FUNCTIONS STARLIKE

WITH RESPECT TO A BOUNDARY POINT

by

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CONTENTS

NOTATIONS

U	the unit disk $\|z\| < 1$
∂U	boundary of the unit disk
H	class of functions f analytic in U and normalized $f(0) = 0 = f'(0) - 1$
S	class of univalent functions in H
Σ	class of analytic and univalent functions $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}$ in the exterior of U
B	bounded functions in U, $\|f(z)\| \le 1$, $f(0) = 0$
C	class of normalized analytic close-to-convex functions in U
$f << g$	g majorizes f
G	class of analytic non-vanishing functions g in U satisfying $g(0) = 1$ and $\text{Re}\left[\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right] > 0$, $z \in U$
G^*	class of analytic functions g in U, normalized so that $g(0) = 1$, $g(1) = \lim_{r \to 1} g(r) = 0$, and $\text{Re}[e^{i\alpha} g(z)] > 0$ for some real α.
$G(\alpha)$	class of analytic non-vanishing functions g in U satisfying $g(0) = 1$ and $\text{Re}\left[\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right] > 0$, $0 \le \alpha < 1$, $z \in U$.
$\mathbb{C}$	the complex plane
arg	argument of a complex number
Im	imaginary part of a complex number
Re	real part of a complex number
K	class of normalized analytic convex functions in U
P	class of normalized analytic functions with positive real part in U
S^*	class of normalized analytic starlike functions in U
$S^*(\alpha)$	class of normalized analytic starlike functions of order α in U

FUNGSI BAK-BINTANG TERHADAP TITIK SEMPADAN

ABSTRAK

Andaikan $U = \{ z: |z| < 1 \}$ sebagai cakera unit terbuka dan S kelas yang terdiri daripada fungsi-fungsi f yang analisis dan univalen pada U dan dinormalkan supaya $f(0) = f'(0) - 1 = 0$. Andaikan S^* sebagai subkelas fungsi-fungsi f dalam S yang memenuhi syarat $Ny\{ zf'(z) / f(z) \} > 0$, $z \in U$. Setiap $f \in S^*$ memetakan U secara univalen dan keseluruh kepada suatu domain bak-bintang terhadap titik asalan. Kelas S^* sudah banyak dikajikan.

Namun tidak banyak yang diketahui tentang kelas fungsi-fungsi analisis yang memetakan U keseluruh domain-domain bak-bintang terhadap titik sempadan. Pengkajian secara sistematik tentang kelas tersebut telah dimulakan oleh Robertson [23].

Lambangkan G sebagai kelas fungsi- fungsi g dimana $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, analisis dan tak- lenyap pada U, dan yang bersifat

$$Ny\left\{\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right\} > 0, \qquad z \in U.$$

Robertson [23] telah membuktikan bahawa fungsi tak-malar g memetakan U pada domain bak bintang terhadap titik sempadan $g(1) = 0$. Hubungan rapat diantara kelas G dan kelas S^* diberikan dimana $g \in G$ jika dan hanya jika wujud suatu fungsi $f \in S^*$ supaya $g^2(z) = (1-z)^2 f(z) / z$. Juga g ialah fungsi malar 1 atau hampir cembung terhadap f.

Rumus Leverenz [17] diperkenalkan pada Bab 2. Rumus tersebut seterusnya diperolehi di dalam bentuk terhingga. Beberapa contoh penggunaan rumus tersebut seterusnya diberikan. Bab 3 membincangkan kepentingan fungsi Koebe $k(z) = z / (1 - z)^2$ dan putarannya terhadap kelas G.

Menggunakan teorem Leverenz [17] kita dapatkan beberapa ketaksamaan yang melibatkan pekali-pekali d_1, d_2 dan d_3 bagi fungsi-fungsi $g \in G$. Seterusnya diperolehi keputusan keherotan bagi kelas G. Anggaran-anggaran ini didapati terbaik mungkin. Ketaksamaan yang diperolehi didapati memberikan beberapa keputusan yang setara dengan beberapa ketaksamaan pekali bagi kelas P dan S^* yang telah diketahui.

Silvia & Silverman [25] seterusnya mengklasifikasikan kelas $G(\alpha)$, $0 \leq \alpha < 1$. Fungsi-fungsi g terletak dalam $G(\alpha)$, jika g analisis dan tak -lenyap pada U serta bersifat

$$\mathrm{Ny}\left\{\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right\} > 0, \qquad z \in U.$$

Jika $\alpha = 1/2$ kita perolehi kelas G. Di dalam Bab 4, kita dapatkan bentuk yang lebih umum bagi keputusan-keputusan yang telah diperolehi di dalam Bab 3 bagi kelas G. Kita juga perolehi keputusan putaran bagi kelas $G(\alpha)$.

ABSTRACT

Let $U = \{ z : |z| < 1 \}$ be the unit disk and S be the class of analytic univalent functions f normalized such that $f(0) = f'(0) - 1 = 0$. Let S^* denote the subclass of functions f in S which satisfy the condition $Re\{ zf'(z) / f(z) \} > 0, \ z \in U$. Functions $f \in S^*$ map U univalently onto domains starlike with respect to the origin. The class S^* has been extensively studied over the last fifty years.

However not much seems to be known about the class of analytic functions that map U onto domains starlike with respect to a boundary point. M.S.Robertson [23] was the first to initiate a systematic study of this class.

Let G denote the class of functions $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ analytic and non-vanishing in U and satisfying

$$Re\left\{ \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \right\} > 0 \ , z \in U.$$

Robertson [23] had shown that nonconstant functions in the class G map U univalently onto domains starlike with respect to the boundary point. By a rotation we may assume that this point is $g(1) = 0$. A close relation between the class G and the class S^* was given such that $g \in G$ if and only if $g^2(z) = (1-z)^2 f(z) / z$, $f \in S^*$. Further, g is either close-to-convex with respect to f or g is the constant function 1.

