A NOVEL DYNAMIC EVOLUTIONARY MODEL INTEGRATING DISCRETE HOPFIELD NEURAL NETWORKS WITH SATISFIABILITY PROBLEMS AND ITS APPLICATIONS IN IMAGE ENCRYPTION AND DECRYPTION

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UNIVERSITI SAINS MALAYSIA

2025

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by

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Thesis submitted in fulfilment of the requirements
for the degree of
Doctor of Philosophy

June 2025

ACKNOWLEDGEMENT

I am deeply grateful to everyone who has helped me along the way. First and foremost, I would like to express my sincere thanks to my supervisor, Prof. Saratha Sathasivam. She has not only provided me with invaluable academic knowledge and guidance but also offered steadfast support and encouragement during moments of difficulty in my research. Her smile, like the warm sunshine in the depths of winter, has given me boundless strength and enabled me to persevere until today.

I would also like to extend my heartfelt thanks to all the lecturers and staff at Universiti Sains Malaysia and the School of Mathematical Sciences. As a Chinese international student, it was their care and assistance over the years that allowed me to successfully complete my Ph.D. studies in a foreign land.

Lastly, I am especially grateful to my family—my husband Cheng Guohua, and my two beloved daughters Cheng Yuying and Cheng Xiaotong， my mother Ou Yulian—as well as my father-in-law Cheng Shaohu and mother-in-law Guan Xiaoying. Their endless love, sacrifice, and encouragement have helped me realize my dream. I also deeply miss my late father Feng Sizong, and I believe he is proud of my achievements from heaven.

Once again, thank you to all my friends who have supported me. May everyone continue to strive in their studies and life, and may they achieve their dreams and happiness.

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LIST OF SYMBOLS

$\wedge$	Conjunction
$\vee$	Disjunction
$\neg$	Negation
$\leftarrow$	Implication
$\cup$	Union
$\cdot$	Dot Product
$\mid \quad \mid$	Absolute Value
σ	Tolerance

LIST OF ABBREVIATIONS

HNN	Hopfield Neural Network
DHNN	Discrete Hopfield Neural Network
SAT	Boolean Satisfiability Problem
WA	Wan Abdullah method
FCM	Fuzzy c-means clustering method
CSA	Crow search algorithm
CSAFC	Crow search-guided fuzzy clustering algorithm
GA	Genetic algorithm
GAFC	Genetic algorithm-guided fuzzy clustering algorithm
DHNN-SAT	Discrete Hopfield-Satisfiability Neural Network
DHNN-SAT-Dy-Ev	Dynamically Evolving Discrete Hopfield-Satisfiability Neural Network
DHNN-SAT-CSAFC	Discrete Hopfield-Satisfiability Neural Network Based on Fuzzy Clustering Hybrid Optimization Guided by Crow Search Algorithm
DHNN-SAT-MD	Modular Dynamic Discrete Hopfield-Satisfiability Neural Network
DHNN-SAT-MD-1	Modular Dynamic Discrete Hopfield Satisfiability Neural Network with a Unique Solution
DHNN-SAT-MD-1-ED	Novel Intelligent Color Image Encryption and Decryption Technology Based on DHNN-SAT-MD-1 Network
GMR	Global minima ratio
PSNR	Peak signal-to-noise ratio
SSIM	Structural similarity index

MODEL NOVEL EVOLUSI DINAMIK MENGINTEGRASIKAN RANGKAIAN NEURAL DISKRET HOPFIELD DENGAN MASALAH KEPUASAN DAN APLIKASINYA DALAM PENYULITAN DAN PENYAHSULITAN IMEJ

ABSTRAK

Dalam tahun-tahun kebelakangan ini, rangkaian hibrid yang menggabungkan rangkaian neural Hopfield diskret dan masalah kepuasan telah mengalami pembangunan yang pesat. Tesis ini membentangkan penghasilan satu siri varian DHNN-SAT yang inovatif dan aplikasinya. Untuk menangani ketidakcekapan rangkaian DHNN-SAT tradisional dalam menyelesaikan masalah SAT dengan konstrain dinamik, rangkaian neural Hopfield-SAT Diskret Evolusi Dinamik dengan seni bina yang fleksibel dan berskala direka khusus. Untuk menangani cabaran yang ditimbulkan oleh rangkaian skala yang berbeza-beza dan kerumitan logik, rangkaian yang dioptimumkan berdasarkan Kaedah Hibrid Berkelompok Fuzzy berpandukan Algoritma Carian Gagak dicadangkan. Untuk meningkatkan lagi kecekapan dalam menyelesaikan masalah SAT dinamik yang berskala besar dan sangat kompleks, rangkaian neural Hopfield-SAT Diskret Dinamik Modular dibangunkan. Justeru itu, bagi memastikan ketepatan hasil penumpuan, rangkaian DHNN-SAT dinamik modular pula dibangunkan bagi memberikan penyelesaian optimum global yang unik. Akhir sekali, teknik penyulitan dan penyahsulitan imej warna pintar baru dengan kualiti tinggi dan keselamatan yang kukuh dibangunkan berasaskan seni bina. Eksperimen intensif pada set data simulasi dan masalah dunia nyata mengesahkan prestasi unggul rangkaian dan teknik yang telah dicadangkan.

