WH FACTORIZATION AND ITS APPLICATION IN GRAPH ENERGIES

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WH FACTORIZATION AND ITS APPLICATION IN GRAPH ENERGIES

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LIST OF ABBREVIATIONS

QIF	Quadrant Interlocking Factorization
WZ	Matrices in W and Z forms
WH	Matrices in W and H forms
QR	Matrices in Q and R forms
AMD	Advanced Micro Devices
Intel	Integrated Electronics
PIE	Parallel Implicit Elimination
MATLAB	Matrix Laboratory
LU	Lower and Upper triangular matrices
GE	Gaussian Elimination
HMO	Hückel Molecular Orbital
Mod	Modulo
LAPACK	Linear Algebra Package

LIST OF SYMBOLS

$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\lceil B \rceil$	Ceiling of B
$\lfloor B \rfloor$	Floor of B
$\prod_{i=1}^{n} X_i$	The finite product of spaces
$\sum_{i=1}^{n} X_i$	The finite sum of spaces
$\|B\|$	Modulus of B
$\Vert B \Vert$	Norm of matrix B
B^T	Transpose of matrix B
B^{-1}	Inverse of matrix B
$\bar{B}$	Conjugate of matrix B
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
$v(G)$	The number of vertices in G or the order of G
$e(G)$	The number of edges in G or the size in G
$d_G(v)$	The degree of vertex v in G
$A(G)$	Adjacency matrix of a graph
$I(G)$	Topological graph indices
$T(n)$	Total number of arithmetic operations
$det(B)$	Determinant of matrix B
α	Alpha

β	Beta
δ	Delta
γ	Gamma
λ	Eigenvalue
$deg^{-}(v)$	In-degree of a vertex
$deg^{+}(v)$	Out-degree of a vertex
$\mathcal{E}_A(G)$	Energy of a graph
$tr(B)$	Trace of B
$tr_a(B)$	Anti trace of B
$\forall$	For all
$\in$	Member of
$>$	greater than
$\leq$	Less than or equal to
$\neq$	Not equal to

PEMFAKTORAN WH DAN APLIKASI DALAM TENAGA GRAF

ABSTRAK

Matriks jam pasir ialah matriks segi empat sama yang diperolehi daripada kuadran saling mengunci pemfaktoran, dikenali sebagai pemfaktoran WH, yang boleh diwakili dalam graf bercampur. Dari kajian terkini perkembangan matriks jam pasir, tidak dibincangkan ciri-ciri matriks yang terlibat dalam pemfaktorannya. Dalam tesis ini, ciri-ciri pemfaktoran WH dibincangkan sebagai fokus utama yang merangkumi kewujudannya bagi setiap matriks pepenjuru dominan yang tegas, yang mana sistem H membentuk matriks songsang segi tiga bawah dan matriks W yang membentuk matriks songsang segi tiga atas. Untuk mengoptimumkan pemfaktoran WH, varian bagi prinsip Cramer digunakan untuk menunjukkan bahawa ia mempunyai kelebihan berbanding teknik tradisional. Dapat disimpulkan bahawa, pemfaktoran WH secara blok boleh memfaktor matriks dengan berkesan. Matriks yang diperoleh daripada pemfaktoran WH diketahui diwakili dengan baik sebagai graf campuran, tetapi bukan sebagai graf indeks topologi. Walaupun terdapat lebih 50 jenis tenaga graf yang diperolehi daripada indeks graf topologi, hanya lapan jenis dipertimbangkan untuk graf jam pasir bercampur: Tenaga Zagreb Pertama dan Kedua, tenaga hiper-Zagreb pertama dan kedua, tenaga Guorava Pertama dan Kedua, tenaga hiper-Guorava Pertama dan Kedua. Tambahan pula, semua tenaga yang dipertimbangkan dibandingkan untuk membuat kesimpulan bahawa tenaga graf memberikan nilai nombor genap manakala nisbah tenaga graf (nisbah antara tenaga Laplacian dan tenaga campuran graf bagi jam pasir bercampur) adalah ganjil. Jumlah tenaga graf Laplacian dan tenaga graf campuran bagi jam pasir bercampur ialah dua kali ganda hasil tambah

darjah bagi bucu-bucu. Didapati bahawa hubungan antara sekurang-kurangnya mana-mana dua topologi indeks graf bagi graf jam pasir bercampur ialah gandaan $(n-1)$, dengan n $(n \geq 3)$ ialah susunan graf jam pasir bercampur. Akhir sekali, matriks-H integer telah dapat dibina daripada pemfaktoran integer WH dengan meletakkan syarat kepada matriks-H integer songsang supaya penentu bagi setiap submatriks berpusat mestilah ± 1.

WH FACTORIZATION AND ITS APPLICATION IN GRAPH ENERGIES

ABSTRACT

Hourglass matrix is a square matrix obtained from quadrant interlocking factorization, known as WH factorization, which can be represented in mixed graph. Recent establishment of hourglass matrix and its factorization technique does not show the all the properties of the matrix. In this thesis, properties of WH factorization were put forward which include its existence for every strict dominant diagonal matrix, that its H_{system} forms a lower triangular invertible matrix and its counterpart W-matrix forms an upper triangular invertible matrix. To optimize WH factorization, a variant Cramer's rule is used to show that it has an advantage over the traditional Cramer's rule. It can be concluded that the block WH factorization technique is able to partition the matrix for effective factorization. The matrix obtained from WH factorization is known to be well represented as mixed graph, but not as topological index graph. There are over 50 types of graph energies obtained from degree based topological indices, only eight types are considered for the mixed hourglass graph: First and Second Zagreb energy, First and Second hyper-Zagreb energy, First and Second Guorava energy, and First and Second hyper-Guorava energy. Furthermore, all considered energies are compared to conclude that the energies give value of even numbers while the graph energy ratio (ratio between Laplacian energy and the mixed energy of mixed hourglass graph) is odd. The sum of the Laplacian energy and the mixed energy of mixed hourglass graph is twice the sum of degree of vertices. It was obtained that the relationship between at least any two topological graph indices of mixed hourglass graph is a multiple of $(n-1)$, where n $(n \geq 3)$ is the order of the mixed hourglass

graph. Finally, the integer H-matrix is established from integer WH factorization with the property that the inverse of integer H-matrix gives integer integer H-matrix provided the determinant of the cantered submatrix is ± 1.

CHAPTER 1

INTRODUCTION

1.1 Background

Quadrant interlocking factorization (QIF) or butterfly factorization of nonsingular matrix B is often called WZ factorization, and was coined by Evans and Hatzopoulos (1978). In 1979, the authors gave details of the factorization as an alternative to LU factorization and the avoidance of breakdown of QIF algorithm. QIF is known for the adaptability of its direct method to solve systems of linear equations. Hence, the factorization leads to the application of Parallel Implicit Elimination (PIE) in solving linear systems. This method allows for the simultaneous computation of two matrix elements (two columns at a time) in a parallel implementation, in contrast to Gaussian Elimination (GE), which computes one column at a time (Bylina and Bylina, 2019). QIF's stability is derived from its nonsingular central submatrices, making it more reliable than any other form of factorization (Rao, 1997). Compared to other factorization techniques such as LU, QR and Cholesky decomposition, QIF has been proven to be the most efficient, parallelizable and accurate algorithm (Rao, 1997; Bylina and Bylina, 2007).