Leverenz theorem [17] is introduced in Chapter 2. An equivalent finite form of the theorem is then obtained. Examples are given to illustrate its application. In Chapter 3, the important role of the Koebe function $k(z) = z / (1-z)^2$ and its rotations as related to the class G is examine.

Using Leverenz theorem sharp inequalities involving the coefficients d_1, d_2 and d_3 are obtain which then leads to a distortion result for the class G. It is found that the results obtained are anologous to some of the well- known inequalities for the class P and the class S*.

Silvia & Silverman [25] have generalized the class G to the class $G(\alpha)$, $0 \leq \alpha < 1$. Specifically $g \in G(\alpha)$ if g is analytic and non-vanishing in U, satisfying

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}+(1-\alpha)\frac{1+z}{1-z}\right\}>0 \ , \ z \in U.$$

The case $\alpha = 1/2$ reduces to the class G. In Chapter 4, we extend the coefficients results found for the class G to the class $G(\alpha)$. A rotation result for this class is also obtained.

CHAPTER ONE

INTRODUCTION

1.1 ANALYTIC AND UNIVALENT FUNCTIONS

This preliminary chapter will review and assemble basic principles of analytic and univalent functions which shall be referred and needed later in this thesis. We start with the definition of a domain. A *domain* in the complex plane $\mathbb{C}$ is an open connected set. A function f is said to be *analytic at a point* z_0 in $\mathbb{C}$ if it is differentiable at every point in a neighbourhood $|z-z_0|<\varepsilon$, where ε is some positive number. A function f is *analytic in a domain* D, $D \subseteq \mathbb{C}$, if it is analytic at each point z_0 in D. A function f is said to be *univalent* in a domain D if it never takes the same value twice, that is $f(z_1) = f(z_2)$ implies that $z_1 = z_2$ for $z_1, z_2 \in D$.

In this thesis we shall consider functions f which are analytic and univalent in the open unit disk $U = \{ z : |z| < 1 \}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. This class of functions will be denoted by S. If $f \in S$, then f has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \quad |z| < 1.$$

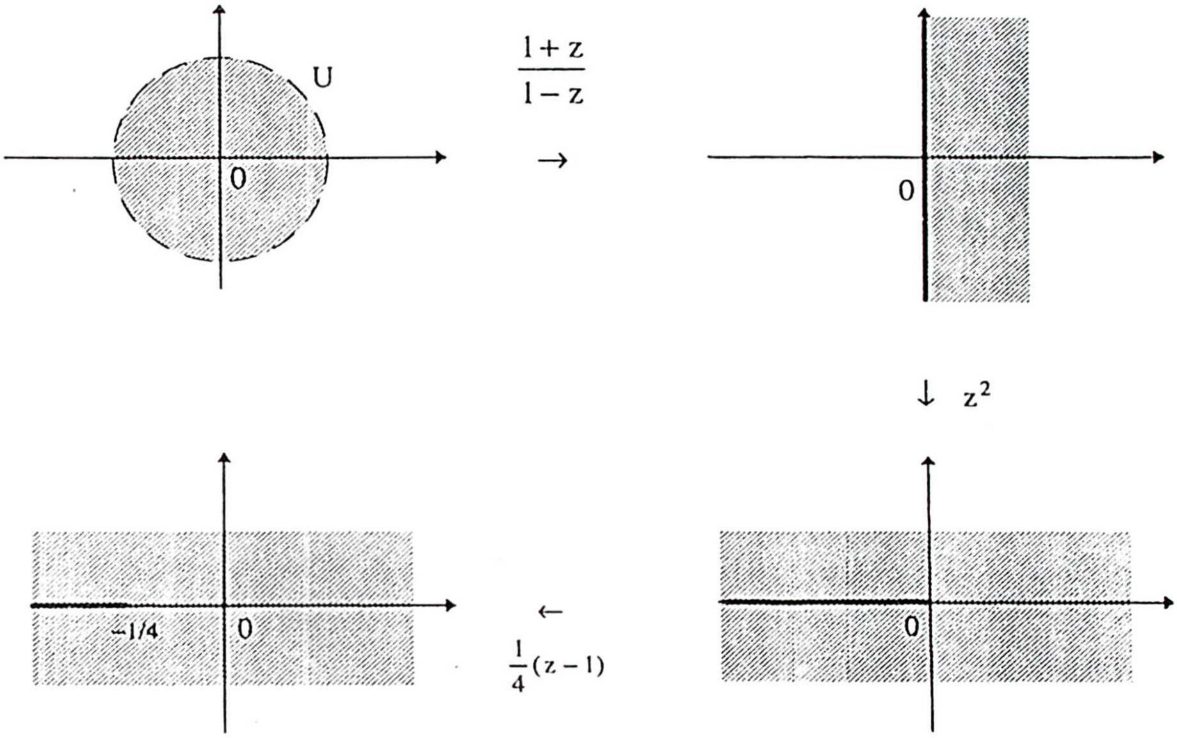
An example of a function in the class S is the *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots. \tag{1.1.1}$$

The function k maps the unit disk U onto the entire complex plane $\mathbb{C}$ minus the negative real axis from $-1/4$ to infinity. To see this, first note

$$\frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right],$$

and so we get the following maps:



Closely related to the class S is the class Σ, where $g \in \Sigma$ if it is analytic and univalent in the exterior of the unit disk $\Omega = \{ z : |z| > 1 \}$ and has a simple pole at infinity with residue 1. Any function $g \in \Sigma$ also has an expansion of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}.$$

Each function $f \in S$ can be transformed to a non-vanishing function $g \in \Sigma$ by the inversion

$$g(z) = \frac{1}{f(1/z)} .$$

Most of the elementary results for the class S were obtained by considering the class Σ. Gronwall [13] in 1914 discovered the *area theorem* which is fundamental to the theory of univalent functions.