A NOVEL DYNAMIC EVOLUTIONARY MODEL INTEGRATING DISCRETE HOPFIELD NEURAL NETWORKS WITH SATISFIABILITY PROBLEMS AND ITS APPLICATIONS IN IMAGE ENCRYPTION AND DECRYPTION

ABSTRACT

In recent years, hybrid networks combining discrete Hopfield neural networks with satisfiability problems have developed rapidly. This thesis proposes a series of innovative DHNN-SAT variants and their applications. To address the inefficiency of traditional DHNN-SAT networks in solving SAT problems with dynamic constraints, a Dynamically Evolving Discrete Hopfield-SAT Neural Network with a flexible and scalable architecture is specifically designed. To tackle challenges posed by varying network scales and logical complexities, an optimized network based on a Crow Search Algorithm-guided Fuzzy Clustering Hybrid Method is proposed. To further improve the efficiency of solving large-scale and highly complex dynamic SAT problems, a Modular Dynamic Discrete Hopfield-SAT Neural Network is developed. Subsequently, to ensure the accuracy of convergence results, a modular dynamic DHNN-SAT network with a unique global optimal solution is designed. Finally, a novel intelligent color image encryption and decryption technique with high quality and strong security is developed based on the architecture. Extensive experiments on both simulated and real-world datasets validate the superior performance of the proposed networks and techniques.

CHAPTER 1

INTRODUCTION

This chapter provides a comprehensive exploration of the research background for this thesis, encompassing neural networks, symbolic neural networks, discrete Hopfield neural networks, the satisfiability problem (SAT), and the development and concept of the discrete Hopfield-satisfiability (DHNN-SAT) neural network. Moreover, this chapter thoroughly discusses the research significance, problem statement, objectives, methodology, scope, and limitations. Finally, an introduction to the layout and structure of the entire thesis will be presented.

1.1 Research Background

Neural network research originated in the 1940s when McCulloch and Pitts first proposed a mathematical model to describe neuronal activity (McCulloch and Pitts 1943), laying the foundation for modern neural network theory. Subsequently, in 1949, Hebb introduced Hebbian learning theory, which explains how neurons strengthen their connections through repeated activation. This theory has profoundly impacted the development of subsequent neural network models (Hebb 1949).

In the 1980s, the introduction of the backpropagation algorithm marked a significant advancement in neural network research, making the training of multi-layer networks feasible (Rumelhart et al. 1986). With the advent of the 21st century, thanks to increased computational power and the widespread availability of big data, deep learning has come to dominate several domains of artificial intelligence, including image recognition, natural language processing, and complex decision-making systems (LeCun et al. 2015).

Against this backdrop, Neural-Symbolic Networks emerged, also known as hybrid connectionist-symbolic networks (Sun and Bookman 1994). These network models combine the logical reasoning capabilities of symbolic systems with the data processing power of neural networks, effectively handling structured data, such as logical rules and knowledge graphs, while optimizing the processing of unstructured data, such as images and texts (Garcez et al. 2008; Besold et al. 2021).

The Discrete Hopfield Network, introduced by John Hopfield in 1982, is a type of recurrent neural network noted for its unique associative memory and pattern recognition capabilities (Hopfield 1982). Its energy minimization property makes it highly effective in optimization computations and problem-solving.

The satisfiability problem (SAT) is a fundamental problem in computer science that involves determining a set of variable assignments that satisfies a given logical formula (Biere et al. 2009; Cook 2023). While traditional algorithms are practical in some scenarios, they often fall short when dealing with large-scale or highly complex problem instances.

With the advancement of neural-symbolic learning techniques, researchers have combined the Discrete Hopfield Network with SAT problems to develop a new neural-symbolic network: the Discrete Hopfield-SAT (DHNN-SAT) network. This network leverages the network's dynamic configuration and optimization of the energy function to efficiently solve SAT problems (Sathasivam et al. 2020a). This network not only demonstrates powerful capabilities in combinatorial optimization and logical reasoning but also provides new solutions for addressing complex SAT problems, as illustrated in Figure 1.1.

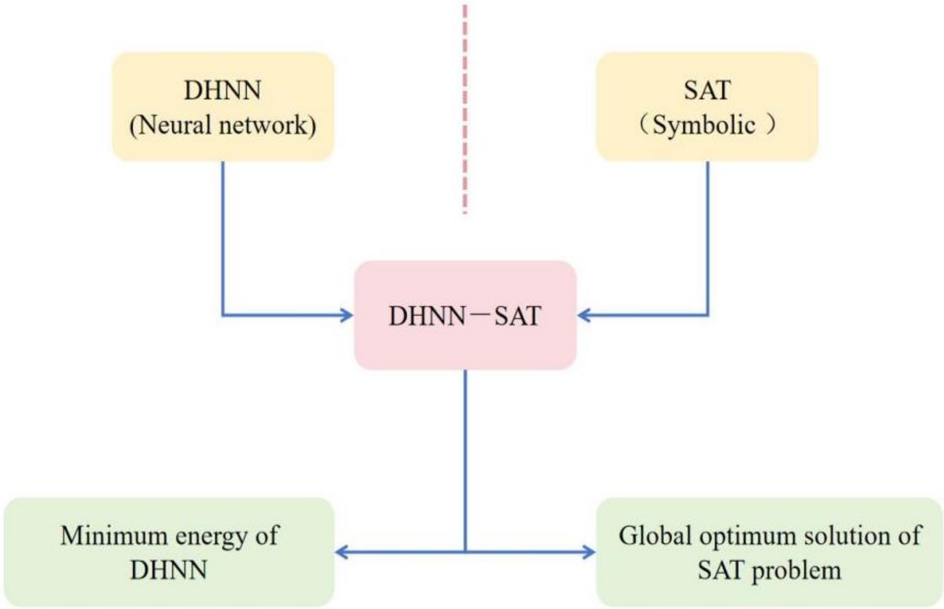

Figure 1.1 Schematic Diagram of New Neural-Symbolic Network DHNN-SAT

1.2 Research Motivation

In the DHNN-SAT network, the SAT problems involve logical rules, variables, and satisfiable solutions. These elements are integrated into the DHNN as synaptic weights through the Wan Abdullah learning method. Driven by dynamic rules, the network evolves towards a state of minimal energy, which is also the globally optimal solution for satisfying the SAT problem. This integration of neural networks with SAT problems forms an innovative neural-symbolic network.

