A WAVELET-BASED APPROACH FOR THE ANALYTICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND THE NUMERICAL SOLUTIONS OF FRACTIONAL INTEGRAL EQUATIONS

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by

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LIST OF ABBREVIATIONS

FDE	fractional differential equation
FIE	fractional integral equation
CWT	continuous wavelet transform
MRA	multiresolution analysis
DFT	discrete Fourier transform
2D DWT	two-dimensional discrete wavelet transform
MSE	mean square error
PDE	partial differential equation
HPM	homotopy perturbation method
ADM	Adomian decomposition method

LIST OF SYMBOLS

α	Fractional order of integral or derivative
$\langle \cdot, \cdot \rangle$	Inner product
$\overline{f(x)}$	Complex conjugate of the function $f(x)$
$\lVert \cdot \rVert$	Norm
j, k, m, n	Integer
$\mathbb{Z}$	Set of integers
c_n	Coefficient
L	Length of interval
$\mathbb{R}$	Set of real numbers
$\hat{f}(\xi)$	Fourier transform of the function f
$\mathcal{F}$	Fourier transform operator
$\mathcal{S}(\mathbb{R})$	Schwartz space
$f^{(n)}(x)$	n-th derivative of the function f
$\mathbb{C}$	Set of complex numbers
$\mathrm{Re}[f(x)]$	Real part of the function f
$\mathrm{Im}[f(x)]$	Imaginary part of the function f
z	Complex variable
$\mathbb{D}$	Unit disk centered at $z = 0$
$\mathcal{H}$	Class of analytic functions defined on $\mathbb{D}$
$\mathcal{B}$	$\{f \in \mathcal{H} \colon \lvert f(z) \rvert \leq 1\}$
$\mathcal{F}_f(r)$	Discrete Fourier transform of the function f with variable r
$\mathscr{H}$	Periodic Hilbert transform
t	Time
$\mathscr{F}$	Windowed Fourier transform operator
$\mathcal{W}$	Continuous wavelet transform operator
ψ	Wavelet
C_ψ	Wavelet admissibility constant

V_j	Multiresolution analysis subspace in $L^2(\mathbb{R})$
ϕ	Scaling function
$P_j(f)$	Orthogonal projection of f onto V_j
W_j	Multiresolution analysis wavelet space
e_k	Basis
H	Hilbert space
φ	Orthonormal basis defined from scaling function ϕ
δ	Kronecker delta
V_j^p	Multiresolution analysis subspace in $L^p(\mathbb{R})$
$\mathbf{C}$	Wavelet decomposition row matrix
$\mathbf{S}$	Two-dimensional discrete wavelet transform bookkeeping matrix
D^α	Fractional differential operator of order α of an arbitrary definition
$\Gamma(\alpha)$	Euler gamma function
$(I_{a+}^\alpha f)(t)$	Left-sided Riemann-Liouville fractional derivative of order α
$(I_{b-}^\alpha f)(t)$	Right-sided Riemann-Liouville fractional derivative of order α
$(\tau_h f)(t)$	Translation operator
$(D_{a+}^\alpha f)(t)$	Left-sided Riemann-Liouville fractional derivative of order α
$[\alpha]$	Integral part of α
$B(\alpha, \beta)$	beta function
$({}^C D_{a+}^\alpha f)(t)$	Left-sided Caputo fractional derivative of order α
$W_\alpha(x)$	Fractional analog of Wronskian
$\det(\mathbf{A})$	Determinant of the matrix $\mathbf{A}$
$G_\alpha(x)$	Fractional analog of the Green function
$\mathcal{L}$	Laplace transform operator
$(\mathcal{W}_n f)(a, b)$	Poisson wavelet transform of order n
$f_n(a, b)$	Translated Poisson transform of order n
$u(x)$	Solution of differential or integral equation
M, J	Positive integers
$L_m(x)$	Legendre polynomial of order m

$\mathbf{\Psi}$	Legendre wavelet matrix
$\mathbf{c}$	Wavelet coefficient matrix
S_n	Partial sum
$\epsilon_{local}, \epsilon_{local}^{J}$	Local error
$\epsilon_{global}, \epsilon_{global}^{J}$	Global error
u_{ex}	Exact solution of a functional equation