Theorem 1.1.1 [13] (Area Theorem)

Let $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \in \dot{\Sigma}$. Then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Proof.

Let E be the set omitted by $g(\Omega)$ and let γ be the image under g of the circle $|z| = r$. Since g is univalent, γ is a simple closed curve which encloses a domain $E_r \supset E$. Denote by A_r the area of E_r. By Green's theorem

$$\begin{aligned} A_r &= \frac{1}{2i}\int_{\gamma} \overline{w}\, dw. \\ &= \frac{1}{2i}\left[\int_{|z|=r} g'(re^{i\theta})\,\overline{g(re^{i\theta})}\, dz\right] \\ &= \frac{1}{2}\int_0^{2\pi}\left\{ re^{-i\theta} + \sum_{n=0}^{\infty}\overline{b_n} r^{-n} e^{in\theta}\right\} \times \left\{ 1 - \sum_{v=1}^{\infty} v b_v r^{-v-1} e^{-i(v+1)\theta}\right\} re^{i\theta}\, d\theta \\ &\doteq \pi\left\{ r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right\} \end{aligned}$$

Since the area A_r cannot be negative, it follows that $\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \leq r^2$. The inequality holds for $r > 1$, therefore it must also hold for $r \to 1$. Letting r decrease to 1, we then obtain $\pi\left\{ 1 - \sum_{n=1}^{\infty} n|b_n|^2 \right\} \geq 0$, hence the result.

The inequality $|b_1| \leq 1$ is sharp, with equality if and only if g has the form $g(z) = z + b_0 + b_1/z$, $|b_1| = 1$.

Theorem 1.1.2 [9] (Bieberbach Theorem)

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, then the following are satisfied:

i) $|a_2^2 - a_3| \leq 1$

ii) $|a_2| \leq 2$.

Equality occurs if and only if f is a rotation of the Koebe function.

Proof.

i) Let

$$g(z) = \frac{1}{f(1/z)}$$
$$= z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots .$$

Then $g \in \Sigma$. From the area theorem (Theorem 1.1.1) for $n = 1$, we get $|b_1| = |a_2^2 - a_3| \leq 1$.

ii) Let

$$g(z) = \left\{ \frac{1}{f(1/z^2)} \right\}^{1/2}$$
$$= \left\{(1/z^2) + a_2(1/z^2)^2 + \cdots \right\}^{-1/2} = z - (a_2/2)z^{-1} + \cdots .$$

Then $g \in \Sigma$. From the area theorem $|b_1| = |a_2/2| \leq 1$ which then gives the result.

Equality occurs if and only if g has the form

$$g(z) = z - \frac{e^{i\theta}}{z},$$

which implies that $f(z) = e^{-i\theta} k(e^{i\theta}z)$, where k is given in (1.1.1).

We shall need a few elementary definitions and theorems on analytic functions.

Definition 1.1.1 [1]

A real-valued function u is said to be *harmonic* in a domain D if the first and second partial derivatives exist and are continous, and if $\Delta u = u_{xx} + u_{yy} = 0$ at all points in the domain D.

Definition 1.1.2 [6]

A function f which in its domain satisfies $|f(z)| \le M$ for some positive constant M, is called a *bounded* function in that domain.

Theorem 1.1.3 [2] (Liouville's Theorem)

An analytic function cannot be single valued and bounded at all finite points of the plane unless it reduces to a constant.

Proof.

If f is analytic and bounded at every point in $\mathbb{C}$, that is $|f(z)| \le M$ for some M, then by Cauchy's inequality

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \le \frac{M}{R}.$$

If R is large enough we can deduce that $f'(z) = 0$ and since z is arbitrary, f then reduces to a constant.

Theorem 1.1.4 [6] (Maximum Modulus principle)

If f is analytic and not a constant in the domain D, then $|f|$ cannot attain its maximum value in D. That is there is no point z_0 such that $|f(z)| \le |f(z_0)|$ for all points $z, z_0 \in D$, unless f reduces to a constant.

Theorem 1.1.5 [1] (Minimum Principle for harmonic functions)

If f is a harmonic function in a domain D, then f cannot attain its minimum in the interior of D, that is there is no point z_0 such that $f(z_0) \leq f(z)$ for all points $z, z_0 \in D$, unless f reduces to a constant.

Lemma 1.1.1 [2] (Schwarz Lemma)

Let f be analytic in the unit disk U, and satisfies the conditions $f(0) = 0$ and $|f(z)| \leq 1$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in U$. If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ where $|c| = 1$.

Proof.

Consider the function $g(z) = f(z)/z$, $z \neq 0$ and $g(0) = f'(0)$ when $z = 0$. Then g is analytic at $z = 0$, and thus analytic throughout the unit disk. If $0 < r < 1$, then from the maximum principle (Theorem 1.1.4) we have

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}, \qquad |z| \leq r.$$

Letting r approach 1 yields $|g(z)| \leq 1$ for all $z \in U$, hence we get $|f(z)| \leq |z|$ and $|g(0)| = |f'(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \in U$, $z \neq 0$ or if $|f'(0)| = 1$, then $|g|$ attains its maximum in U and by Theorem 1.1.4, g must reduce to a constant c with $|c| = 1$, that is $f(z) = cz$.