The WZ factorization divides the matrix into organizational forms which are then divided into smaller groups and solved (Huang et al., 2010). W-matrix and Z-matrix exist together in WZ factorization of invertible matrix B (Bylina, 2003), provided that

$$B = WZ, \tag{1.1}$$

where Z-matrix and W-matrix are specifically defined for order $n\,(n \geq 3)$ as in (Evans and

Hatzopoulos, 1979).

$$Z=\begin{cases}(\underbrace{0,...,0}_{i-1},z_{i,i},...,z_{i,n-i+1},0,...,0)^T, & i=1,...,\lfloor\frac{(n+1)}{2}\rfloor;\\ \\ (\underbrace{0,...,0}_{n-i},z_{i,n-i+1},...,z_{i,i},0,...,0)^T, & i=\lfloor\frac{(n+1)}{2}\rfloor+1,...,n.\end{cases} \tag{1.2}$$

$$W=\begin{cases}[\underbrace{0,\dots,0}_{i-1},w_{i+1,i},\dots,w_{n-i,i},0,\dots,0]^T \quad i=1(1)\left\lfloor\frac{n-1}{2}\right\rfloor;\\ \\ [\underbrace{0,\dots,0}_{i-1},1,0,\dots,0]^T \quad i=\begin{cases}\frac{n}{2}\\ \\ \frac{n+2}{2}\end{cases},n\text{-even} \quad i=\frac{n+1}{2},n\text{ -odd };\\ \\ [\underbrace{0,\dots,0}_{n-i+1},w_{n-i+2,i},\dots,w_{i-1,i},1,0,\dots,0]^T \quad i=\left\lfloor\frac{n+4}{2}\right\rfloor(1)n.\end{cases} \tag{1.3}$$

The QIF has been known for awhile with many variations. WH factorization, a kind of QIF was recently detailed in (Babarinsa and Kamarulhaili, 2018b), which has several potential applications in scientific computing due to the structure of the matrix. The matrix was named after its resemblance with hourglass device. Although (Demeure, 1989) coined the term "hourglass matrix" to describe the matrix obtained from factorizing a square matrix via QIF, it was Babarinsa and Kamarulhaili (2018b) that established an algorithm to obtain nonsingular hourglass matrix from a given matrix. Thus, an hourglass matrix is a nonsingular matrix of order $n\,(n\geq 3)$ that has nonzero entries between the ith to the $(n-i+1)$ element of the ith and $(n-i+1)$ row of the matrix, otherwise 0's, for $i=1,2,...,\lfloor\frac{n+1}{2}\rfloor$, see Figure 1.1. In WH factorization, $\sum_{k=1}^{\lfloor\frac{n}{2}-1\rfloor}(n-2k)$ of 2×2 systems of linear equations are to be solved via Cramer's rule which account for the elements in W-matrix and H-matrix (Babarinsa et al., 2020a).

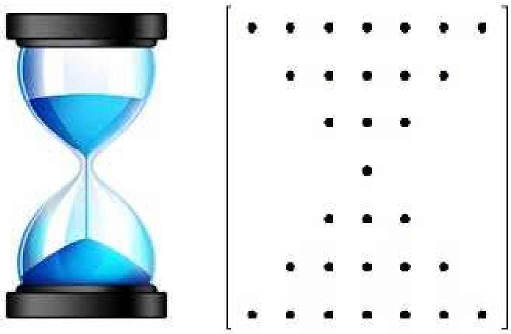

Figure 1.1: Respective hourglass device and hourglass matrix.

Hailiza and Babarinsa (2019) established mixed hourglass graph and some of its energies such as mixed energy and Laplacian energy. However, the mixed hourglass graph has not been previously considered as part of a degree-based topological indices. The hourglass graph, so named because of its shape, is a planar, undirected graph that is made up of at least two triangles that overlap at a single vertex. The hourglass graph that is being discussed here, however, has an underline complete graph named after its mixed adjacency matrix called mixed hourglass graph. Initially, the mixed hourglass graph represented directly from the hourglass matrix (known as weighted hourglass-adjacency matrix) form a mixed graph with weights and loops. Replace all of the weights in the weighted hourglass adjacency matrix with 1's to produce an unweighted mixed hourglass graph that has loops but no multiple arcs. Swapping the diagonal of the unweighted hourglass-adjacency matrix with 0's to create a mixed hourglass graph devoid of loops and numerous arcs. Thus, unweighted mixed hourglass graph $\mathscr{G}$ is a weighted mixed hourglass graph of weight 1 such that $\mathscr{G} = (V, E, A)$ is an ordered triple consisting of a set of vertices $V\mathscr{G} = \{v_1, v_2, ..., v_n\}$, a set of unweighted edges $E\mathscr{G} = \{e_1, e_2, ..., e_n\}$, a set of unweighted arcs $A\mathscr{G} = \{a_1, a_2, ..., a_n\}$.

1.2 Problem Statement

The hourglass matrix was recently established from WH factorization for which Babarinsa and Hailiza concentrated on its factorization technique. Unfortunately, few results from hourglass matrix and its factorization technique were only considered. The block WH factorization was not considered in the previous research work. Though several results have been established on mixed hourglass graph including its mixed and Laplacian energy, but excluding the First and second Zagreb (including its hyper-Zagreb energy), and the First and second Guoarava energy (including its hyper-Guorava energy) of mixed hourglass graph. Besides, integer hourglass matrix from WH factorization has not been established which possess potential application in cryptography.

1.3 Research Objectives

The objectives of this study are as follows:

1. To establish some new results on WH factorization and on its variant (Block WH factorization).

2. To compute and investigate the relationships between the First and Second Zagreb energies, First and Second hyper-Zagreb energies, First and Second Guorova energies, First and Second hyper-Guorova energies of a mixed hourglass graph.

3. To construct an integer hourglass matrix from WH factorization.

1.4 Motivation of the Study

The motivation of this research was not only to obtain a new variant of WH factorization but also to establish graph energies of mixed hourglass graph, as well to establish method and technique to obtain integer hourglass matrix. The importance of graph energy is to study the strength of the eigenvalues of its adjacency matrix which can be linearly transformed while the integer hourglass matrix could serve as the private key in cryptography.

1.5 Significance of the Study

Hourglass matrix is a type of matrix with potential application to the scientific world. The significance of this study lies in the new results of the hourglass matrix. Establishing integer hourglass matrix from the WH factorization will have a great potential impact on area of cryptography, information security, and algorithmic number theory where integers are the main playing field.

1.6 Thesis Organization

Chapter 1 provides an introduction on hourglass matrix and its block factorization algorithm, as well as on its graph energies. Chapter 2 provides the literature review on hourglass matrix and its WH factorization, and graph energies. Chapter 3 considers a variant of WH factorization. This chapter establishes a variant of Cramer's rule (VCR) to solve the linear systems in WH factorization, and some results were deduced such as block WH factorization. Chapter 4 extends results on mixed hourglass graph. It compares mixed energy, Laplacian energy, First and Second Zagreb energy, First and Second Hyper-Zagreb energy, First and Second Guorava energy, and First and Second Hyper-Guorava energy of mixed hourglass graph with unique results. Chapter 5 establishes an integer hourglass matrix. Chapter 6 gives the concluding part of the thesis.