LIST OF APPENDICES

PENDEKATAN BERASASKAN ANAK GELOMBANG UNTUK PENYELESAIAN ANALISIS BAGI PERSAMAAN PEMBEZAAN PECAHAN DAN PENYELESAIAN BERANGKA BAGI PERSAMAAN KAMIRAN PECAHAN

ABSTRAK

Berikutan kebangkitan model tertib pecahan dan akibatnya, persamaan pentadbiran pecahan, kaedah analisis dan berangka baharu telah dibangunkan dan dipelajari secara meluas. Walau bagaimanapun, dalam pelbagai kaedah jelmaan kamiran, belum terdapat kajian yang dibuat mengenai aplikasi jelmaan anak gelombang selanjar dalam menyelesaikan persamaan pembezaan pecahan secara analisis. Bagi mengisi jurang tersebut, disertasi ini menerbitkan kaedah analisis berdasarkan jelmaan kamiran ini dengan mengaplikasikannya pada kamiran dan terbitan Riemann-Liouville, dan terbitan Caputo. Dengan bantuan teorem dan teknik dalam kalkulus dan kalkulus pecahan berserta keputusan yang penting dalam analisis fungsian dan sifat jelmaan anak gelombang selanjar, ia adalah didapati bahawa jelmaan anak gelombang Poisson berperingkat $n = 1$ berupaya menghasil keputusan bermakna yang sesuai untuk menyelesaikan persamaan tertib pecahan. Untuk mendemonstrasikannya, skema ini diaplikasikan untuk menyelesaikan dua persamaan pembezaan pecahan yang ditakrifkan berdasarkan setiap terbitan pecahan yang disebutkan di atas, yang mana penyelesaian tepat berjaya diperoleh. Sebaliknya, kebanyakan analisis berangka bagi persamaan pecahan ditumpukan pada persamaan pembezaan dan oleh demikian, algoritma untuk persamaan kamiran pecahan didapati tidak mencukupi. Didorong oleh kebolehan anak gelombang Legendre dalam menyelesaikan persamaan pembezaan pecahan secara berangka, tesis ini berusaha untuk membina skema berangka berdasarkan anak gelombang ini untuk persamaan kamiran pecahan. Dengan aplikasi analisis berbilang resolusi dan teori penghampiran, kaedah kolokasi menggunakan anak gelombang Legendre telah dibangunkan untuk menghampiri penyelesaian persamaan kamiran pecahan. Perisian MATLAB digunakan untuk membantu dalam proses pengiraan dan penyediaan grafik. Suatu analisis penumpuan

juga disertakan untuk memastikan hampiran yang dihasilkan memang menumpu kepada penyelesaian sebenar. Sebagai akibatnya, algoritma ini ditunjukkan untuk mempunyai ketepatan yang lebih tinggi berbanding kaedah anak gelombang Haar apabila digunakan untuk menyelesaikan persamaan kamiran pecahan Volterra jenis kedua, dan juga mampu menghasilkan hampiran bagi persamaan kamiran pecahan dengan penyelesaian yang tidak diketahui.

A WAVELET-BASED APPROACH FOR THE ANALYTICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND THE NUMERICAL SOLUTIONS OF FRACTIONAL INTEGRAL EQUATIONS

ABSTRACT

Following the rise of fractional-order models and consequently, their fractional governing equations, new analytical and numerical methods have been developed and studied extensively. However, amidst the multitude of integral transform methods, there have been no research done on the application of the continuous wavelet transform to analytically solve fractional differential equations. To fill this gap, the present dissertation derives an analytical method based on this integral transform by applying it to the Riemann-Liouville integral and derivative, and the Caputo derivative. With the help of theorems and techniques in calculus and fractional calculus, important results in functional analysis and properties of the continuous wavelet transform, it is found that Poisson wavelet transform of order $n = 1$ is able to yield meaningful results, suitable for solving fractional-order equations. To demonstrate this, the scheme is applied to solve two fractional differential equations, defined based on each of the aforementioned fractional derivatives, wherein the exact solutions were successfully obtained. On the other hand, much of the numerical analysis of fractional equations have been focused on that of differential equations and thus, algorithms for fractional integral equations have been found wanting. Motivated by the prowess of the Legendre wavelet in solving fractional differential equations numerically, this thesis strives to construct a numerical scheme based on this wavelet for fractional integral equations. By the application of multiresolution analysis and approximation theory, a collocation method using the Legendre wavelet was developed to approximate the solutions of fractional integral equations. The software MATLAB was used to help with computation and graphics. This was also accompanied by a convergence analysis to ensure that the approximation produced indeed converges to the actual solution. As a result, the algorithm was shown to have greater accuracy than the Haar wavelet method when applied to solve fractional