Definition 1.1.3 [5]

A family F of functions analytic in a domain D is called a *normal family* if every sequence of functions $F_n \in F$ has a subsequence which converges uniformly on each compact subset of D. A normal family F is *compact* if the limits of all converging sequences of functions in F are also functions of F.

Theorem 1.1.6 [5] (Hurwitz's Theorem)

Let f_n be sequence of analytic and univalent functions and $f_n \to f$ uniformly on every compact subset of D. Then either f is a constant or f is univalent in D. Moreover $f_n' \to f'$ on every compact subset of D.

The class S of normalized analytic univalent functions is a compact normal family. The class S and its subclasses have been extensively studied for the last fifty years. We will discuss a few important subclasses of S which will be needed in this thesis. Definitions and related theorems will also be given.

1. 2 STARLIKE AND CONVEX FUNCTIONS

Definition 1.2.1 [4]

A set E in a complex plane is *starlike with respect to a point* $w_0 \in E$ if the linear segment joining w_0 to any other point $w \in E$ lies entirely in E. That is if we denote the line segment by $[w_0, w]$, then $[w_0, w] \subset E$.

If $w_0 = 0$, then the set E is simply referred to as a starlike set. For $w_0 \in \partial E$, (w_0 a boundary point of E), set E is said to be *starlike with respect to the boundary point* w_0 if $[w, w_0) \subset E$.

Definition 1.2.2 [3]

A *starlike function* is a function f that maps U onto a domain starlike with respect to the origin.

We shall denote by S* the class of analytic functions f which are starlike with respect to the origin and normalized such that $f(0) = f'(0) - 1 = 0$.

We now give an analytic description of the class S*.

Theorem 1.2.1 [6]

Let $f \in S$. Then the following are equivalent:

(i) $f \in S^*$,

(ii) the domain $D_r = f(U_r)$ where $U_r = \{ z : |z| < r \}$, $0 < r < 1$, is starlike with respect to the origin,

(iii) $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in U.$

Proof.

(i) $\Rightarrow$ (ii). Since $f(U)$ is starlike, this implies that all points $t f(z) \in f(U)$ for every $z \in U$ and $0 \le t \le 1$. This can be expressed in the form $tf(z) = f[w(z)]$ where w is analytic in U, and $|w(z)| \le 1$. Further $w(0) = 0$. It follows from Schwarz Lemma (Lemma 1.1.1), that

$$(1.2.1) \qquad t f(z) = f[w(z)], \qquad |w(z)| \le |z|, \qquad 0 \le t \le 1.$$

This is a necessary and sufficient condition for $w = f(z)$ to map U onto a starlike domain.

Let $w_0 \in D_r$ and $0 \le t \le 1$. Then $|z_0| = |f^{-1}(w_0)| < r$ and (1.2.1) gives

$$|f^{-1}(tw_0)| = |w(z_0)| \le |z_0| < r.$$

This shows that $tw_0 \in D_r$ for $0 \le t \le 1$. Hence D_r is starlike.

(ii) $\Rightarrow$ (i). This is clear since every point of $f(U)$ are in D_r for some r, $0 < r < 1$.

(ii) $\Leftrightarrow$ (iii). From (ii) the image of the circle $|z| = r$, $r < 1$ is starlike. Hence if the point $z = re^{i\theta}$ describes this circle in the positive sense, the argument ϕ of the image point $f(z) = Re^{i\phi}$ must always vary in the same direction. If this were not the case, there would be rays emanating from $w = 0$, which intersect this curve more than once. Since this direction is necessarily the positive one, we have

$$0 \le \frac{\partial}{\partial\theta}\left[\arg f(re^{i\theta})\right] = \frac{\partial\phi}{\partial\theta}.$$

Conversely if this condition is satisfied, the curve above is starlike.

Note that the condition can be written as

$$0 \le \frac{\partial}{\partial\theta}\left[\arg f(re^{i\theta})\right] = \frac{\partial}{\partial\theta}\left[\operatorname{Im}\log f(re^{i\theta})\right] = \operatorname{Im}\left[\frac{izf'(z)}{f(z)}\right]$$
$$= \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

Let $g(z) = zf'(z)/f(z)$. Since $g(0) = 1$, the minimum principle for harmonic functions (Theorem 1.1.5) implies that $\operatorname{Re}\{zf'(z)/f(z)\} > 0$.

M.S. Robertson [22] had extended the class S^* to the class $S^*(\alpha)$, $0 \le \alpha < 1$. We give preliminary definitions and theorems concerning this class.

Definition 1.2.3 [22]

Let $f \in S$ and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\,,\quad 0 \le \alpha < 1\,,\quad z \in U.$$

Then f is said to be a *starlike function of order* α.

The class of functions satisfying the above conditions shall be denoted by $S^*(\alpha)$. The case $\alpha = 0$ reduces to the class S^*.

Theorem 1.2.2 [22]

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$ and $0 \le \alpha < 1$. Then

$$\left| \arg \frac{f(z)}{z} \right| \le 2(1-\alpha)\sin^{-1}|z|.$$

This inequality is sharp for $f(z) = z/(1-xz)^{2(1-\alpha)}$, $|x| = 1$.

We now give another important subclass of S which is contained in the class S^*. This class is called the class of convex functions.

Definition 1.2.4 [6]

A set E in a complex plane $\mathbb{C}$ is *convex* if the linear segment joining any two points of E lies entirely in E.

Clearly a convex domain is starlike with respect to each of its points.

Definition 1.2.5 [2]

A function f is a *convex function* if it maps U onto a convex domain.