CHAPTER 2

LITERATURE REVIEW AND PRELIMINARIES

2.1 Introduction

For over half a century, various techniques for decomposing matrices have been utilized, including LU, QR and Cholesky factorizations (Bornemann, 2018). LU factorization is a highly accurate method that involves inverting a matrix utilizing Gaussian elimination (Bunch and Hopcroft, 1974). Its efficiency is heightened by the conjunction of low computational complexity and the utilization of partial pivoting techniques. It is important to note, however, that without proper ordering of the matrix, LU factorization can fail. This issue can be resolved by rearranging the rows of the matrix to ensure the first element is nonzero (Quintana-Ortí and Van De Geijn, 2008; Murota, 1983). This shows that a suitable rearrangement of either rows or columns can make LU factorization numerically stable (Wang et al., 2016). To take advantage of contemporary software library architectures, LU factorization is implemented in the LAPACK library (Dongarra et al., 2014). An alternative to LU factorization that is more efficient and suitable for parallel computing is Quadrant Interlocking Factorization (QIF) (Evans and Hatzopoulos, 1978). The factorization of a matrix is difficult to compute and the application of various optimization techniques along with the parallelism of modern computers makes QIF factorization very effective.

It was the reconstruction of QIF that gave path to WH factorization which yielded hourglass matrix and its representation in graph theory (Babarinsa and Kamarulhaili, 2018b). The type of graph considered for hourglass matrix is mixed hourglass graph. The study of graphs of mixed hourglass matrix was established due to its representation of the matrix in a graph. This brought a new development in the area.

2.2 Quadrant Interlocking Factorization

Quadrant Interlocking Factorization, QIF, is a method that decomposes the coefficient matrix B into two interlocking matrix quadrants (Evans, 2004). The interlocking matrix quadrants are in butterfly forms such that

$$\begin{bmatrix} b_{1,1} & b_{1,j} & b_{1,n} \\ b_{i,1} & B_{i,j} & b_{i,n} \\ b_{n,1} & b_{n,j} & b_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -w_{i,1} & \mathbb{I}_{n-2} & -w_{i,n} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,j} & b_{1,n} \\ 0 & B_{n-2} & 0 \\ b_{n,1} & b_{n,j} & b_{n,n} \end{bmatrix}, \tag{2.1}$$

where B_{n-2} is a matrix of size $(n-2)\times(n-2)$ to update B'_{n-2} as $B'_{n-2} = -w_{i,1}\,b_{1,j} + B_{n-2} - w_{i,n}\,b_{n,j}$, for $i,j = 2,3,...,n-1$.

The QIF is suitable for parallel computing. Several calculations (the computation can be broken down into smaller subproblems) are carried out simultaneously in a process known as parallel computing (Bylina and Bylina, 2016). A single instruction stream controls the execution of a parallel algorithm, assigning its subproblems to several processors and directing their input and output to the proper processors. QIF has its limitation to breakdown, however the solution to ensure the factorization does not break down has been proffered (Evans, 2002). First, if the determinant of any submatrix of B is zero, then can fail to produce the desired interlocking matrix quadrants. If the denominators are zero at any stage of the factorization procedure, the entire process will fail. To avoid breakdown, there must be row or column interchange. The factorization procedure will not fail as long as

(i) $b^{(0)}_{1,1} \neq 0$ or $b^{(0)}_{n,n} \neq 0$,

(ii) $b^{(0)}_{n,n}\,b^{(0)}_{1,1} - b^{(0)}_{1,n}\,b^{(0)}_{n,1} \neq 0$.

It is well noted that the factorization will not fail even if the necessary column interchanges are not used provided that matrix B is either real symmetric and positive definite or diagonally dominating (Rao, 1997). The two types of QIF that, among others, differ in applications and attributes but are identical in algorithm and computation are WZ factorization and WH factorization.

2.2.1 WZ Factorization

This section gives the fundamental definition of WZ factorization and the diagrammatical representation of W-matrix and Z-matrix.

Definition 2.2.1. *(Bylina and Bylina, 2013) A Z-matrix is defined as*

$$Z = \begin{cases} (\underbrace{0,\dots,0}_{i-1}, z_{i,i},\dots,z_{i,n-i+1},0,\dots,0)^T, & i = 1,\dots,\lfloor\frac{(n+1)}{2}\rfloor; \\ (\underbrace{0,\dots,0}_{n-i}, z_{i,n-i+1},\dots,z_{i,i},0,\dots,0)^T, & i = \lfloor\frac{(n+1)}{2}\rfloor+1,\dots,n. \end{cases}$$

for $z_{i,j} \in \mathbb{R}$.

It is a necessary and sufficient condition for matrix B to be WZ factorization is that the central submatrices are nonsingular, where $B = [b_{i,j}]_{i,j=1}^{n}$ and n is even order of matrix (the assumption also holds for odd order). Z-matrix and W-matrix are well-known as interlocking quadrant factors of B having butterfly shape for the even order of the form:

$$
W = \begin{bmatrix}
1 & & & & & & & \circ \\
\bullet & 1 & & & & & & \bullet \\
\bullet & \circ & 1 & & & & \circ & \bullet \\
\bullet & \circ & \circ & 1 & & \circ & \circ & \bullet \\
\bullet & \circ & \bullet & & 1 & \bullet & \circ & \bullet \\
\bullet & \bullet & & & & 1 & \bullet & \bullet \\
\bullet & & & & & & 1 & \bullet \\
\circ & & & & & & & 1
\end{bmatrix}
\quad \text{and} \quad
Z = \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \circ & \circ & \circ & \circ & \circ & \bullet & \\
 & & \circ & \circ & \circ & \bullet & & \\
 & & & \circ & \bullet & & & \\
 & & & \bullet & \circ & & & \\
 & & \bullet & \circ & \circ & \circ & & \\
 & \bullet & \circ & \circ & \circ & \circ & \circ & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{bmatrix},
$$

where $\bullet$ specifically means nonzero entries and $\circ$ may represent either zero or nonzero entries. Then the direct method to solve the 2×2 linear systems in WZ factorization under the nonsingularity constraint presumed for their determinants solely depends on a conventional method called Cramer's rule. Gaussian elimination method is another direct method that can be used to solve linear systems, however it cannot be applied to WZ factorization because it does not indicate if the subcentral matrices is invertible or not (Babarinsa, 2019). Cramer's rule is shown to be reliable and efficient for solving system of n linear equations in n variables.