Theoretically, the DHNN-SAT network not only retains the logical reasoning capabilities of symbolic systems but also leverages neural networks' learning and optimization capacities, significantly enhancing the network's overall performance. As a variant of DHNN, the DHNN-SAT network treats symbolic knowledge as valuable training data, providing superior generalization ability over existing neural networks. Moreover, unlike traditional DHNNs, which are prone to getting trapped in local optima, the DHNN-SAT network uses logical rules to guide the direction of convergence, increasing the likelihood of reaching a global optimum. Additionally, as tools for solving SAT problems, the DHNN-SAT network offers an explicit SAT-solving process, which holds substantial importance in Explainable AI. In practical applications, as the logical rules embedded within the DHNN-SAT network continue to expand and evolve, these networks have been applied across various fields, including large-scale integration, model satisfiability, and logic mining.

However, current DHNN-SAT networks are primarily designed for static problems and lack the capability to handle dynamically changing constraints, making them unsuitable for real-world tasks where variables or rules evolve over time. Meanwhile, when dealing with large-scale and high-complexity problems, these networks often suffer from poor convergence, high computational cost, and limited optimization capacity, which significantly hinders their wider adoption in practical applications. In addition, few studies have explored the application of DHNN-SAT networks in intelligent image encryption and decryption, and their feasibility, efficiency, and security in image-related scenarios have not yet been systematically validated.

The motivation of this study is to overcome the limitations of existing DHNN-SAT networks in terms of convergence, scalability, optimization capability, and image security applications, and to construct a novel intelligent neural encryption model that balances theoretical innovation with practical engineering utility.

1.3 Problem Statement

However, as a novel type of symbolic neural network, the DHNN-SAT network currently faces several challenges. The first challenge lies in the fact that existing DHNN-SAT networks are primarily designed for solving static SAT problems, where the Boolean variables and constraint conditions are predefined and fixed. This static network architecture is ineffective in handling dynamically evolving SAT problems.

The second challenge concerns the impact of network scale and the logical complexity of SAT problems on overall performance. When dealing with large-scale and logically complex satisfiability problems, the performance of existing DHNN-SAT networks tends to decline. Therefore, it is essential to develop effective optimization methods to enhance the network's problem-solving capability.

The third challenge arises from the fully connected structure of the existing DHNN-SAT networks, which leads to high computational costs when processing large-scale data and complex tasks. This highlights the need to explore new network architectures or computational mechanisms.

The fourth challenge involves the lack of guaranteed convergence in current DHNN-SAT networks. This is due to the possibility that SAT problems may have multiple consistent satisfying assignments, while discrete Hopfield neural networks inherently possess multiple attractors and pseudo-attractors, resulting in unstable convergence. At present, no research has demonstrated that DHNN-SAT networks can converge to a unique global optimum.

The fifth challenge is the lack of research into applying DHNN-SAT networks for image encryption and decryption. This is because the existing DHNN-SAT network often does not have a definitive convergence result, which prevents them from guaranteeing convergence to the correct image state. Moreover, due to its fully connected architecture, the DHNN-SAT network suffers from low computational efficiency, making it incapable of quickly handling larger-scale image tasks.

1.4 Research Objectives

The primary objective of this study is to develop more efficient and broadly applicable networks and technologies addressing the current issues of the existing DHNN-SAT network. The specific objectives are as follows:

1. To design a new dynamic network architecture and decision-making mechanism to better solve dynamically evolving SAT problems.

2. To introduce optimization algorithms to improve the retrieval accuracy and speed of the network to handle large-scale and logically complex satisfiability problems

3. To design a new modular dynamic architecture and modular computing mechanism to significantly improve efficiency and speed when dealing with large-scale problems.

4. To introduce an implementation strategy capable of constructing CNFs with unique solutions, thereby adapting to SAT problems in specific application scenarios.

5. To design an intelligent colour image encryption and decryption technology based on the DHNN-SAT-MD-1 network (DHNN-SAT-MD-1-ED).

Figure 1.2 provides an illustration of the newly developed networks and the underlying technologies.