Volterra integral equations of the second kind, and is also able to produce approximations for fractional integral equations with unknown solutions.

CHAPTER 1

INTRODUCTION

1.1 Motivation

> ...fractional calculus can be viewed nowadays as the probability theory before 1933, the year in which Kolmogorov laid the foundations of probability theory by his three axioms.

Such was a depiction of modern fractional calculus by [78]. Indeed, in its current fractured state, this extension of traditional calculus is not blessed with unification. Where, in other corners of mathematics, definitions serve as an unambiguous description of an object or concept, definitions of fractional derivatives and integrals take many names and forms, each with their own faces and characteristics. Yet despite its shattered nature, fractional calculus has garnered the attention of many researchers and forthwith, the cogs of fractional-order differential and integral equations were set into motion.

Among the many revelations brought forth to the field of fractional calculus by mathematicians, scientists and engineers, the first of its applications to real world problems was the integral equation due to Abel in solving the tautochrone problem in 1823 [4, 48]. So monumental was his research that [149] attributed it to have laid a "complete framework for fractional-order calculus". In the years following this publication leading up to the present day, many more studies have been done to explore fractional-order models; yet beyond what was contributed by Abel, these models have simply been extensions of integer-order to fractional. However, one should not exclaim the current state of affairs as disappointing, as the differential and integral equations of fractional calculus has led to a great many discoveries and improvements to existing models. This could be seen in areas involving viscoelasticity [27, 195], medical and health sciences [105, 168], signal processing and control [46, 123], as well as solid mechanics, fluid dynamics and image processing [53]. Within the realm of physics, this also led to new possibilities in the modelling of certain systems and processes, such as electromagnetism [64], fractional quantum mechanics [110] and even the fractional Brownian motion [140].

As the application of fractional calculus to real world problems increases, so too does the need to solve these fractional differential and integral equations. To this end, researchers strode tirelessly to develop algorithms both analytical and numerical. In an effort to circumvent the difficulty of computing fractional integrals and derivatives (in most cases), mathematicians developed analytical schemes centered around integral transforms, allowing for one to bypass this obstacle altogether [98]. These methods were developed for several transformations, among which are notably the Laplace transform and the Fourier transform. This provided an inspiration to probe deeper and investigate whether such results hold similarly for other integral transforms that were hitherto unexplored. This led to our encounter with the continuous wavelet transform and wavelet theory, wherein a further curiosity was aroused – the extent of the applicability of wavelets in both analytical and numerical methods concerning fractional differential equations (FDEs) and fractional integral equations (FIEs).

1.2 Problem Statement

Integral transform methods have left an undeniable mark in fractional calculus, particularly on obtaining the exact analytical solutions of FDEs. This interested many researchers to study the limits of these transforms in the aforementioned area, leading to a multitude of advancements in this direction. This is evident in the recent works surrounding the application of integral transforms to FDEs, for instance the Fourier transform in [121, 188] and the Laplace transform in [20, 32]. Yet amidst the various results established by mathematicians, we were not able to find any account related to the continuous wavelet transform (CWT). This propelled the question of whether we can obtain results that are useful in solving FDEs by applying the CWT to fractional derivatives of functions and hence, provide a new avenue for obtaining the analytical solutions to FDEs.