2.2.1(a) Importance of Cramer's Rule in WZ Factorization

Consider n linear equations with n unknowns $x_1, x_2, x_3, ..., x_n$ is defined by

$$
\begin{cases}
b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 + \cdots + b_{1,n}x_n = c_1 \\
b_{2,1}x_1 + b_{2,2}x_2 + b_{2,3}x_3 + \cdots + b_{2,n}x_n = c_2 \\
b_{3,1}x_1 + b_{3,2}x_2 + b_{3,3}x_3 + \cdots + b_{3,n}x_n = c_3 \\
\quad \vdots \; + \; \vdots \; + \; \vdots \; + \cdots + \; \vdots \; = \vdots \\
b_{n,1}x_1 + b_{n,2}x_2 + b_{n,3}x_3 + \cdots + b_{n,n}x_n = c_n.
\end{cases}
\tag{2.2}
$$

Thus, the unique solution $(x_1, x_2, \ldots, x_n)$ furnishes a solution to the system via Cramer's rule

$$Bx_i = c. \tag{2.3}$$

Theorem 2.2.1. *(Anton, 2013)[Cramer's rule] Let $Bx = c$ be an $n \times n$ system of linear equations and B is an $n \times n$ matrix of x such that $det(B) \neq 0$, then the unique solution $(x_1, x_2, \ldots, x_n)$ to the system in Equation (2.3) is given by*

$$x_i = \frac{det(B_{i|c})}{det(B)}, \tag{2.4}$$

where $B_{i|c}$ is derived from B by replacing the vector column of c in the ith column of matrix B. For $i = 1, 2, \ldots, n$, $det(B) \neq 0$, $x_i = (x_1, \ldots, x_n)^T$, $c = (c_1, \ldots, c_n)^T$, $x, c \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$.

Recently, Cramer's rule has been modified to optimize the factorization process, see Corollary 2.2.2 and Corollary 2.2.3.

Corollary 2.2.2. *(Babarinsa and Kamarulhaili, 2018a) Let $Bx = c$ be an $n \times n$ system of linear equations and B a square matrix of x. Then the ith entry x_i of the unique solution $x = (x_1, x_2, \ldots, x_n)^T$ to the linear system is given by*

$$x_i = \frac{\det\left(B^{\alpha_i}_{i+c}\right)}{\det\left(B^{\alpha_i}\right)} - 1,$$

where $B^{\alpha_i}_{i+c}$ is the matrix obtained from B^{α_i} by adding the column vector c to the ith column of B^{α_i} and B^{α_i} is the matrix obtained from B with its ith column being replaced by the row sum of B, for $i = 1, 2, \ldots, n$.

Corollary 2.2.3. *(Babarinsa and Kamarulhaili, 2018a) Let $Bx = c$ be an $n \times n$ system of linear equations and B a square matrix of x, then the ith entry x_i of the unique solution*

$x = (x_1, x_2, \ldots, x_n)^T$ to the linear system is given by

$$x_i = 1 - \frac{\det\left(B_{i-c}^{\alpha_i}\right)}{\det\left(B^{\alpha_i}\right)},$$

where $B_{i-c}^{\alpha_i}$ is the matrix obtained from B^{α_i} by subtracting the column vector c from the ith column of B^{α} and B^{α_1} is the matrix obtained from B with its ith column being replaced by the row sum of B, for $i = 1, 2, \ldots, n$.

Cramer's rule has undergone numerous modifications compared to all other direct methods combined, primarily because it can determine if a system is incompatible or indeterminate without fully solving it. Many previous works on Cramer's rule have utilized properties of determinants, particularly cofactors, in their proofs Whitford and Klamkin (1953); Watkins (2004). Though, Cramer's rule is least consider for practical use but has some theoretical applications and work where other direct solvers may fail (Habgood, 2011). Inverse matrix method for solutions of linear system has similar computation and result as Cramer's rule, because both are applicable when the coefficient matrix is invertible, otherwise a unique solution is not available. Some of the reasons why Cramer's rule are less considered are: When the determinant of the coefficient matrix is zero, the method fails. The approach necessitates the computation of $n+1$ determinants, each of size $n \times n$. In addition, significant round-off errors may occur in large problems involving non-integer coefficients. (Debnath, 2013; Vein and Dale, 1999).

For any numerical method, the goal is to produce precise results, but the required level of accuracy varies depending on the specific application of the algorithm. Besides accuracy and stability, a well-designed algorithm should also efficiently manage workloads and optimize memory usage (Habgood, 2011). In scientific computation, when dealing with problems that exhibit significant variations in magnitude, the relative error is attributed to independent scaling. Errors affecting accuracy and causing instability fall into two categories: round-off error and truncation error, also known as discretization errors (Collins, 1990). While round-off errors cannot be entirely avoided, they can be mitigated by using higher precision data storage, such as moving from single to double floating-point values. Truncation errors, on the other hand, arise from the limitations of representing the solution due to a finite number of steps in the computational process (Quarteroni et al., 2010). An algorithm is considered stable if it can maintain small computational errors even as the problem size increases, leading to increased solution accuracy. It's important to note that an algorithm can be stable for solving a particular problem but may exhibit instability when applied to another problem. To prove the stability of an algorithm, presenting a convincing argument is essential. Two common approaches are forward analysis and backward analysis (Higham, 2002). Forward analysis involves bounding the disparity lies in the solution produced by the algorithm and the actual correct solution. Nevertheless, this approach necessitates a known, accurate solution for comparison. In contrast, backward analysis anticipates the potential disruptions that the algorithm might introduce to the initial data (Habgood and Arel, 2012).

For optimizing Cramer's rule, the accuracy, stability and efficiency depends on the algorithm used. The stability of an algorithm ensures computational error remains minimal as the size of the problem grows (Habgood and Arel, 2012). To show an algorithm is stable, consider its forward and backward error analysis. The most common numerical error encounter in WZ factorization using Cramer's rule is the round-off error. Round-off error is the error encounter during approximation of obtained output (Higham, 2002). Though the error is inevitable but using high precision can reduce its impact. The round-off error of Cramer's rule has been debated for over fifty years with conclusion that the alteration in the Cramer's rule algorithm yields better results either for well-conditioned or ill-conditioned systems (Dunham, 1980). Thus, the numerical stability of Cramer's rule depends on the accurate method used to evaluate its determinant (Habgood, 2011). Backward error is a unit of measurement for errors in approximating a solution to a problem. The backward error, as contrast to the forward error, measures how much data must be altered in order for the approximate solution to be produced (Higham, 2002). Backward error analysis is the study of backward stable algorithms, which always yield a little backward error.