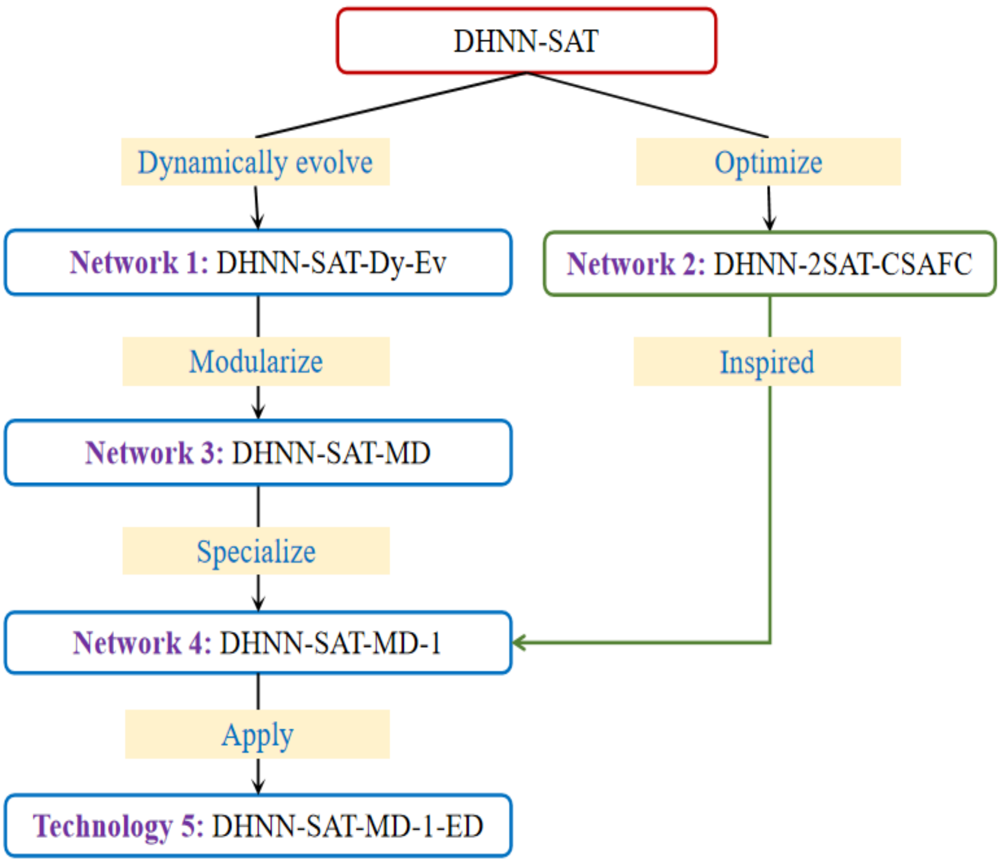

Figure 1.2 Diagram of New Networks and Technologies Being Developed

1.5 Research Significance

This study holds significant value at both theoretical and application levels. Theoretically, it addresses the limitations of current DHNN-SAT networks in handling dynamic constraints, increasing logical complexity, low computational efficiency on large-scale data, and lack of convergence guarantees. By introducing a modular, dynamic, and optimization-driven architecture—featuring dynamic evolution mechanisms, Crow Search algorithm-guided hybrid optimization, and modular structural design—this research enriches the integration of symbolic neural networks with satisfiability problem-solving. It provides new perspectives for neural computation models in dynamic logical reasoning and optimization. Moreover, the development of the DHNN-SAT-MD-1 network with a provably unique global optimal solution helps fill the theoretical gap concerning convergence in discrete Hopfield networks, contributing to the theoretical robustness of such models.

Practically, this work is the first to explore the application of DHNN-SAT neural networks to colour image encryption and decryption. The proposed DHNN-SAT-MD-1-based intelligent image encryption and decryption system (DHNN-SAT-MD-1-ED) systematically demonstrates the model's feasibility, stability, and resistance to attacks in image security scenarios. This not only introduces a novel neural network-based paradigm for image information protection but also provides a foundation for building high-performance, scalable, and low-cost image security systems. The outcomes are expected to play a critical role in fields requiring high levels of image data confidentiality, such as national defense, finance, healthcare, and communications, offering strong real-world applicability and broad prospects for adoption.

1.6 Thesis Organization

Chapter 1 details the research background and significance, problem statement, objectives, and methodology. Chapter 2 reviews the progress in related fields, including symbolic neural networks, DHNN, SAT problems, DHNN-SAT network, and image encryption and decryption technologies. Chapter 3 comprehensively elaborates on the core theories and key technologies involved in this study, including DHNN, DHNN-SAT network, the Wan Abdullah learning method, and systematically introduces heuristic algorithms such as the crow search algorithm and fuzzy C-means clustering method. Chapter 4 systematically explains the working principles, research significance, and implementation steps of the newly developed networks and technology, which include DHNN-SAT-Dy-Ev, DHNN-SAT-CSAFC, DHNN-SAT-

MD, DHNN-SAT-MD-1 networks, and the DHNN-SAT-MD-1-ED technology. Chapter 5 presents the simulation results of these three new networks, DHNN-SAT-Dy-Ev, DHNN-SAT-CSAFC, and DHNN-SAT-MD, on simulated datasets and comprehensively evaluates their performance and advantages by comparing them with other traditional networks. Chapter 6 validates the effectiveness and advancement of the proposed networks and technology on real datasets. Chapter 7 summarizes the results of this study and provides an outlook on future research directions and suggestions.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter conducts a systematic literature review covering topics such as symbolic neural networks, Hopfield neural networks (DHNN), Boolean satisfiability problems (SAT), Discrete Hopfield-Satisfiability Neural Networks (DHNN-SAT), and image encryption and decryption technologies. DHNN-SAT, the focus of this research, represents a specialized form of symbolic neural networks. It combines the associative memory characteristics of discrete Hopfield neural networks with the logical rigor of SAT problem solving, showcasing significant potential in addressing complex optimization issues and encryption tasks. This review aims to provide a comprehensive background and theoretical foundation for further research into DHNN-SAT neural networks by tracing these key technologies' development, theoretical bases, and applications across various fields.

2.2 Neural-Symbolic Networks

Neural-symbolic networks trace their origins to two primary schools of thought in artificial intelligence: symbolism and connectionism. Symbolism, or logicism, is an approach to intelligent reasoning based on symbols and rules. This theory relies on explicit symbol systems and logical rules to simulate cognitive processes, emphasizing rule-based, interpretable ways to express and process knowledge. The core of symbolism lies in explicitly representing knowledge as symbols and structured rules for logical reasoning and decision-making. In contrast, connectionism is typically implemented through artificial neural networks, simulating brain neurons' connections and information processing methods. Connectionism does not rely on explicit rules or symbol systems but processes information through numerous neurons and their connections. These neural networks can learn and recognize patterns from input data, optimizing performance by adjusting the network's connection weights, and are especially adept at processing perceptual data like images and sounds.

Neural-symbolic networks combine symbolism's logical reasoning capabilities with connectionism's perceptual learning abilities, aiming to create computational networks capable of handling both complex logical reasoning and rich perceptual tasks (Valiant 2006; Garcez et al. 2015, 2019; Marcus 2020). In the 1990s, Sun and Bookman first proposed combining neural and symbolic systems, which led to the hybrid model known as the neural-symbolic or hybrid connectionist-symbolic model (Sun and Bookman 1994).