On the other hand, although the literature for fractional calculus has grown richer over time, further examination transpired that, while a decent amount of work has been done to obtain the solutions of FDEs, research on solving FIEs is much more lacking in comparison, even numerically. Nonetheless, wavelet-based numerical methods have been applied by numerical analysts to solve these equations, with the Haar wavelet being a particularly popular wavelet basis (see, for example, [3, 191]). Many of these algorithms also incorporated other techniques from numerical analysis, such as the pseudospectral method in [49] and operational matrix in [96]. Still, the problem of inadequacy remains. Thus in our effort to contribute to the (wavelet-based) numerical analysis of FIEs, we beg the question: how can we construct a numerical scheme to accurately approximate the solutions of FIEs by utilizing wavelet bases?

1.3 Research Objectives

1. To obtain the continuous wavelet transform of the Riemann-Liouville fractional integral with order $\alpha > 0$.

2. To formulate expressions pertaining to the continuous wavelet transform of the Riemann-Liouville and Caputo fractional derivatives.

3. To apply the results obtained from the previous objectives to analytically solve a fractional differential equation.

4. To develop a numerical algorithm based on orthogonal wavelet bases and multiresolution analysis, and apply it for the numerical solution of fractional Volterra integral equations.

5. To analyze the convergence of the solution approximated by the wavelet-based algorithm to the actual solution of the FIE.

1.4 Research Scope

It is foremost to mention that in the current thesis, only wavelet-based methods will be considered, wherein only those wavelets and orthogonal wavelet bases that already exist will be used. While it is definitely possible to construct new wavelets, orthogonal or not, we believe it best to first exhaust established avenues before turning to the construction of new ones, so as to not bloat the pool of wavelets with unnecessary inventions. Particularly for the results concerning the CWT of fractional operators, an analysis on the suitability (or lack thereof) of certain wavelets will accompany the findings of our study. This, as one will come to find out, leads to only the Poisson wavelet transform being applied in our research. On the other hand, the wavelet-based numerical algorithm will consider only the Legendre wavelet due to its notability in the solution of FDEs. The scheme will be performed in MATLAB and so, all limitations and inaccuracies of said program will apply to our computations and approximations. We will provide an account for all such failings encountered in the relevant chapter.

We end this section by highlighting two further limitations:

1. All considerations in our research will be confined to the one-dimensional case.

2. We will consider only fractional derivatives and integrals of real order $\alpha > 0$.

1.5 Significance of the Study

The present study will establish a new method of solving FDEs by applying the CWT and, in turn, fill the gap in the application of integral transforms in fractional calculus. As this develops a new-found purpose for the wavelet transform, it could motivate the construction of new wavelets or wavelet-like functions that are more effective in achieving this goal. Thus, this wavelet transform method, and what spawns of its extensions and/or generalizations, could lead to the exact solutions of FDEs that are hitherto unknown. This purpose proves to be important due to the usefulness of FDEs in modeling physical and engineering systems [121]. Moreover, this application of the CWT to solve systems governed by fractional calculus operators also partially answers the question posed by [88], especially when taking into account the freedom in the choice of wavelets.

In addition, our research will also devise a wavelet-based numerical algorithm that yields better accuracy that some other published methods, yet is more straightforward to implement comparatively to approximate solutions of FIEs whose exact solutions are "inaccessible" [19]. Perhaps more importantly, however, is its contribution to the numerical analysis of FIEs in wavelet bases, which has seen significantly less attention than its differential equivalent. This study aspires to attract more research in this direction by demonstrating that simpler methods could also yield desirable approximations.

1.6 Thesis Organization

This dissertation investigates the application of wavelet analysis in fractional calculus in an effort to establish further linkage between the two. This thesis is organized as follows. Foremost, the present chapter discusses the motivation and outlines the direction of this research. Subsequently, Chapter 2 serves as exposition to the concepts and theories necessary for the reader to understand the remainder of this thesis, with particular emphasis placed on important definitions in Fourier analysis, the CWT, multiresolution analysis (MRA), and an introduction to fractional calculus and some of its operators. Chapter 3 then serves as an appetizer to the main course of this thesis by providing a comprehensive literature review on published works and studies surrounding the this research's interests. The next two chapters contain the main results of this thesis. First, Chapter 4 explores the interaction of the CWT with fractional operators of the Riemann-Liouville and Caputo sense, as well as the ability of the results obtained thereof to analytically solve FDEs. Then, Chapter 5 sees the construction of a numerical algorithm based on the collocation method and the use of a wavelet basis through MRA, and utilizes this algorithm to approximate the solutions of several FIEs. Finally, Chapter 6 concludes the current work and examines possible future studies.