2.2.1(b) WZ Factorization Algorithm

In WZ factorization, matrix B is dense nonsingular. Thus, a dense matrix is a matrix with more nonzero entries than the zero entries (Scott and Tuuma, 2023). However, if central matrix of B is singular then interchange columns or rows of the matrix by suitable permutation is required to avoid breakdown of the factorization method, else the factorization breakdown. For the establishment of elements in W-matrix, column ith and $(n-1)$th are obtained by solving simultaneous equation via Cramer's rule which requires matrix B to be successfully updated and this update changes matrix B to Z-matrix (Bylina and Bylina, 2014; Levin and Evans, 1991). The matrix update of WZ factorization indicates the most time consuming part of the factorization. The steps to obtain Z-matrix is as follows:

Step 1: Let $B^{(0)} = Z^{(0)}$for initial update and obtain the first and last rows of Z-matrix as $b_{1,1}^{(0)} = z_{1,1}^{(0)}$, $b_{1,i}^{(0)} = z_{1,i}^{(0)}$, $b_{1,n}^{(0)} = z_{1,n}^{(0)}$, $b_{n,1}^{(0)} = z_{n,1}^{(0)}$, $b_{n,i}^{(0)} = z_{n,i}^{(0)}$, $b_{n,n}^{(0)} = z_{n,n}^{(0)}$, where $i = 2,\ldots,n-1$. Now, compute $w_{i,1}^{(1)}$ and $w_{i,n}^{(1)}$ from $(n-2)$ sets of 2×2 linear system in Equation (2.5) of matrix B using Cramer's rule

$$\begin{cases} z_{1,1}^{(0)} w_{i,1}^{(1)} + z_{n,1}^{(0)} w_{i,n}^{(1)} = -z_{i,1}^{(0)} \\ z_{1,n}^{(0)} w_{i,1}^{(1)} + z_{n,n}^{(0)} w_{i,n}^{(1)} = -z_{i,n}^{(0)}. \end{cases} \tag{2.5}$$

The values of $w_{i,1}^{(1)}$ and $w_{i,n}^{(1)}$ are put in matrix form as:

$$W^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ w_{2,1}^{(1)} & 1 & \ddots & \vdots & w_{2,n}^{(1)} \\ \vdots & 0 & \ddots & 0 & \vdots \\ w_{n-1,1}^{(1)} & \vdots & \ddots & 1 & w_{n-1,n}^{(1)} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Step 2: Update matrix B (let $B^{(1)} = Z^{(1)}$ for the first update) and compute:

$$Z^{(1)} = W^{(1)} Z^0.$$

Proceed analogously for the inner square matrices of $Z^{(1)}$ of size $(n-2)$ and so on.

Step 3: Compute $w_{i,k}^{(k)}$ and $w_{i,n-k+1}^{(k)}$ from Equation (2.6) by solving its 2×2 linear equations using Cramer's rule, where $k = 1, 2, ..., \frac{n}{2} - 1;\ i = k+1, ..., n-k$.

$$\begin{cases} z_{k,k}^{(k-1)} w_{i,k}^{(k)} + z_{n-k+1,k}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,k}^{(k-1)} \\ z_{k,n-k+1}^{(k-1)} w_{ik}^{(k)} + z_{n-k+1,n-k+1}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,n-k+1}^{(k-1)}. \end{cases} \tag{2.6}$$

Then, put the values of $w_{i,k}^{(k)}$ and $w_{i,n-k+1}^{(k)}$ in a matrix form as:

$$W^{(k)} = \begin{bmatrix} 1 & & & & \\ w_{k+1,k}^{(k)} & \ddots & & & w_{k+1,n-k+1}^{(k)} \\ \vdots & & \ddots & & \vdots \\ w_{n-1,k}^{(k)} & & & \ddots & w_{n-k,n-k+1}^{(k)} \\ & & & & 1 \end{bmatrix}.$$

Step 4: Compute for kth such successful steps as:

$$Z^{(k)} = W^{(k)} Z^{(k-1)}.$$

To arrive at the Z-matrix, let $Z^{(k)} = Z$. Thus,

$$Z = \begin{bmatrix} z_{1,1}^{(0)} & z_{1,2}^{(0)} & \cdots & \cdots & \cdots & z_{1,n-1}^{(0)} & z_{1,n}^{(0)} \\ 0 & \ddots & \vdots & \cdots & \vdots & \iddots & 0 \\ 0 & 0 & z_{k,k}^{(k-1)} & & z_{k,n+1-k}^{(k-1)} & 0 & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & z_{n+1-k,k}^{(k-1)} & & z_{n+1-k,n+1-k}^{(k-1)} & 0 & 0 \\ 0 & \iddots & \vdots & \cdots & \vdots & \ddots & 0 \\ z_{n,1}^{(0)} & z_{n,2}^{(0)} & \cdots & \cdots & \cdots & z_{n,n-1}^{(0)} & z_{n,n}^{(0)} \end{bmatrix}.$$

A complete one-stage in WZ factorization is when $Z^{(k-1)}$ is computed. However, the factorization requires $\lfloor \frac{(n-1)}{2} \rfloor$ stages to compute all the elements of the matrix W and Z (Evans and Hatzopoulos, 1979).

Theorem 2.2.4. *Factorization Theorem (Rao, 1997). Let $B \in \mathbb{R}^{n \times n}$ be a nonsingular matrix that has a unique QIF factorization, then $B = WZ$ if and only if the submatrices of B are invertible.*

2.2.1(c) Numerical Example of WZ Factorization

By WZ factorization Z-matrix can be obtained from the given dense nonsingular matrix B.

$$B = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 5 & 10 & -7 & 8 & 11 & 4 \\ 0 & -12 & 9 & 6 & 18 & 1 \\ -13 & 12 & 8 & -20 & 14 & 17 \\ 3 & 1 & 1 & -1 & 1 & 4 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

Step 1: Let $b_{i,j}^{(0)} = z_{i,j}^{(0)}$, for $i, j = 1, ..., 6$. Compute the set of 2×2 system of linear equations from

$$\begin{cases} z_{k,k}^{(k-1)} w_{i,k}^{(k)} + z_{n-k+1,k}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,k}^{(k-1)}, \\ z_{k,n-k+1}^{(k-1)} w_{i,k}^{(k)} + z_{n-k+1,n-k+1}^{(k-1)} w_{i,n-k+1}^{(k)} = -z_{i,n-k+1}^{(k-1)}, \end{cases}$$

for $k = 1, ..., \lfloor \frac{n-1}{2} \rfloor = 2$.

If $k = 1$, then

$$\begin{cases} z_{1,1}^{(0)} w_{i,1}^{(1)} + z_{n,1}^{(0)} w_{i,n}^{(1)} = -z_{i,1}^{(0)} \\ z_{1,n}^{(0)} w_{i,1}^{(1)} + z_{n,n}^{(0)} w_{i,n}^{(1)} = -z_{i,n}^{(0)}. \end{cases}$$

When $i = 2$, then

$$\begin{cases} z_{1,1}^{(0)} w_{2,1}^{(1)} + z_{6,1}^{(0)} w_{2,6}^{(1)} = -z_{2,1}^{(0)} \\ z_{1,6}^{(0)} w_{2,1}^{(1)} + z_{6,6}^{(0)} w_{2,6}^{(1)} = -z_{2,6}^{(0)} \end{cases} \Rightarrow \begin{cases} 2w_{2,1}^{(1)} + 10w_{2,6}^{(1)} = -5 \\ -w_{2,1}^{(1)} + 14w_{2,6}^{(1)} = -4 \end{cases} \Rightarrow \begin{cases} w_{2,1}^{(1)} = -\frac{15}{19} \\ w_{2,6}^{(1)} = -\frac{13}{38} \end{cases}$$

When $i = 3$, then

$$\begin{cases} z_{1,1}^{(0)} w_{3,1}^{(1)} + z_{6,1}^{(0)} w_{3,6}^{(1)} = -z_{3,1}^{(0)} \\ z_{1,6}^{(0)} w_{3,1}^{(1)} + z_{6,6}^{(0)} w_{3,6}^{(1)} = -z_{3,6}^{(0)} \end{cases} \Rightarrow \begin{cases} 2w_{3,1}^{(1)} + 10w_{3,6}^{(1)} = 0 \\ -w_{3,1}^{(1)} + 14w_{3,6}^{(1)} = -1 \end{cases} \Rightarrow \begin{cases} w_{3,1}^{(1)} = \frac{5}{19} \\ w_{3,6}^{(1)} = -\frac{1}{19} \end{cases}$$

