3 Two-Dimensional Element

The two-dimensional elements will be briefly described in this section. The complete polynomials in two dimensions can be easily developed by using Pascal triangle shown in Figure 2.3 (Hilton and Owen, 1979). Thus a complete n^{th} order polynomial may be written as

$$\phi (x,y) = \sum_{r=1}^m \alpha_r x^i y^j \quad ; \quad i + j \leq n \quad (2.18)$$

where

$$m = (n + 1) (n + 2) / 2 \quad . \quad (2.19)$$

These polynomials of different orders may be chosen carefully following the basic concepts developed earlier for the one-dimensional elements to form two-dimensional elements with appropriate interpolation properties, which

will be briefly stated below. Details regarding these element may be referred to the references cited earlier.

					<u>Order</u>	<u>Number of term (m)</u>
		1			Constant	1
	x		y		Linear	3
	x ²	xy	y ²		Quadratic	6
x ³	x ² y	xy ²	y ³		Cubic	10
x ⁴	x ³ y	x ² y ²	xy ³	y ⁴	Quartic	15

Figure 2.3 The Pascal triangle

2.3.1 Linear Triangular Element

Figure 2.4 shows a 3-node triangular C⁰ element with interpolation polynomial of

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y \quad . \quad (2.20)$$

When a local coordinate system (ξ , η) is used to define the element geometry, the shape functions are given as

$$\begin{aligned} N_i &= 1 - \xi - \eta \\ N_j &= \xi \\ N_k &= \eta \quad . \end{aligned} \quad (2.21)$$

For all the two-dimensional C^0 element, the shape functions satisfy the condition

$$\sum N_i^{(e)}(\xi, \eta) = 1 \quad (2.22)$$

and

$$N_i^{(e)}(\xi_j, \eta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (2.23)$$