CHAPTER 2

PRELIMINARIES

In order to have a fruitful discussion pertaining to the newfound results of our research, we need first indulge ourselves in some very important preliminaries. Thus, this chapter shall lay the groundwork by providing a rudimentary exposition of the theories and definitions for concepts that are necessary for the sequel.

2.1 Basic Measure Theory and Functional Analysis

We begin the present chapter with expository descriptions for the technical notions relating to measure theory and functional analysis that are later used in this thesis. It shall be important to highlight that the explanations hereafter are merely rudimentary, as they are only intended to give the reader who is unfamiliar with real and functional analysis a basic idea of the definitions relevant to this work. We start by defining set functions and measures.

Definition 2.1.1 ([192]). *Let* $S = \{A_1, A_2 \ldots\}$ *be a collection of subsets of some interval* $[a, b] \subset \mathbb{R}$ *such that* $\bigcup_{n=1}^{\infty} A_n \in S$*. A function assigning a real number to each set* $A \in S$ *is called a set function. A set function* μ *on* S *is a measure if*

(i) $0 \leq \mu(A) \leq b - a$ *for all* $A \in S$*,*

(ii) $\mu(\emptyset) = 0$*,*

(iii) $\mu(A_1) \leq \mu(A_2)$ *whenever* $A_1 \subset A_2$ *for* $A_1, A_2 \in S$*, and*

(iv) for $A = \bigcup_{n=1}^{\infty} A_n$*,*

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n),$$

where $A_n \in S$ *for* $n = 1, 2, \ldots$ *and* $A_n \cap A_m = \emptyset$ *whenever* $n \neq m$*.*

Hereafter, μ will always be a measure. What follows naturally is the measure for sets, followed by a definition of how one would describe a set as being *measurable*.

Definition 2.1.2 (Outer and inner measures [192]). *Let E be an open set and $A \subset E$. Then, the outer measure $\mu^*(A)$ of A is defined as*

$$\mu^*(A) = \mathrm{glb}\{\mu^*(G) \colon A \subset G,\, G \text{ open in } E\},$$

where glb *denotes the greatest lower bound; while its inner measure $\mu_*(A)$ is given by $1 - \mu^*(E \setminus A)$.*

Remark. *For further understanding on how the sets G are constructed and measured, see the reference cited in this definition.*

Definition 2.1.3 (Measurable sets [192]). *We say that a set $A \subset E$ is (Lebesgue) measurable if $\mu^*(A) = \mu_*(A)$. For any such set A, its measure is given by $\mu(A) = \mu^*(A) = \mu_*(A)$.*

After establishing the idea of measurability of sets, it is now possible to delve into the particularity of measure zero sets and the notion of "almost everywhere". Let $A \subset E$. If there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ (that is, intervals of the form (a, b)) such that for every $\epsilon > 0$,

$$A \subset \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} |I_n| < \epsilon,$$

where $|I_n|$ is the length of the interval I_n, then A is said to have *measure zero*. Any one element set has measure zero. Moreover, we say that two functions f and g are equal *almost everywhere* if $f(x) = g(x)$ for all values of x except at sets of measure zero.

To define the L^p spaces, the notion of *measurable functions* is needed; we shall give its definition here using measurable sets.

Definition 2.1.4 ([192]). *Let $A \subset \mathbb{R}$ be a bounded measurable set. The function $f \colon A \to \mathbb{R}$ is said to be measurable on A if, for every $c \in \mathbb{R}$, the set $\{x \in A \colon c < f(x)\}$ is measurable.*

Indeed, the spaces $L^p(\mathbb{R})$, for $1 \leq p < \infty$, consist of measurable functions f

satisfying

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The objects of primary space here are the spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, with the former serving as a starting point to define the Fourier transform, as one will come to meet. The latter, on the other hand, exhibits many useful properties leading it to becoming the star of many works in Fourier (or harmonic) analysis; this comes from the fact that it is a *Hilbert space*. However, to acquaint ourselves with this concept, we need first grasp the ideas of metrics and completeness.