When $i = 4$, then

$$\begin{cases} z_{1,1}^{(0)} w_{4,1}^{(1)} + z_{6,1}^{(0)} w_{4,6}^{(1)} = -z_{4,1}^{(0)} \\ z_{1,6}^{(0)} w_{4,1}^{(1)} + z_{6,6}^{(0)} w_{4,6}^{(1)} = -z_{4,6}^{(0)} \end{cases} \Rightarrow \begin{cases} 2w_{4,1}^{(1)} + 10w_{4,6}^{(1)} = 13 \\ -w_{4,1}^{(1)} + 14w_{4,6}^{(1)} = -17 \end{cases} \Rightarrow \begin{cases} w_{4,1}^{(1)} = \frac{176}{19} \\ w_{4,6}^{(1)} = -\frac{21}{38} \end{cases}$$

When $i = 5$, then

$$\begin{cases} z_{1,1}^{(0)} w_{5,1}^{(1)} + z_{6,1}^{(0)} w_{5,6}^{(1)} = -z_{5,1}^{(0)} \\ z_{1,6}^{(0)} w_{5,1}^{(1)} + z_{6,6}^{(0)} w_{5,6}^{(1)} = -z_{5,6}^{(0)}. \end{cases} \Rightarrow \begin{cases} 2w_{5,1}^{(1)} + 10w_{5,6}^{(1)} = & -3 \\ -w_{5,1}^{(1)} + 14w_{5,6}^{(1)} = & -4. \end{cases} \Rightarrow \begin{cases} w_{5,1}^{(1)} = & -\frac{1}{19} \\ w_{5,6}^{(1)} = & -\frac{11}{38} \end{cases}$$

Hence, record the values of $w_{i,1}^{(1)}$ and $w_{i,n}^{(1)}$ in a matrix form as

$$W^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{15}{19} & 1 & 0 & 0 & 0 & -\frac{13}{38} \\ \frac{5}{19} & 0 & 1 & 0 & 0 & -\frac{1}{19} \\ \frac{176}{19} & 0 & 0 & 1 & 0 & -\frac{21}{38} \\ -\frac{1}{19} & 0 & 0 & 0 & 1 & -\frac{11}{38} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Update $z_{i,j}^{0}$ to $z_{i,j}^{1}$ by computing the entries as

$$z_{i,j}^{(k)} = z_{i,j}^{(k-1)} + w_{i,k}^{(k)} z_{k,j}^{(k-1)} + w_{i,n-k+1}^{(k)} z_{n-k+1,j}^{(k-1)} \Rightarrow z_{i,j}^{(1)} = z_{i,j}^{(0)} + w_{i,1}^{(1)} z_{1,j}^{(0)} + w_{i,6}^{(1)} z_{6,j}^{(0)}.$$

When $i = 2$ and $j = 2,3,4,5$, then

$$z_{2,2}^{(1)} = z_{2,2}^{(0)} + w_{2,1}^{(1)} z_{1,2}^{(0)} + w_{2,6}^{(1)} z_{6,2}^{(0)} = 10 + \left(-\frac{15}{19}\right)(0) + \left(-\frac{13}{38}\right)(6) = \frac{151}{19}$$

$$z_{2,3}^{(1)} = z_{2,3}^{(0)} + w_{2,1}^{(1)} z_{1,3}^{(0)} + w_{2,6}^{(2)} z_{6,3}^{(0)} = -7 + \left(-\frac{15}{19}\right)(2) + \left(-\frac{13}{38}\right)(9) = -\frac{443}{38}$$

$$z_{2,4}^{(1)} = z_{2,4}^{(0)} + w_{2,1}^{(1)} z_{1,4}^{(0)} + w_{2,6}^{(4)} z_{6,4}^{(0)} = 8 + \left(-\frac{15}{19}\right)(4) + \left(-\frac{13}{38}\right)(-13) = \frac{353}{38}$$

$$z_{2,5}^{(1)} = z_{2,5}^{(0)} + w_{2,1}^{(1)} z_{1,5}^{(0)} + w_{2,6}^{(3)} z_{6,5}^{(0)} = 11 + \left(-\frac{15}{19}\right)(3) + \left(-\frac{13}{38}\right)(10) = \frac{99}{19}$$

When $i = 3$ and $j = 2,3,4,5$, then

$$z_{2,2}^{(1)} = z_{3,2}^{(0)} + w_{3,1}^{(1)} z_{1,2}^{(0)} + w_{3,6}^{(1)} z_{6,2}^{(0)} = -12 - (\tfrac{5}{19})(0) + \left(-\tfrac{1}{19}\right)(6) = -\tfrac{234}{19}$$

$$z_{3,3}^{(1)} = z_{3,3}^{(0)} + w_{3,1}^{(1)} z_{1,3}^{(0)} + w_{3,6}^{(1)} z_{6,3}^{(0)} = 9 - (\tfrac{5}{19})(2) + \left(-\tfrac{1}{19}\right)(9) = \tfrac{172}{19}$$

$$z_{3,4}^{(1)} = z_{3,4}^{(0)} + w_{3,1}^{(1)} z_{1,4}^{(0)} + w_{3,6}^{(1)} z_{6,4}^{(0)} = 6 - (\tfrac{5}{19})(4) + \left(-\tfrac{1}{19}\right)(-13) = \tfrac{147}{19}$$

$$z_{3,5}^{(1)} = z_{3,5}^{(0)} + w_{3,1}^{(1)} z_{1,5}^{(0)} + w_{3,6}^{(1)} z_{6,5}^{(0)} = 18 - (\tfrac{5}{19})(3) + \left(-\tfrac{1}{19}\right)(10) = \tfrac{347}{19}$$

When $i = 4$ and $j = 2, 3, 4, 5$, then

$$z_{4,2}^{(1)} = z_{4,2}^{(0)} + w_{4,1}^{(1)} z_{1,2}^{(0)} + w_{4,6}^{(1)} z_{6,2}^{(0)} = 12 + \left(\tfrac{176}{19}\right)(0) + \left(-\tfrac{21}{38}\right)(6) = \tfrac{165}{19}$$

$$z_{4,3}^{(1)} = z_{4,3}^{(0)} + w_{4,1}^{(1)} z_{1,3}^{(0)} + w_{4,6}^{(1)} z_{6,3}^{(0)} = 8 + \left(\tfrac{176}{19}\right)(2) + \left(-\tfrac{21}{38}\right)(9) = \tfrac{819}{38}$$

$$z_{4,4}^{(1)} = z_{4,4}^{(0)} + w_{4,1}^{(1)} z_{1,4}^{(0)} + w_{4,6}^{(1)} z_{6,4}^{(0)} = -20 + \left(\tfrac{176}{19}\right)(4) + \left(-\tfrac{21}{38}\right)(-13) = \tfrac{921}{38}$$

$$z_{4,5}^{(1)} = z_{4,5}^{(0)} + w_{4,1}^{(1)} z_{1,5}^{(0)} + w_{4,6}^{(1)} z_{6,5}^{(0)} = 14 + \left(\tfrac{176}{19}\right)(3) + \left(-\tfrac{21}{38}\right)(10) = \tfrac{689}{19}$$