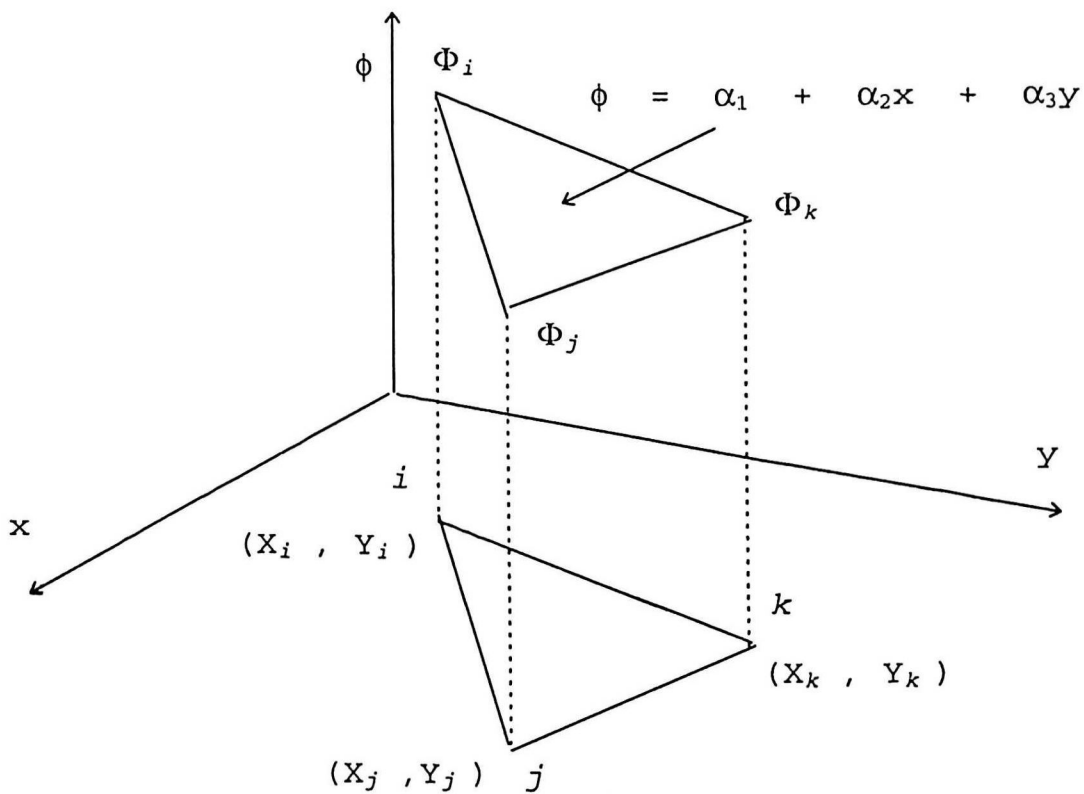


Figure 2.4 Linear triangular element

2.3.2 Bilinear Rectangular Element

For the 4-node element shown in Figure 2.5, the interpolation functions is

$$\phi = c_1 + c_2x + c_3y + c_4xy \quad (2.24)$$

The shape functions on the reference element $ijklm$, relative to the coordinate system (ξ, η) are given as

$$N_i = \frac{1}{4} (1 - \xi) (1 - \eta)$$

$$N_j = \frac{1}{4} (1 + \xi) (1 - \eta)$$

$$N_k = \frac{1}{4} (1 + \xi) (1 + \eta)$$

$$N_m = \frac{1}{4} (1 - \xi) (1 + \eta) \quad (2.25)$$

A reference element is an element with the simplest node coordinates, for example $(1,1)$. Henceforth the same terminology is applied elsewhere.