Definition 2.1.5 (Metric and metric space [102])**.** *Let X be a set and $x, y, z \in X$. A real-valued, finite and nonnegative function d defined on $X \times X$ satisfying*

(i) $d(x, y) = 0$ *if and only if* $x = y$*,*

(ii) $d(x, y) = d(y, x)$*, and*

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

is called a metric on X. The pair (X, d) is then called a metric space.

Definition 2.1.6 ([102])**.** *A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ (dependent on ϵ) such that $d(x_m, x_n) < \epsilon$ for every $m, n > N$. The space X is complete if every Cauchy sequence in X converges in X.*

We are now ready to define Hilbert spaces and the associated concept of inner product spaces.

Definition 2.1.7 ([102])**.** *Let X be a vector space, and let $x, y, z \in X$ and α be a scalar. An inner product $\langle \cdot, \cdot \rangle$ on X is a function on $X \times X$ which assigns to each pair of vectors $x, y \in X$ a scalar, and satisfies*

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$*,*

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$*,*

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, *and*

(iv) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$*,*

where $\overline{g}$ *denotes the complex conjugate of* g*. An inner product on* X *induces a norm on* X *defined by* $\|x\| = \sqrt{\langle x, x\rangle}$ *and a metric such that* $d(x, y) = \|x-y\| = \sqrt{\langle x - y, x - y\rangle}$*. An inner product space (sometimes called a pre-Hilbert space) is a vector space equipped with an inner product. A Hilbert space is a complete inner product space with respect to the metric induced by the inner product.*

We have used the notation $\|\cdot\|$ in the above definition; this refers to a *norm*, which we shall define subsequently.

Definition 2.1.8 ([102])**.** *A norm on a vector space* X *is a real-valued function, denoted by* $\|\cdot\|$*, such that for any* $x, y \in X$ *and scalar* α*, one has*

(i) $\|x\| \geq 0$*,*

(ii) $\|x\| = 0$ *if and only if* $x = 0$*,*

(iii) $\|\alpha x\| = |\alpha|\|x\|$*, and*

(iv) $\|x + y\| \leq \|x\| + \|y\|$*.*

The metric d *on* X *defined by a norm is given by* $d(x, y) = \|x - y\|$ *and is called the metric induced by the norm. A normed space* $(X, \|\cdot\|)$ *is a vector space equipped with a norm.*

Much like the case of inner product spaces, a complete normed space is also given a unique name and is known as a *Banach space*.

Equipped with this knowledge, we are now ready to venture forth into the theories in harmonic analysis and fractional calculus that serve as the backbone of this dissertation.

2.2 The Fourier Transform

A formalism of the Fourier transform is in order. We start with the space $L^1(\mathbb{R})$ of absolutely integrable functions and work our way towards other relevant function spaces.

Definition 2.2.1 (Fourier transform [67])**.** *Let* $f \in L^1(\mathbb{R})$*. Then, the Fourier transform of* f *is given by*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}. \tag{2.1}$$

We denote the Fourier transform operator by $\mathcal{F}$*, so that* $(\mathcal{F} f)(\xi) = \hat{f}(\xi)$*.*

We remark that we chose this expression of the Fourier transform for the sake of convenience; other variations of this transform and their respective explanations can be found in [67]. We now provide a property of the Fourier transform in $L^1(\mathbb{R})$ whose proof we will omit as it involves techniques and theorems in real analysis that do not fit in this thesis; the interested reader can refer to [182].

Proposition 2.2.1 ([182])**.** *If* $f \in L^1(\mathbb{R})$*, then* $\hat{f}$ *is continuous and bounded.*

What naturally follows in this discussion is then the Fourier inversion theorem (sometimes called the inverse Fourier transform), whose proof can be found in [164, 182].

Theorem 2.2.1 (Fourier inversion [67])**.** *If* $f, \hat{f} \in L^1(\mathbb{R})$*, then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi}\, d\xi, \quad x \in \mathbb{R} \tag{2.2}$$

almost everywhere.