We shall denote by K the class of analytic convex functions, normalized such that $f(0) = f'(0) - 1 = 0$. Every convex function is starlike but a starlike ($K \subset S^*$) function is not necessarily convex.

Theorem 1.2.3 [6]

Let $f \in S$. Then the following are equivalent :

(i) $f \in K$,

(ii) the domain $D_r = f(U_r)$ where $U_r = (z : |z| < r)$, $0 < r < 1$ is convex,

(iii) $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad |z| < 1,$

(iv) $zf'(z) \in S^*$. (Alexander's theorem)

Proof.

(i) $\Rightarrow$ (ii). The map yielded by $w = f(z)$ is convex if and only if, the function $w_1 = f(z) - f(\varsigma)$ maps U onto a starlike domain for any $\varsigma \in U$. It follows therefore that w_1 maps U_r onto a starlike domain provided $|\varsigma| < r$. Since ς is otherwise arbitrary, the image of U_r by $w = f(z)$ is thus a convex domain.

(ii) $\Rightarrow$ (i). This is clear since any two points in $f(U)$ are in D_r for some r, $0 < r < 1$.

(ii) $\Leftrightarrow$ (iii). Let γ denote the curve $w(\theta) = f(re^{i\theta})$. The angle between the tangent to the curve γ at the point w and the positive real axis is $\arg(\partial f(re^{i\theta}) / \partial\theta)$. The curve γ is convex if and only if this angle grows monotonically as z traverses the circle $|z| = r$, that is

$$0 \le \frac{\partial}{\partial\theta}\left\{ \arg\frac{\partial}{\partial\theta}(f(re^{i\theta})) \right\}$$

$$= \frac{\partial}{\partial\theta}\left(\arg(izf'(z))\right) = \operatorname{Im}\left(\frac{\partial}{\partial\theta}\log(izf'(z))\right) = \operatorname{Im} i\left(\frac{f'(z)+zf''(z)}{f'(z)}\right)$$

$$= \operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right).$$

Let $g(z) = zf''(z)/f(z)$. Since $g(0) = 1$, the minimum principle for harmonic functions ensures that $\operatorname{Re}\{1 + zf''(z)/f(z)\} > 0$.

(iii) $\Leftrightarrow$ (iv) The function f satisfies (iii) if and only if $g(z) = z f'(z)$ satisfies (iii) in Theorem 1.2.1.

Directly related to the class of starlike functions and several other important classes of univalent functions is the class of functions with positive real part, which is denoted by P. This along with the class of close-to-convex functions will be discussed in this next section.

1.3 FUNCTIONS WITH POSITIVE REAL PART AND CLOSE -TO -CONVEX FUNCTIONS

Definition 1.3.1 [4]

An analytic function $p(z) = 1 + p_1z + p_2z^2 + \cdots$, is called *function with positive real part* if $\operatorname{Re} p(z) > 0$ and $p(0) = 1$ for $z \in U$.

The class of functions satisfying the above conditions shall be denoted by P. If $p \in P$, then there exists a sequence of functions $\{p_n\}$ so that P_n has the form

$$p_n(z) = \sum_{k=1}^{n}\delta_k \frac{1+\varepsilon_k z}{1-\varepsilon_k z}$$

and $|\varepsilon_k| = 1$, $\delta_k \geq 0$, $\sum_{k=1}^{n} \delta_k = 1$ and $p_n \to p$ uniformly on a compact subsets of U.

An example of a function in P is $p(z) = (1+z)/(1-z)$ where p maps the unit disk U onto the right-half plane.The above function is extremal in many problems in the class P.

Theorem 1.3.1 [4]

The mapping from S* to P defined by $p(z) = zf'(z)/f(z)$ is a one-to-one and f is uniquely defined by

$$f(z) = z\exp\left(\int_0^z \frac{p(t)-1}{t}dt\right).$$

Definition 1.3.2 [2]

A *Linear transformation* is a transformation of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a,b,c,d are complex constants and $ad - bc \neq 0$.

Theorem 1.3.2 [4]

Linear transformation maps circles onto circles or lines.

Theorem 1.3.3 [10]

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P$. Then

$$|p_n| \leq 2, \quad n \geq 1.$$

This coefficient bound is sharp for each n.

Proof.

Let p be analytic in $|z| \le 1$ and consider

$$I = \frac{1}{2\pi i} \int_{|z|=1} p(z) \left(2 - z^n - \frac{1}{z^n}\right) \frac{dz}{z}.$$

The residue theorem and the fact that $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ gives

$$I = 2 - p_n$$

Let $z = e^{i\theta}$. Then

$$I = \frac{2}{\pi} \int_0^{2\pi} p(e^{i\theta}) \,[\, 2(1 - \cos n\theta) \,]\, d\theta = \frac{2}{\pi} \int_0^{2\pi} p(e^{i\theta}) \sin^2 \frac{n\theta}{2} \, d\theta.$$

Hence

$$\operatorname{Re} I = \frac{2}{\pi} \int_0^{2\pi} \operatorname{Re}\{ p(e^{i\theta})\} \sin^2 \frac{n\theta}{2} \, d\theta .$$

Since $\operatorname{Re} p(e^{i\theta}) \ge 0$, it follows therefore that $\operatorname{Re} I \ge 0$ or

(1.3.1) $$\operatorname{Re}(p_n) \le 2 .$$

If $\operatorname{Re} p(z) \ge 0$, then it is clear that $\operatorname{Re} p(e^{i\theta} z) \ge 0$, $(0 \le \theta < 2\pi)$. Also the transformation $z \to ze^{i\theta}$ is a rotation of the unit disk about the origin.The n-th coefficient for $p(ze^{i\theta})$ is then $e^{in\theta} p_n$. Hence by (1.3.1)

$$\operatorname{Re}(e^{in\theta} p_n) \le 2, \qquad 0 \le \theta < 2\pi.$$

Taking in particular a value θ such that $n\theta + \arg\{p_n\} = 0$, we obtained $e^{in\theta} p_n = |p_n|$ and therefore $|p_n| \le 2$.