The origins and evolution of neural-symbolic networks span several key historical stages of artificial intelligence, from the initial explorations of symbolism and connectionism to their integrated application. In the 1960s and 70s, symbolism dominated the AI reasoning era, with the "Logical Theorist" program introduced by Newell and Simon at the 1956 Dartmouth Conference (Newell and Simon 1956), successfully proving mathematical theorems, marking the beginning of the AI reasoning era. In the subsequent decades, particularly the 1970s to the 1990s, as domain-specific knowledge was incorporated, AI gradually managed to tackle more complex issues, exemplified by the first expert system "Dendral" created by Feigenbaum and Lederberg in 1965 (Buchanan and Feigenbaum 1981), signalling the era of AI knowledge integration.

Entering the 1990s, with the rise of neural networks, AI entered the machine learning era dominated by neural networks (Rissati et al. 2020). The triumph of deep convolutional neural networks (CNNs) in the 2012 ImageNet competition, and AlphaGo's victory over the world Go champion in 2016, marked the peak of this era. However, neural networks still fall short in tasks requiring complex logic and advanced reasoning abilities, such as in question-answering systems (Gupta et al. 2022), autonomous driving (Wen and Jo 2022), and medical diagnostics (Salahuddin et al. 2022). Furthermore, the issues of logical reasoning capability and interpretability of neural networks remain to be addressed, underscoring the importance of explainable artificial intelligence (Ratti and Graves 2022).

In recent years, neural-symbolic networks have rapidly developed with the fusion of symbolism and connectionism (Manhaeve et al. 2018a; Dong et al. 2019; Mao et al. 2019). This fusion leverages the ability of neural networks to handle complex data, while using the precision of symbolic reasoning to manage logical reasoning and abstract thinking. The core idea of neural-symbolic networks is to embed symbolic systems (such as logical rules) within neural network structures, or through hybrid networks, combining symbolic reasoning with neural computation, effectively addressing complex cognitive tasks. Neural-symbolic networks typically consist of two main components: a symbolic component for handling high-level abstractions and rule-based reasoning and a neural component for handling perceptual tasks and data-driven learning. This fusion enables the network to excel in both perceptual and reasoning tasks.

The significance of neural-symbolic networks lies in their ability to address the limitations of traditional symbolic methods and neural networks. Regarding efficiency, traditional symbolic approaches often rely on search algorithms to process solution spaces, but as the search space expands, so does the computational complexity. The introduction of neural-symbolic networks significantly reduces this complexity and enhances reasoning speed (Zhang et al. 2020); in terms of generalization, by incorporating symbolic knowledge into training data, neural-symbolic networks have an advantage over standalone neural networks, significantly boosting the network's generalizability (Kampffmeyer et al. 2019). In terms of interpretability, unlike neural networks, which are often seen as "black box" systems, neural-symbolic learning systems represent "grey box" systems that use symbolic knowledge to provide explicit computational processes, such as tracing reasoning processes or chains of evidence, offering more transparent decision-making processes, which positions them crucially within the field of explainable AI (Yang and Song 2020).

As shown in Figure 2.1, the research on neuro-symbolic networks can be divided into three main directions: learning for reasoning, where neural networks replace traditional symbolic reasoning algorithms, effectively reducing the search space of symbolic systems, and improving computational efficiency. For example, PLogicNet and ExpressGNN accelerate the search process of solution spaces using neural networks (Qu and Tang 2019). Secondly, the reasoning for learning, where symbolic systems provide additional knowledge or constraints in a format suitable for neural networks, guiding the learning process and enhancing performance and interpretability (Hu et al. 2020; Xie et al. 2019a; Xu et al. 2018; Diligenti et al. 2017). Lastly, the combination of learning and reasoning, where neural networks generate hypotheses or predictions, and symbolic reasoning components use these hypotheses for logical reasoning, and the results are fed back to the neural network to improve predictions. This bidirectional interaction allows neural and symbolic systems to work collaboratively and complement each other (Manhaeve et al. 2018b; Zhou 2019; Gupta et al. 2020; Cai et al. 2021; Tian et al. 2022a).

Neural-symbolic networks have a broad range of applications, demonstrating outstanding performance across various fields (Yu et al. 2023), including computer vision (Donadello et al. 2017; Wang et al. 2018; Li et al. 2019; Xie et al. 2019b; Chen et al. 2020), natural language processing (Liang et al. 2016; Yi et al. 2018; Tian et al. 2022b), knowledge graphs (Neelakantan et al. 2015; Das et al. 2017; Xiong et al. 2018; Meilicke et al. 2020; Zhang et al. 2021), and reinforcement learning (Garnelo et al. 2016), among others. Additionally, they show promising applications in emerging fields such as advanced robotics (Conti et al. 2020) and information extraction related to the COVID-19 pandemic (Ngan et al. 2022), showcasing their extensive application potential.