Before proceeding, we first introduce Fubini's theorem [172], which will be used in subsequent proofs both within and beyond this chapter.

Theorem 2.2.2 (Fubini [172])**.** *Let* $-\infty \le a < b \le \infty$ *and* $-\infty \le c < d \le \infty$*, and let* f *be a measurable function defined on* $[a, b] \times [c, d]$*. Then,*

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, d(x, y)$$

if at least one of the integrals is absolutely convergent. In particular,

$$\int_a^b \int_a^x f(x, y)\, dy\, dx = \int_a^b \int_y^b f(x, y)\, dx\, dy$$

if at least one of them is absolutely convergent.

In our journey to extend the Fourier transform to the space $L^2(\mathbb{R})$, it is a must that we speak of the Schwartz space $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. To start, we note that if a function f is such that its n-th derivative, $f^{(n)}(x)$ is $O(x^{-m})$ as $x \to \pm\infty$ for $m, n \in \mathbb{Z}$ with $m \geq 1$, $n \geq 0$, then the equality (2.2) holds for all $x \in \mathbb{R}$ [181]. This is precisely the Schwartz space $\mathcal{S}(\mathbb{R})$, and f is said to be *rapidly decreasing at infinity*.

Now, equip $\mathcal{S}(\mathbb{R})$ with the Hermitian inner product given by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}).$$

Then,

$$\left\langle \hat{f}, \hat{g}\right\rangle = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\int_{-\infty}^{\infty} g(x)e^{-ix\xi}\, dx}\, d\xi. \tag{2.3}$$

We give a lemma that is essential to continue the above computation.

Lemma 2.2.3. *Let $f\colon \mathbb{R} \to \mathbb{C}$. Then,*

$$\overline{\int_{-\infty}^{\infty} f(x)\, dx} = \int_{-\infty}^{\infty} \overline{f(x)}\, dx.$$

Proof. Let $\mathrm{Re}[f(x)]$ and $\mathrm{Im}[f(x)]$ denote the real and imaginary parts of $f(x)$, respectively. Observe that

$$\begin{aligned}
\overline{\int_{-\infty}^{\infty} f(x)\, dx} &= \overline{\int_{-\infty}^{\infty} \mathrm{Re}[f(x)] + i\,\mathrm{Im}[f(x)]\, dx} \\
&= \overline{\int_{-\infty}^{\infty} \mathrm{Re}[f(x)]\, dx + i\int_{-\infty}^{\infty} \mathrm{Im}[f(x)]\, dx} \\
&= \int_{-\infty}^{\infty} \mathrm{Re}[f(x)]\, dx - i\int_{-\infty}^{\infty} \mathrm{Im}[f(x)]\, dx \\
&= \int_{-\infty}^{\infty} \mathrm{Re}[f(x)] - i\,\mathrm{Im}[f(x)]\, dx \\
&= \int_{-\infty}^{\infty} \overline{f(x)}\, dx.
\end{aligned}$$

□

Hence, (2.3) becomes

$$\left\langle \hat{f}, \hat{g}\right\rangle = \int_{-\infty}^{\infty} \hat{f}(\xi)\int_{-\infty}^{\infty} \overline{g(x)e^{-ix\xi}}\, dx\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)\int_{-\infty}^{\infty} \overline{g(x)}e^{ix\xi}\, dx\, d\xi,$$

which gives

$$\left\langle \hat{f}, \hat{g} \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi}\, d\xi\, \overline{g(x)}\, dx = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = 2\pi\langle f, g\rangle.$$

by Theorem 2.2.1 and Fubini's theorem. So, $(2\pi)^{-1/2}\mathcal{F}\colon\ \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ forms an isometry. Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ [68], we can extend $\mathcal{F}$ to $L^2(\mathbb{R})$. Consequently, for $f \in L^2(\mathbb{R})$, we define its Fourier transform by a limiting process, that is,

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{-R}^{R} f(x) e^{-ix\xi}\, dx$$

for almost every $x \in \mathbb{R}$. The almost everywhere convergence of the limit is due to Carleson's theorem (or rather, its integral analogue) [35]. As a consequence of this extension of the Fourier transform, its properties that take place in $\mathcal{S}(\mathbb{R})$ can similarly be generalized to the space $L^2(\mathbb{R})$.