When $i = 5$ and $j = 2, 3, 4, 5$, then

$$z_{5,2}^{(1)} = z_{5,2}^{(0)} + w_{5,1}^{(1)} z_{1,2}^{(0)} + w_{5,6}^{(1)} z_{6,2}^{(0)} = 1 + \left(-\tfrac{1}{19}\right)(0) + \left(-\tfrac{11}{38}\right)(6) = -\tfrac{14}{19}$$

$$z_{5,3}^{(1)} = z_{5,3}^{(0)} + w_{5,1}^{(1)} z_{1,3}^{(0)} + w_{5,6}^{(1)} z_{6,3}^{(0)} = 1 + \left(-\tfrac{1}{19}\right)(2) + \left(-\tfrac{11}{38}\right)(9) = -\tfrac{65}{38}$$

$$z_{5,4}^{(1)} = z_{5,4}^{(0)} + w_{5,1}^{(1)} z_{1,4}^{(0)} + w_{5,6}^{(1)} z_{6,4}^{(0)} = -1 + \left(-\tfrac{1}{19}\right)(4) + \left(-\tfrac{11}{38}\right)(-13) = \tfrac{97}{38}$$

$$z_{5,5}^{(1)} = z_{5,5}^{(0)} + w_{5,1}^{(1)} z_{1,5}^{(0)} + w_{5,6}^{(1)} z_{6,5}^{(0)} = 1 + \left(-\tfrac{1}{19}\right)(3) + \left(-\tfrac{11}{38}\right)(10) = -\tfrac{39}{19}$$

Thus,

$$Z^{(1)} = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 0 & \frac{151}{19} & \frac{-443}{38} & \frac{353}{38} & \frac{99}{19} & 0 \\ 0 & \frac{-234}{19} & \frac{172}{19} & \frac{147}{19} & \frac{347}{19} & 0 \\ 0 & \frac{165}{19} & \frac{819}{38} & \frac{921}{38} & \frac{689}{19} & 0 \\ 0 & \frac{-14}{19} & \frac{-65}{38} & \frac{97}{38} & \frac{-39}{19} & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

Step 3: Compute the next set of 2×2 systems of linear equations from the entries in $z^1_{i,j}$.

Let $k = 2$, then

$$\begin{cases} z^{(1)}_{2,2} w^{(2)}_{i,2} + z^{(1)}_{n-1,2} w^{(2)}_{i,n-1} = -z^{(1)}_{i,2} \\ z^{(1)}_{2,n-1} w^{(2)}_{i,2} + z^{(1)}_{n-1,n-1} w^{(2)}_{i,n-1} = -z^{(1)}_{i,n-1}. \end{cases}$$

When $i = 3$, then

$$\begin{cases} z^{(1)}_{2,2} w^{(2)}_{3,2} + z^{(1)}_{5,2} w^{(2)}_{3,5} = -z^{(1)}_{3,2} \\ z^{(1)}_{2,5} w^{(2)}_{3,2} + z^{(1)}_{5,5} w^{(2)}_{3,5} = -z^{(1)}_{3,5} \end{cases} \Rightarrow \begin{cases} \frac{151}{19} w^{(2)}_{3,2} + (-\frac{14}{19}) w^{(2)}_{3,5} = \frac{234}{19} \\ \frac{99}{19} w^{(2)}_{3,2} + (-\frac{39}{19}) w^{(2)}_{3,5} = -\frac{347}{19} \end{cases} \Rightarrow \begin{cases} w^{(2)}_{3,2} = \frac{13985}{4503} \\ w^{(2)}_{3,5} = \frac{75563}{4503} \end{cases}$$

When $i = 4$, then

$$\begin{cases} z^{(1)}_{2,2} w^{(2)}_{4,2} + z^{(1)}_{5,2} w^{(2)}_{4,5} = -z^{(1)}_{4,2} \\ z^{(1)}_{2,5} w^{(2)}_{4,2} + z^{(1)}_{5,5} w^{(2)}_{4,5} = -z^{(1)}_{4,5} \end{cases} \Rightarrow \begin{cases} \frac{151}{19} w^{(2)}_{4,2} + (-\frac{14}{19}) w^{(2)}_{4,5} = -\frac{165}{19} \\ \frac{99}{19} w^{(2)}_{4,2} + (-\frac{39}{19}) w^{(2)}_{4,5} = -\frac{689}{19} \end{cases} \Rightarrow \begin{cases} w^{(2)}_{4,2} = \frac{3211}{4503} \\ w^{(2)}_{4,5} = \frac{87704}{4503} \end{cases}$$

$$W^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{13984}{4503} & 1 & 0 & \frac{75563}{4503} & 0 \\ 0 & \frac{3211}{4503} & 0 & 1 & \frac{87704}{4503} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 4: Proceed to update $z^1_{i,j}$ to $z^2_{i,j}$ by computing the entries as

$$z^{(k)}_{i,j} = z^{(k-1)}_{i,j} + w^{(k)}_{i,k} z^{(k-1)}_{k,j} + w^{(k)}_{i,n-k+1} z^{(k-1)}_{n-k+1,j} \Rightarrow z^{(2)}_{i,j} = z^{(1)}_{i,j} + w^{(2)}_{i,2} z^{(1)}_{2,j} + w^{(2)}_{i,5} z^{(1)}_{5,j}.$$

When $i = 3$ and $j = 3,4$, then

$$z^{(2)}_{3,3} = z^{(1)}_{3,3} + w^{(2)}_{3,2} z^{(1)}_{2,3} + w^{(2)}_{3,5} z^{(1)}_{5,3} = \tfrac{172}{19} + (\tfrac{13984}{4503})(\tfrac{-443}{38}) + (\tfrac{75563}{4503})(\tfrac{-65}{38}) = -\tfrac{9557475}{171114}$$

$$z^{(2)}_{3,4} = z^{(1)}_{3,4} + w^{(2)}_{3,2} z^{(1)}_{2,4} + w^{(2)}_{3,5} z^{(1)}_{5,4} = \tfrac{147}{19} + (\tfrac{13984}{4503})(\tfrac{353}{38}) + (\tfrac{75563}{4503})(\tfrac{97}{38}) = \tfrac{13589845}{171114}$$