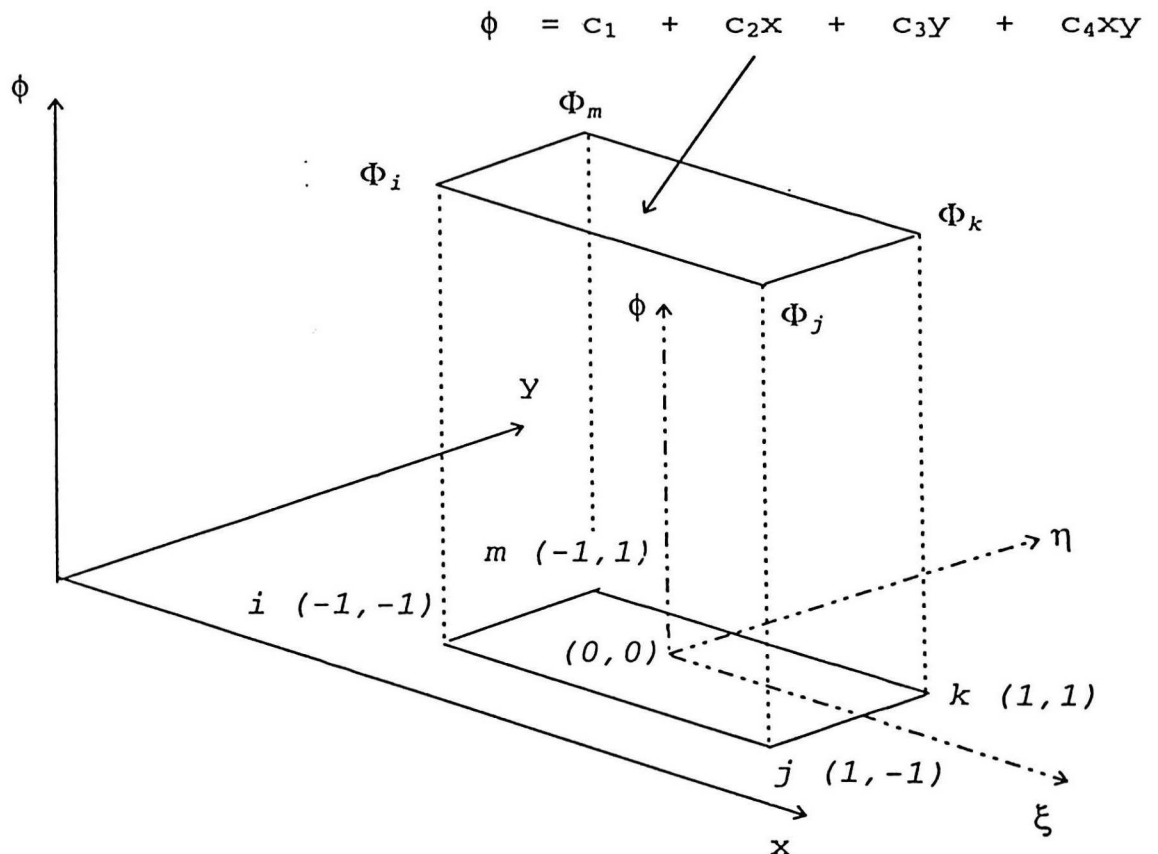


Figure 2.5 Bilinear rectangular element

Elements of higher order may also be developed, the details of which are available from Hilton and Owen, 1979.

2.4 Three-Dimensional Element

For three dimensions, Figure (2.6) describes the terms in a complete polynomial of n^{th} order which may be written as (Hinton and Owen, 1979)

$$\phi(x, y, z) = \sum_{r=1}^m \alpha_r x^i y^j z^k \quad ; \quad i+j+k \leq n \quad (2.26)$$

where

$$m = (n + 1)(n + 2)(n + 3) / 6 . \quad (2.27)$$

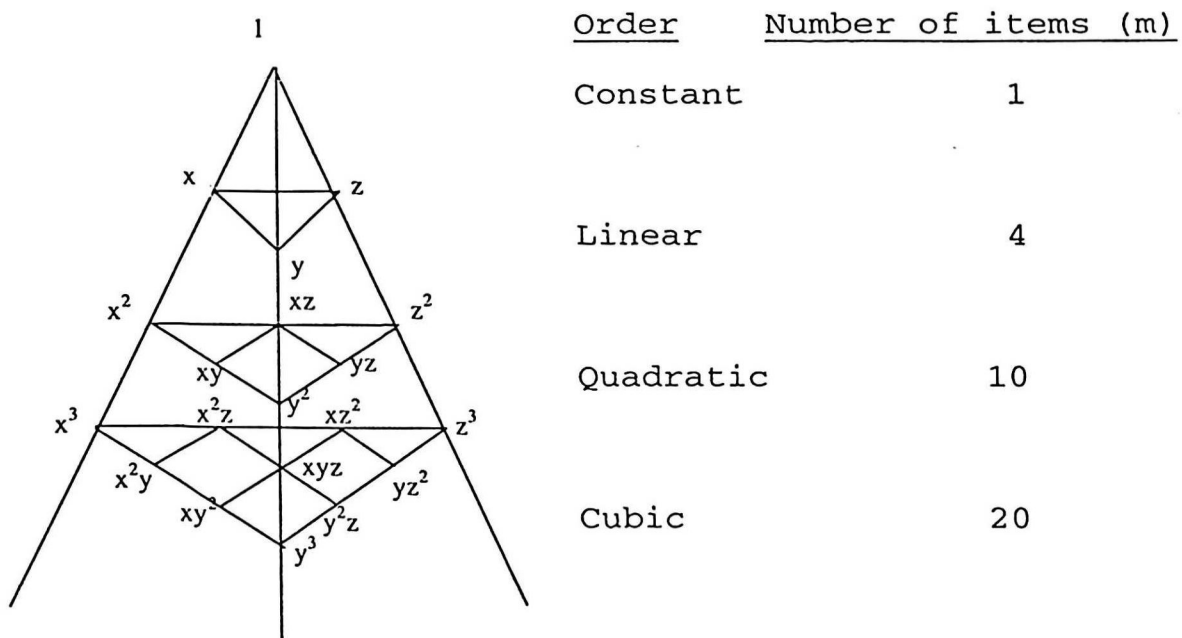


Figure 2.6 Complete polynomials in three dimensions