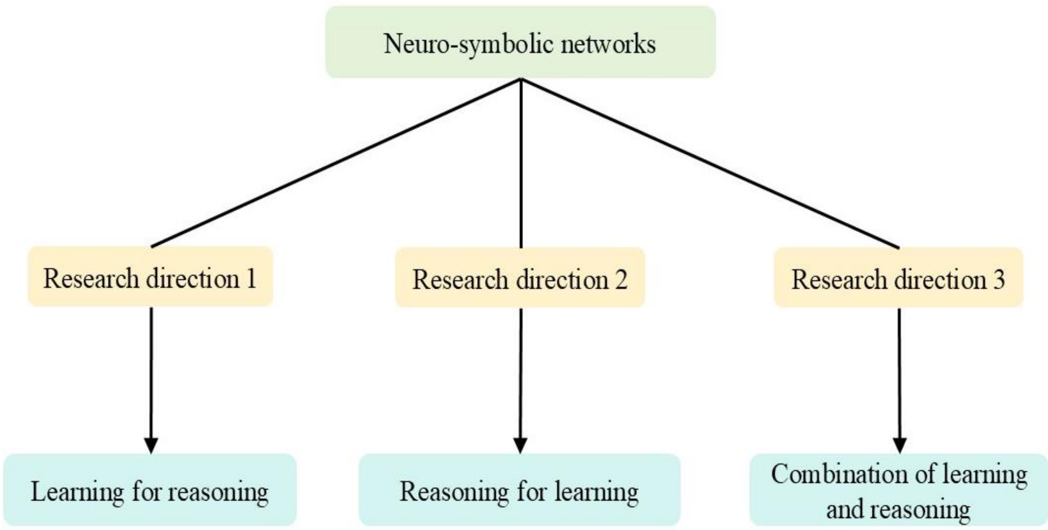

Figure 2.1 Research Directions of Neuro-Symbolic Networks

2.3 Hopfield Neural Networks (HNN)

The Hopfield neural network is a classic network of associative memory. Physicist John Hopfield first introduced a network composed of binary-state neurons known as the discrete Hopfield neural network (DHNN) in 1982, demonstrating the capability to store and retrieve information (Hopfield 1982). In 1984, he extended this network to the continuous Hopfield neural network (CHNN) (Hopfield 1984). He proved that the network's dynamic behaviour could minimize a Lyapunov energy function, thereby demonstrating the network's convergence properties. Theoretically, a Hopfield neural network can minimize any function with proper adjustment of network parameters. In 1985, Hopfield and Tank demonstrated the potential of using the Hopfield neural network to solve the problematic NP-hard optimization problem, the traveling salesman problem (TSP) (Hopfield and Tank 1985). However, in 1988, Wilson and Pawley critiqued the method of Hopfield and Tank for solving optimization problems (Wilson and Pawley 1988), highlighting that in most cases, the Hopfield neural network returned invalid results and lacked systematic methods to adjust a large number of parameters to mitigate this issue. Subsequently, many studies focused on enhancing the reliability of network performance (Aiyer et al. 1990; Chu 1992; Gee 1993; Smith 1999) , yielding valid solutions, including the introduction of hybrid strategies, such as combining the Hopfield neural network with genetic algorithms (Salcedo-Sanz and Yao 2004; Kumar and Singh 2010), simulated annealing (Chen et al. 1998; Salcedo-Sanz et al. 2004), and tabu search (Li et al. 2011), to facilitate escaping local minima and enforcing convergence to better local minima or even global minima.

Originally designed to solve associative memory problems, Hopfield networks achieve stable recall of stored patterns through converging network states. The discrete Hopfield neural network specifically deals with binary pattern storage and recall, iterating initial input patterns into stable stored patterns (attractors). This recall mechanism makes it an efficient tool for associative memory.

Due to its strong noise resistance and implementation simplicity, the discrete Hopfield network played a pivotal role in the early research of neural networks and laid a solid theoretical foundation for subsequent studies. In the academic field, the discrete Hopfield network served as an efficient associative memory network, driving the widespread application of neural networks. It also provided new perspectives for understanding the nonlinear dynamics of complex systems. This, in turn, advanced the theory of neural computation and inspired the development of energy networks such as the Boltzmann machine.

Researchers have continuously worked on enhancing the discrete Hopfield network's storage capacity, noise resistance, and training algorithms. For instance, in 2022, Song et al. optimized the weight matrix of the Hopfield network by introducing a quantum perceptron to replace the traditional Hebbian rule, significantly improving memory capacity and retrieval capabilities. This approach addressed the classic Hopfield network's issues of limited memory capacity and false attractors (Song et al. 2022). In 2023, Krotov proposed integrating modern learning techniques to enhance the performance of Hopfield networks in complex tasks (Krotov 2023). In 2024, Danca and Chen utilized the parameter switching (PS) algorithm to approximate and decompose the attractors of Hopfield neural networks, demonstrating that each attractor could be represented as a convex combination of other attractors (Danca and Chen 2024).

The discrete Hopfield neural network has found extensive applications across various fields (Wang et al. 2020; Xu et al. 2022; Lin et al. 2023), particularly excelling in areas such as handwritten digit recognition (Belyaev and Velichko 2020), image recognition (Liu et al. 2020), face recognition (Soni et al. 2016), image restoration (Sun et al. 1995; Hong et al. 2020), and image encryption and decryption (Lakshmi et al. 2020a; Zhang and Yang 2022).

2.4 Boolean Satisfiability Problem (SAT)

The Boolean Satisfiability Problem (SAT) is fundamental in computer science, with its origins tracing back to the development of mathematical logic, particularly within the realm of propositional logic. With the advancement of computer science, especially the rise of computational complexity theory, the SAT problem has gradually become a research focal point. In 1971, Stephen Cook proved that the SAT problem was the first NP-complete problem, a discovery that established the central role of SAT in computational complexity theory (Cook 2023). Richard Karp elaborated on the NP-completeness of the SAT problem in 1972 and highlighted its importance in other combinatorial problems (Karp 2010).

The SAT problem involves determining whether there exists an assignment of Boolean variables that can make a logical formula composed of Boolean variables and logical operators such as AND, OR, and NOT, evaluates to true. Typically, Boolean logic formulas are expressed in Conjunctive Normal Form (CNF) to facilitate the solving of SAT problems. If an assignment of variables makes the formula true, the problem is considered satisfactory; otherwise, it is unsatisfiable.