When $i = 4$ and $j = 3,4$, then

$$z^{(2)}_{4,3} = z^{(1)}_{4,3} + w^{(2)}_{4,2} z^{(1)}_{2,3} + w^{(2)}_{4,5} z^{(1)}_{5,3} = \tfrac{819}{38} + (\tfrac{3211}{4503})(\tfrac{-443}{38}) + (\tfrac{87704}{1171})(\tfrac{-65}{38}) = -\tfrac{3435276}{171114}$$

$$z^{(2)}_{4,4} = z^{(1)}_{4,4} + w^{(2)}_{4,2} z^{(1)}_{2,4} + w^{(2)}_{4,5} z^{(1)}_{5,4} = \tfrac{921}{38} + (\tfrac{3211}{4503})(\tfrac{353}{38}) + (\tfrac{87704}{4503})(\tfrac{97}{38}) = \tfrac{13788034}{171114}$$

$$Z = \begin{bmatrix} 2 & 0 & 2 & 4 & 3 & -1 \\ 0 & \frac{151}{19} & -\frac{413}{38} & \frac{503}{38} & \frac{99}{19} & 0 \\ 0 & 0 & -\frac{9557475}{171114} & \frac{13589845}{171114} & 0 & 0 \\ 0 & 0 & -\frac{3435276}{171114} & \frac{13788034}{171114} & 0 & 0 \\ 0 & -\frac{14}{19} & -\frac{65}{38} & \frac{97}{38} & -\frac{39}{19} & 0 \\ 10 & 6 & 9 & -13 & 10 & 14 \end{bmatrix}.$$

Thus,

$$B = \left(W^{(2)} \cdot W^{(1)}\right)^{-1} \cdot Z$$

2.2.1(d) Z_{system} and Block WZ factorization

This section discovered the existence of Z_{system} and the Block WZ factorization.

Definition 2.2.2. *(Bylina, 2018) Z_{system} is the partitioning of Z-matrix into blocks of similar sizes.*

A partitioned matrix is a division of a matrix into smaller rectangular submatrices or blocks (Beaumont et al., 2018). A submatrix of a matrix B is a matrix obtained from B by removing any number of rows or columns from B (Bronson et al., 2023). Therefore, Z-matrix of even order (also applicable to odd order) can be partitioned to form structured Z_{system} of 2×2 triangular block matrices which is defined as

$$Z_{system} = \begin{cases} \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{bmatrix} & \text{if } n \text{ is even;} \\ \begin{bmatrix} Z_{1,1} & x_1 & Z_{1,2} \\ 0 & x & 0 \\ Z_{2,1} & x_2 & z_{2,2} \end{bmatrix} & \text{if } n \text{ is odd.} \end{cases} \tag{2.7}$$

Each block contains $\frac{n}{2} \times \frac{n}{2}$ block size if n has an even dimension, accompanied by an additional column vector, $\tilde{x}$, position at $\frac{n+1}{2}$th column of the matrix if n has an odd dimension (Bylina, 2018). This column vector, $\tilde{x}$, can be further partitioned into x_1, x and x_2. This partition produced the Schur complement.

Definition 2.2.3. *(Zhang, 2006) The Schur complement factorizes a block matrix into a product of simpler block matrices.*

Given a block matrix M where

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

the nonsingular matrix P is the leading submatrix of the partitioned matrix such that Schur complement for the matrix is

$$S - RP^{-1}Q.$$

Then, the Schur complement of a matrix block in Equation (2.7) is defined as follows

$$Z_{system}/Z_{1,1} = Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}. \tag{2.8}$$

Theorem 2.2.5. *(Bylina, 2018) If the Z_{system} and the matrix $Z_{1,1}$ are invertible then the matrix $\left(Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}\right)$ is a lower triangular invertible matrix.*

The block WZ factorization is discussed in (Benaini and Laiymani, 1994; Heinig and Rost, 2005; Golpar-Raboky, 2012; Bylina, 2018) where the Z-matrix is divided into p^2 block each of the size $m \times m$ and $n = m \times p$ as follows"

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,p+1} & B_{1,p} \\ B_{2,1} & B_{2,2} & \dots & B_{2,p+1} & B_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p+1,1} & B_{p+1,2} & \dots & B_{p+1,p+1} & B_{p+1,p} \\ B_{p,1} & B_{p,2} & \dots & B_{p,p+1} & B_{p,p} \end{bmatrix},$$

where each block of $m \times m$ is given as

$$B_{i,j} = \begin{bmatrix} b_{(i-1)m+1,(j-1)m+1} & \dots & b_{(i-1)m+1,jm} \\ \vdots & \ddots & \vdots \\ b_{im,(j-1)m+1} & \dots & b_{im,jm} \end{bmatrix}.$$

This factorization produces block Z-matrix of p^2 blocks with each of the size $m \times m$.

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & \dots & \dots & Z_{1,(p-1)} & Z_{1,p} \\ 0 & \ddots & \dots & \dots & \vdots & 0 \\ \vdots & 0 & Z_{\frac{p}{2},\frac{p}{2}} & Z_{\frac{p}{2},\frac{p}{2}+1} & 0 & \vdots \\ \vdots & 0 & Z_{\frac{p}{2}+1,\frac{p}{2}} & Z_{\frac{p}{2}+1,\frac{p}{2}+1} & 0 & \vdots \\ 0 & \vdots & \dots & \dots & \ddots & 0 \\ Z_{p,1} & Z_{p,2} & \dots & \dots & Z_{p,(p-1)} & Z_{p,p} \end{bmatrix}.$$

Definition 2.2.4. *Bylina (2018) A strictly dominant diagonal matrix is a square matrix in which the element in the diagonal entry of a row surpasses the sum of the elements in all non-diagonal entries of that row. That means*

$$|b_{i,i}| > \sum_{j=1, j \neq i}^{n} |b_{i,j}|. \tag{2.9}$$

Corollary 2.2.6. *(Rao, 1997). Every symmetric positive definite and strictly diagonally dominant matrix has a QIF.*

Theorem 2.2.7. *(Bylina, 2018) The block WZ factorization exists if the matrix B has a strict dominant diagonal.*

More so, matrix B being factorize into Z-matrix may or may not be integer. First, a matrix with just integer elements is known as an integer matrix. Although real and integer WZ factorizations by using the null space generators of some special nested submatrices of a matrix was adopted by Golpar-Raboky and Babolian (2022), the initial matrix (symmetric and unimodular) used cannot account for all integer Z-matrices if other predefined matrices are formulated. Thus, the smallest possible magnitude of the determinant of an invertible integer matrix is one and they do not become excessively large when inverses exist (Hanson, 1982). The determinant of a matrix with integer entries is integer since the determinant of a matrix with integer values is a linear combination of integers, it must also be an integer (Komlós, 1967). There is no other variants of QIF that ensures the factorization will yield integer matrix. The trace and anti-trace of matrix B and Z-matrix play important role in factorization process in terms of central submatrices.

Definition 2.2.5. *(Rao, 1997) Central submatrix or centro-nonsingular* $B_{n+2-2l} = \left[b_{j,k}\right]_{j,k=1}^{n+1-l}$ *are nonsingular matrix obtained by deleting the first and last row as well as the first and last column of matrix B, for* $l = 1, \ldots, m = \frac{n}{2}$.

Definition 2.2.6. *(Anton, 2013) Let A be* $n \times n$ *matrix, then the trace of A, denoted by* $tr(A)$*, is defined to be the sum of the entries on the main diagonal of A,*

$$tr(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n} = \sum_{i=1}^{n} a_{i,i}.$$

For instance, the trace of matrix $A = \begin{bmatrix} -5 & 3 \\ 7 & 2 \end{bmatrix}$ is $tr(A) = -5 + 2 = -3$.