This proof is valid if p is analytic in a close unit disk. If p is analytic only in U, we note that $p_1(z) = p(rz)$, $0 < r < 1$, is also analytic in $\overline{U}$, and has positive real part. The n-th coefficient is $r^n p_n$. Hence $r^n |p_n| \le 2$, where r is any value between 0 and 1. As r approaches

1, we then obtained $|p_n| \le 2$. The bound is sharp for function

$p(z) = (1+z)/(1-z) = 1+2z+2z^2+\cdots$.

Theorem 1.3.4 [6]

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P$. Then

$$\frac{1-r}{1+r} \le \operatorname{Re} p(z) \le |p(z)| \le \frac{1+r}{1-r}, \quad |z| = r$$

The above result is best possible, as demonstrated by $p(z) = (1+z)/(1-z)$.

Theorem 1.3.4 yields the following theorem.

Theorem 1.3.5 [4]

Let $f \in S^*(\alpha)$ and let

$$f_1(z) = \frac{f(z)}{(1-z)^{2\alpha}} \text{ and } f_2(z) = z\left(\frac{f(z)}{z}\right)^{1/1-\alpha}.$$

Then f_1, f_2 belong to S^*.

Proof.

We note that

$$\frac{zf'(z)}{f(z)} = \frac{zf_1'(z)}{f_1(z)} - \frac{2\alpha z}{1-z},$$

that is

$$\operatorname{Re}\left\{\frac{zf_1'(z)}{f_1(z)}\right\} = \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} + \frac{2\alpha z}{1-z}\right\}.$$

Theorem 1.3.4 then gives

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \ge \frac{1+2\alpha z-|z|}{1+|z|}$$

and hence $\operatorname{Re}\{zf_1'(z)/f_1(z)\} \ge [1-|z|]/[1+|z|] \ge 0$. Also

$$\text{Re}\left\{\frac{zf_2'(z)}{f_2(z)}\right\} = \frac{1}{1-\alpha}\,\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > 0 \,.$$

since $f \in S^*(\alpha)$.

Let B denote the class of functions w that are analytic in U and satisfiy $|w(z)| \leq 1$, $z \in U$. The class B and P are related through the linear transformation

$$p(z) = \frac{1-w(z)}{1+w(z)}\,, \quad w \in B.$$

Theorem **1.3.6** [6]

If $f(z) = \sum_{n=0}^{\infty} b_n z^n \in B$, then

(i) $|b_0| \leq 1$,

(ii) $|b_1| \leq 1 - |b_0|^2$.

Proof.

(i) Schwarz Lemma (Lemma 1.1.1) yields $|b_0| = |f'(0)| \leq 1$.

ii) Let

$$g(z) = \frac{f(z) - f(\varsigma)}{1 - \overline{f(\varsigma)}f(z)}\,, \quad \varsigma \in U.$$

Since g is a composition of linear transformations from U onto U, then $g \in B$ and $g(\varsigma) = 0$.

The function

$$h(z) = \frac{g(z)}{\dfrac{z-\varsigma}{1-\bar{\varsigma}z}} = \left(\frac{f(z)-f(\varsigma)}{z-\varsigma}\right)\left(\frac{1-\bar{\varsigma}z}{1-\overline{f(\varsigma)})f(z)}\right), \tag{1.3.2}$$

is analytic at $z = \varsigma$ and also at all other points of U. Furthermore $h(z)$ is bounded in U. Indeed $\overline{\lim}_{|z|\to 1}|g(z)| \leq 1$ and $|(z-\varsigma)/(1-\bar{\varsigma}z)| = 1$ for $|z| = 1$.Hence by maximum principle $|h(z)| \leq 1$ in U. Setting $z = \varsigma$ in (1.3.2) we thus have

$$|f'(\varsigma)|\left(\frac{1-|\varsigma|^2}{1-|f(\varsigma)|^2}\right) \leq 1,$$

and hence

$$|f'(\varsigma)| \leq \frac{1-|f(\varsigma)|^2}{1-|\varsigma|^2}.$$

In particular at $\varsigma = 0$ we obtain the result .

This will again be proven using Leverenz[17] theorem in chapter two.

From the coefficients estimates of functions in the class P, we can now proceed to obtain coefficients estimates for functions in the class S* and K.

Theorem 1.3.7 [3]

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$. Then

$$|a_n| \leq n, \quad n \geq 2.$$

The result is sharp for each n.

Proof.

If $f \in S^*$, then from Theorem 1.2.1 (iii), Re $\{z f'(z) / f(z)\} > 0$. Let

$$\frac{zf'(z)}{f(z)} = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots.$$

Theorem 1.3.3 then gives the coefficients bound $|p_n| \leq 2$ for $n = 1, 2, \cdots$, but

$$zf'(z) = \frac{zf'(z)}{f(z)} f(z)$$

$$= \left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(\sum_{n=1}^{\infty} a_n z^n\right), \quad a_1 = 1,$$

$$= z + \sum_{n=2}^{\infty} n a_n z^n .$$

Comparing coefficients of z^n, we obtained

$$na_n = a_n + \sum_{k=1}^{n-2} a_{n-k}p_k + p_{n-1} \tag{1.3.3}$$

for $n = 3,4, \cdots$. In particular for $n = 2$, we get $2a_2 = a_2 + p_1$. Hence $|a_2| = |p_1| \le 2$.