Consider a Boolean logic formula, which contains N Boolean variables and M logical clauses . Each clause combines Boolean variables connected by logical operators (AND, OR, NOT). The 2-SAT problem requires each clause to contain precisely two variables, whereas the 3-SAT problem requires each clause to contain three variables.

The SAT problem has many simple examples in everyday life. For instance, they determine a schedule for a series of games that resolves the availability of players and venues or find a seating arrangement for a dinner that satisfies various rules imposed by the host. Although these applications may seem different, they share a core need to determine the values of variables (e.g., who sits in a specific seat at dinner) and ensure these variables satisfy certain constraints (e.g., the seating rules of the host) (Malik and Zhang 2009). A variable refers to a Boolean variable, which takes the values "True" or "False," and is a core term in SAT problems. For example, "Person A sitting at position 1 can be represented as the variable $x_{A,1} = \text{True}$."

The practical applications of the SAT problem are extensive, spanning fields from hardware and software verification to automated planning, software testing, artificial intelligence, and cryptanalysis. For example, in hardware verification, SAT solvers help designers detect potential logical errors by networking the behaviour of complex circuit designs as Boolean formulas and determining their satisfiability (Biere et al. 1999). In cryptanalysis, SAT solvers are used to generate tests for Boolean expressions (Arcaini et al. 2011). Additionally, SAT solvers excel in quickly solving automated planning problems by converting planning tasks into SAT problems, thereby efficiently addressing complex planning challenges (Bard et al. 2007). In knowledge representation and reasoning, the SAT problem is also a core technology (Armando et al. 2005).

Although the SAT problem has significant applications across various fields, its complexity cannot be overlooked. As the problem size increases, the search space grows exponentially, making traditional exhaustive search methods ineffective for large-scale problems. Consequently, researchers have focused on developing efficient algorithms to solve SAT problems, including thorough search, the DPLL algorithm (Davis et al. 1962), the CDCL algorithm (Marques-Silva and Sakallah 1999), branch and bound (Marques-Silva et al. 2021), and learning and backtracking methods. In recent years, specially designed SAT solvers have developed substantially and succeeded in practical applications (Fleury and Heisinger 2020).

2.5 Discrete Hopfield -Satisfiability Neural Network (DHNN-SAT)

In recent years, with the advancements in neural-symbolic learning techniques and the introduction of the Wan Abdullah learning method, a hybrid network combining discrete Hopfield neural networks with SAT problems has been developed. This network, which falls under the category of neural-symbolic networks, integrates symbolic logic with neural network technology. The core mechanism involves embedding the logical rules of SAT problems into the discrete Hopfield neural network through synaptic weights, using dynamic regulations to drive the network toward a minimum energy state, thereby converging to the global optimal solution of the SAT problem (Kasihmuddin et al. 2016a).

In 1991, Pinkas first proposed embedding SAT problems as symbolic language into artificial neural networks, remarkably symmetric neural networks, to manage neurons more effectively. He pointed out that finding the minimum of an energy function using Hopfield networks and Boltzmann Machines through gradient descent is equivalent to finding the truth assignments that satisfy the SAT problem (Pinkas 1991). This discovery provided a new approach to explaining the workings of black-box networks like DHNN, where the satisfiability interpretation of SAT illustrates the network's hidden units and constraints.

In 1992, Abdullah introduced a method for embedding Horn formulas into DHNN, converting logic programming into symbolic rules within neural networks (Abdullah 1992). This method determines the optimal synaptic weights by comparing the energy function with the network's cost function, and in 2011, it was formally named the Wan Abdullah learning method (Sathasivam and Wan Abdullah 2011). This method established the rationality and effectiveness of using the Conjunctive Normal Form (CNF) as the clause representation in neural-symbolic integration (Sathasivam 2010). The logical rules of SAT problems are typically represented in CNF, with the truth values of logical variables represented by neuron states and combinations of logical clauses expressed through different neuron state combinations. The optimal weight matrix representing CNF is learned and stored in the discrete Hopfield neural network by comparing the cost function with the energy function.

In 2014, Sathasivam et al. successfully embedded higher-order Horn SAT into DHNN, further expanding the network's application scope (Sathasivam et al. 2014). Different logical rules typically represent constraints or conditions in various real-world scenarios related to SAT. In recent years, the DHNN-SAT network has undergone multiple expansions. Different SAT logical rules have been successfully embedded into DHNN, successfully yielding the required solutions. For instance, in 2017, Kasihmuddin and others proposed a DHNN embedded with 2-SAT containing two literals (Kasihmuddin et al. 2017), while Mansor and colleagues introduced a DHNN embedded with 3-SAT containing three literals, which represents higher-order logical rules (Mansor et al. 2017). It was reported that these two rules could retrieve final neuron states with global minimum energy in 90% of cases, demonstrating the effectiveness and compatibility of different logical rules in managing neuron connections within DHNN.

Following this, Sathasivam et al. proposed a more flexible DHNN-RAN-2-SAT network in 2020, which includes first-order and second-order logical rules (Sathasivam et al. 2020b). In 2021, Karim et al. proposed a higher-order Random 3-Satisfiability

(RAN3SAT) network (Karim et al. 2021). In 2022, Alway et al. introduced the ratio of significant clauses, proposing Major 2-Satisfiability (MAJ2SAT), composed of high-ratio 2-SAT clauses (Alway et al. 2022). In 2023, Azizan and Sathasivam presented a DHNN combined with 3-SAT fuzzy logic (Azizan and Sathasivam 2023). They collectively refer to this hybrid network, which embeds SAT logical rules into discrete Hopfield neural networks, as the Discrete Hopfield-SAT Neural Network, abbreviated as DHNN-SAT or DHNN-SAT-WA.