Assume that $|a_k| \le k$, for $k = 2,3,\cdots,n-1$. Then (1.3.3) gives

$$(n-1)\,|a_n| = \left|\sum_{k=1}^{n-2} a_{n-k}p_k + p_{n-1}\right| \le \sum_{k=1}^{n-2} 2\,|a_{n-k}| + 2 \le 2\left(1 + \sum_{k=2}^{n-1} k\right)$$

$$= (n-1)\,n.$$

The result then follows by induction. Sharpness can be seen by taking f to be the Koebe function in (1.1.1).

Theorem 1.3.8 [26] (Trimble Inequality)

If $f \in K$, the following inequality is satisfied:

$$\left|a_2^2 - a_3\right| \le \frac{(1-|a_2|^2)}{3}.$$

Proof.

Let

$$h(z) = 1 + \frac{zf''(z)}{f'(z)}, \qquad \phi(z) = \frac{1-h(z)}{1+h(z)}. \tag{1.3.4}$$

Then ϕ is analytic in U and $|\phi(z)| \le 1$, hence $\phi \in B$. Expanding (1.3.4) we get $\phi(z) = -a_2z + 3(a_2^2 - a_3)z^2 + \cdots$, which gives $3|a_2^2 - a_3| \le (1-|a_2|^2)$ from Theorem 1.3.6(ii) applied to $\phi(z)/z$. This then implies the result.

Theorem 1.3.9 [14]

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$. Then

$$|a_n| \leq 1 , \quad n \geq 2 .$$

Proof.

Let $g(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$. Since $f \in K$, from Alexander's theorem (Theorem 1.2.3(iv)), $g \in S^*$. By Theorem 1.3.7 we get $|na_n| \leq n$ or $|a_n| \leq 1$. Sharpness is obtained for the function

$$f(z) = 1 + z + z^2 + \cdots + z^n + \cdots = \frac{z}{1-z},$$

which maps U onto the half-plane $\operatorname{Re} f(z) > -1/2$.

Kaplan [16] in 1952 introduced another class of functions in S analytic and univalent and containing the class S*. This class is called the class of close- to -convex functions .

Definition 1.3.3 [4]

A function f analytic in U is said to be *close- to- convex* if there exists a function $g \in S^*$ and β a real number such that

$$\operatorname{Re} e^{i\beta} \frac{zf'(z)}{g(z)} > 0, \quad z \in U.$$

The class of normalized close-to-convex functions will be denoted by C. Every starlike function is close-to-convex and the containment of the classes of univalent functions introduced so far is such that

$$K \subset S^* \subset C \subset S.$$

Theorem 1.3.10 [2] (Noshiro-Warschawski Theorem)

If f is analytic in a convex domain D and $\operatorname{Re} f'(z) > 0$, then f is univalent.

Proof.

Let $z_1, z_2 \in D$ and $z_1 \neq z_2$. Let ξ denote the closed line segment from z_1 to z_2 parameterized by $z = z_1 + t(z_2 - z_1)$, $0 \leq t \leq 1$. Since D is a convex domain we may write

$$f(z_2) - f(z_1) = \int_{\xi} f'(z)\, dz = \int_0^1 f'(z)(z_2 - z_1)\, dt$$

$$= (z_2 - z_1) \int_0^1 f'(z)\, dt \neq 0,$$

that is, $f(z_1) \neq f(z_2)$. This shows that f is univalent in D.

Theorem 1.3.11 [4]

Every close- to-convex function is univalent.

Proof

Let $f \in C$. Then f is analytic and from Definition 1.3.3 $\operatorname{Re}\{ e^{i\beta} f'(z) / h'(z) \} > 0$, for some convex function $h \in K$. Let E be the range of h, and consider the function $T(w) = f(h^{-1}(w))$, $w \in E$. Then

$$T'(w) = \frac{f'(h^{-1}(w))}{h'(h^{-1}(w))} = \frac{f'(z)}{h'(z)},$$

and so $\operatorname{Re}\{ e^{i\beta} T'(w) \} > 0$ in E and hence from Theorem 1.3.10 f is univalent.

Theorem 1.3.12 [10]

Let $f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$. Then

$$|c_n| \le n.$$

The result is sharp for each n.

Proof.

Since $f \in C$, from Definition 1.3.3 and Theorem 1.2.3(iv) there exists a function $h \in K$ and a real number β such that

$$\operatorname{Re}\left\{e^{-i\beta}\frac{f'(z)}{h'(z)}\right\} > 0.$$

Let

$$q(z) = e^{-i\beta}\frac{f'(z)}{h'(z)} = \sum_{n=0}^{\infty} q_n z^n = e^{-i\beta} + \cdots,$$

and

$$p(z) = \frac{q(z) + i\sin\beta}{\cos\beta} = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Since Re $p(z) > 0$, therefore from Theorem 1.3.3, $|p_n| \le 2$ and $q_n = p_n \cos\beta$ implies that $|q_n| \le 2$. If $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \le 1$ by Theorem 1.3.9. As $q_o = e^{-i\beta}$, the relation $e^{-i\beta} f'(z) = h'(z)q(z)$ implies that

$$nc_n = e^{i\beta}\left[nb_n e^{-i\beta} + (n-1)b_{n-1}q_1 + (n-2)b_{n-2}q_2 + \cdots + q_{n-1}\right].$$

Hence

$$n|c_n| \le n|b_n| + (n-1)|b_{n-1}|\,|q_1| + (n-2)\,|b_{n-2}|\,|q_2| + \cdots + |q_{n-1}|$$

$$\le n + 2(n-1) + 2(n-2) + \cdots + 2.2 + 2 = n^2,$$

that is

$$|c_n| \le n.$$

Since the Koebe function is in C, the result is sharp for each n.

1.4 FUNCTIONS STARLIKE WITH RESPECT TO A BOUNDARY POINT

The class $S^*(\alpha)$ and its subclasses have been extensively investigated. However not much seems to be known about the class of analytic functions that map U onto *domains that are starlike with respect to a boundary point.* Egerváry [12] seems among the early researchers to have come across such functions in his investigations on the Cesàro partial sums of the geometric series $\sum_{n=1}^{\infty} z^n$.

M.S. Robertson [22] in 1981 was the first to initiate a systematic study of this class. We now give definitions and known results for this class of functions which will be further discussed in chapters three and four.

A domain Δ is *starlike with respect to a boundary point* w_0 if $t\,w + (1-t)\,w_0 \in \Delta$ for $0 < t \leq 1$, $w \in \Delta$. A function g which maps U onto Δ is said to be *starlike with respect to a boundary point.* Without loss of generality, by a rotation (the domain being starlike is invariant under rotation) we may assume that $w_0 = g(1) = \lim_{r\to 1} g(r)$. This limit is a radial limit as r approaches 1 from left. We may also assume that $g(0) = 1$.

Definition 1.4.1 [22]

Let G denote the class of functions $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, analytic and non-vanishing in U, normalized so that $g(0) = 1$ and such that

$$\operatorname{Re}\left[\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right] > 0\ , \quad z \in U.$$

The class G is closely related to the class S^*, as demonstrated by the following theorem.

Theorem 1.4.1 [23]

Let g be analytic in U with $g(0) = 1$. Then $g \in G$ if and only if, there is a function $f \in S^*(1/2)$ such that

$$g(z) = (1-z)\frac{f(z)}{z}.$$

Proof.

Let $g(z) = (1-z)\, f(z)/z$, $f \in S^*(1/2)$. Then g is analytic and non- vanishing in U, $g(0) = 1$ and $z\,g'(z) + g(z) = (1-z)\, f'(z) - f(z)$. Hence

$$\mathrm{Re}\left\{\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right\} = \mathrm{Re}\left\{\frac{2zf'(z)}{f(z)} - 1\right\} > 0, \quad z \in U.$$

Thus $g \in G$.

Conversely, if $g \in G$, let $f(z) = z\,g(z)/(1-z)$. Then $f(0) = 0$ and $f'(0) = 1$. Further

$$\mathrm{Re}\left[\frac{2zf'(z)}{f(z)} - 1\right] = \mathrm{Re}\left[\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right] > 0,$$

which implies that $f \in S^*(1/2)$.

We note that the above condition is equivalent to saying $g \in G$, if and only if there exists an $f \in S^*$ such that

$$g^2(z) = \frac{f(z)\,(1-z)^2}{z}.$$

Theorem 1.4.2 [23]

Let $g \in G$. Then either g is a univalent and close- to -convex function in U or g is the constant 1.

Proof.

Let $g \in G$ and assume that $g(z)$ is not the constant function 1. Let $f(z) = zg(z)/(1-z)$. Then $f \in S^*(1/2)$ by Theorem 1.4.1. Let $0 < r < 1$ and

(1.4.1) $$g_r(z) = (f(rz)/rz)(1-z).$$

Then as $r \to 1$, $g_r(z) \to g(z)$. Since $f(rz)/r \in S^*(1/2)$ it follows from Theorem 1.4.1 that $g_r \in G$. From (1.4.1) we note that

$$\frac{-r(1-z)\; zg'_r(z)}{f(rz)} = \frac{-(1-z)^2}{z}\, rz\, \frac{f'(rz)}{f(rz)} - (1-z^{-1})$$

which is analytic in $|z| \le 1$. For $z = e^{i\theta}$, we have

$$\mathrm{Re}\left[\frac{zg_r'(z)}{-h_r(z)}\right] = 2\,(1-\cos\theta)\left[\mathrm{Re}\left\{\frac{rzf'(rz)}{f(rz)}\right\} - \frac{1}{2}\right] \ge 0,$$

where $h_r(z) = f(rz)/r(1-z) \in S^*$ from Theorem 1.3.5. Hence for $z \in U$, $\mathrm{Re}\{-zg'/h\} \ge 0$ where $g(z) = \{f(z)/z\}(1-z)$ and $h(z) = f(z)/(1-z) \in S^*$. Since $g(z)$ is not a constant, by the minimum principle (Theorem 1.1.5), $\mathrm{Re}\{-zg'/h\} > 0$ in U. This means that g is univalent and close- to -convex in U. We see that $g(z) \equiv 1$ only when $f(z) = z/(1-z)$.

Definition 1.4.2 [2]

Let $\{D_1, D_2, D_3, \cdots\}$ be a sequence of domains in $\mathbb{C}$, all containing the origin and none coinciding with $\mathbb{C}$. The *kernel* of the sequence $\{D_n\}$ is the largest domain D containing the origin and having the property that each compact subset of D lies in all but a finite number of domains D_n.

The sequence $\{D_n\}$ is said to *converge* to its kernel if every subsequence of $\{D_n\}$ has the same kernel. This is indicated by $D_n \to D$.

Theorem 1.4.3 [2] (Caratheodory Kernel Theorem)

Let $\{D_n\}$ be a sequence of simply connected domains with $0 \in D_n \subsetneq \mathbb{C}$, $n = 1,2,\cdots$. Let f_n map the unit disk U conformally onto D_n and satisfies $f_n(0) = 0$ and $f_n'(0) > 0$. Let