The DHNN-SAT neural-symbolic network is an innovative tool for solving SAT problems by embedding them as symbolic language into the DHNN. It transforms complex logical problems into a format that can be processed by neural networks, leveraging the computational power of neural networks to solve these logical problems without requiring additional logical reasoning steps.

In the DHNN-SAT network, each variable and constraint in the logical formula is mapped to neuron states and synaptic weights. Each neuron corresponds to a logical variable in the SAT problem, with its activation state (+1 or -1) reflecting the truth value of that variable in the logical formula. This mapping ensures a tight correlation between neuron states and the logical variables of the SAT problem. By adjusting the synaptic weights, the network systematically optimizes neuron states to satisfy the SAT problem's logical constraints gradually. This direct mapping makes the network more orderly and structured when dealing with logical issues and improves the efficiency of neuron management.

In the DHNN-SAT network, logical rules are transformed into specific synaptic weight configurations, with each rule corresponding to a set of weights that collectively determine the shape of the energy function and the optimization path. The network can progressively lower its energy by continuously adjusting neuron states, eventually reaching a stable state. Suppose this stable state represents the network's minimum energy state. In that case, it corresponds to the global optimal solution of the SAT problem, which satisfies the entire CNF (Conjunctive Normal Form), making all clauses in the CNF true and resulting in a cost function value of zero.

The DHNN-SAT is also a significant variant of the discrete Hopfield neural network, effectively addressing the issue where traditional discrete Hopfield networks often return invalid results in most cases. By embedding SAT problems as symbolic language into the discrete Hopfield network, neurons can be effectively managed, making the network's energy function more transparent and more structured, ensuring that neuron states are efficiently guided during the optimization process to automatically avoid invalid states that do not meet the logical constraints. Consequently, neurons are constrained to search within a state space that adheres to the logical rules, significantly reducing the search space and improving convergence efficiency and accuracy, making achieving a globally optimal solution that satisfies all logical rules easier.

Embedding different SAT logical rules into DHNN typically does not lead to over fitting because DHNN aims to find solutions that meet specific logical constraints rather than generalizing to unseen input data. However, embedding complex logical rules can introduce some challenges. While the network may converge to a stable state, this state may only be a local optimum rather than a global optimum, a problem that becomes more pronounced when dealing with complex SAT problems. Additionally, as the size of the SAT problem increases, the network's complexity and computational resource demands also increase significantly, leading to slower convergence or even failure to find a suitable solution within a limited number of iterations. This issue is particularly acute when dealing with large-scale problems, significantly increasing memory usage and computation time.

To enhance network performance, researchers have explored various meta-heuristic optimization methods. Meta-heuristic methods are commonly used in multiple optimization tasks because they can find near-optimal solutions within a reasonable computation time (Mostafa et al. 2023). In 2019, Kho et al. proposed a DHNN-2SAT network combined with an ant colony algorithm, significantly improving solution efficiency (Kho et al. 2019). In 2020, Mansor et al. proposed a hybrid optimization algorithm for the DHNN-P 2SAT network based on the artificial bee colony algorithm, further enhancing the network's learning ability and stability (Mansor et al. 2020). In the same year, Sathasivam et al. introduced the election algorithm (EA) to enhance the learning performance of the DHNN-Random KSAT network (Sathasivam et al. 2020c).

In 2021, Mansor et al. proposed the DHNN-3SAT network using the grey wolf optimization algorithm, which improved the network's performance on complex SAT problems through the optimization algorithm (Mansor et al. 2021). Bazuhair et al. further proposed a random 3SAT network combined with the election algorithm, significantly improving solution accuracy (Bazuhair et al. 2021). Karim et al. proposed a novel multi-objective hybrid election algorithm to optimize DHNN performance in high-order random SAT problems (Karim et al. 2022).

In addition to the meta-heuristic mentioned above methods, recent research has also seen the gradual development of hybrid approaches. In 2022, Guo et al. introduced a mixed satisfiability logical rule (YRAN2SAT) that allows for random enumeration based on clauses' first-order, second-order, or combined structure (Guo et al. 2022). In 2023, Azizan et al. proposed a DHNN-3SAT optimization scheme that combines fuzzy logic with genetic algorithms, significantly enhancing the network's effectiveness (Azizan et al. 2023). In 2024, Guo et al. further improved the learning and retrieval processes of the DHNN-Y type random 2SAT network by integrating a hybrid differential evolution algorithm with swarm mutation, achieving a 100% global minimum rate (Guo et al. 2024). These studies have yielded promising results in smaller-scale neural networks, particularly demonstrating high accuracy in global minimum rates (Mohd Kasihmuddin et al. 2023).

However, the specific impacts of the logical complexity and size of SAT problems on networks remain under-explored. To date, only Sathasivam has investigated the application of fuzzy logic to solve logic programs with lower logical complexity (Sathasivam 2012).

In recent years, the DHNN-SAT network has been widely applied in fields such as logic programming and constraint solving, particularly excelling in scenarios requiring the efficient resolution of complex logical problems. As the DHNN-SAT network evolves, its logic rules have been enriched, and its application areas have expanded. The network integrates logic mining with satisfiability problems to address various optimization and classification challenges. For instance, the network has been successfully implemented in commodity price forecasting (Alzaeemi and Sathasivam 2021), medical data analysis (Abdullahi et al. 2020), logic mining in football matches (Kho et al. 2020), as well as VLSI circuit configuration (Mansor et al. 2016a), Bezier curve network optimization (Kasihmuddin et al. 2016b), and student performance analysis (Kasihmuddin et al. 2019) among other fields (Mansor et al. 2016b; Zamri et al. 2020, 2024; Kasihmuddin et al. 2023), demonstrating its formidable capability in handling complex logic challenges. Despite its impressive performance across various