Remark. *Note that an isometry is a mapping $T\colon X \to X'$ between two metric spaces $X = (X, d)$ and $X' = (X', d')$ such that $d'(Tx, Ty) = d(x, y)$ for all $x, y \in X$. An isometry is useful as it is a* distance-preserving *transformation.*

It shall be meaningful to delve into a shallow introduction of the discrete Fourier transform (DFT), whose strength lies in the analysis of discrete objects, such as sequences and datasets with distinct values, wherein the (continuous) Fourier transform is rendered inapplicable. To meet our needs in the sequel, we define $f\colon \{0, 1, \dots, N-1\} \to \mathbb{C}$ and denote $\{z_n\}_{n=0}^{N-1} = \{z_0, z_1, \dots, z_{N-1}\} \coloneqq \{f(0), f(1), \dots, f(N-1)\}$.

Definition 2.2.2 (Discrete Fourier transform [22])**.** *Let (z_n) be a sequence of N complex numbers. The discrete Fourier transform of (z_n) is given by $\mathcal{F}[(z_n)] = (x_k)$, where*

$$x_k = \sum_{n=0}^{N-1} z_n e^{-2\pi ink/N} = \sum_{n=0}^{N-1} z_n \left[\cos\left(\frac{2\pi nk}{N}\right) - i\sin\left(\frac{2\pi nk}{N}\right)\right],$$

for $k = 0, 1, \dots, N = 1$.

Recent papers took on the task of studying the application of integral operators to complex analytic functions and a famous inequality surrounding it. The first of such articles is perhaps the one written by Kayumov et al. [93] involving the Cesáro operators. Further discussions around the Bohr phenomenon of integral operators subsequently arose, such as the analysis done in [107] for the β-Cesáro operator with $\beta > 0$, the Bernardi operator, the Libera operator and the Alexander operator. This prompted our interest in an investigation of the Bohr inequality of the Fourier transform when applied to analytic functions, and its corresponding Bohr radius.

Recall that a function f defined on $\mathbb{C}$ is analytic at a point if it is differentiable in a neighborhood of that point, and that it is analytic in a domain $D \subset \mathbb{C}$ if it is analytic at every point in that domain [10]. Our main consideration lies in the family $\mathcal{H}$ of functions defined on $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ that are analytic. Notably, any $f \in \mathcal{H}$ can be written as [180]

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

where $a_n \in \mathbb{C}$.

Our analysis on the Bohr inequality of the Fourier Transform (namely, the DFT) of analytic functions yielded the following result, whose proof can be found in Appendix A.

Theorem 2.2.4. *Let* $\mathcal{B} = \{f \in \mathcal{H} \colon |f(z)| \leq 1\}$. *Then for any* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$,

(i) $\mathcal{F}_f(r) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left| a_k e^{-\frac{2\pi i n k}{n+1}} \right| \right) r^n \leq \frac{1}{1-r}$ *whenever* $r \leq 1/3$, *and*

(ii) $\mathcal{F}_f(r) \leq \frac{1}{1-r}\sqrt{\frac{1}{1-r^2}}$,

where $r = |z|$ *and* $\mathcal{F}_f(r)$ *denotes the DFT of* f *with respect to the variable* r. *These inequalities are sharp.*

2.3 The Wavelet Transform

In what follows we will present only the one-dimensional argument, as our work does not involve higher dimensions; however, most of the results can be extended to the multi-dimensional case, albeit with different notations. In a textbook-like fashion, we start the section with a definition of the CWT.

Definition 2.3.1 (Continuous wavelet transform [50]). *Let $f \in L^2(\mathbb{R})$, and let $a, b \in \mathbb{R}$ be such that $a > 0$. The continuous wavelet transform of f is given by*

$$(\mathcal{W} f)(a, b) = a^{-1/2} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\, dt = \langle f, \psi_{a,b} \rangle, \tag{2.4}$$

where $\overline{g(x)}$ denotes the complex conjugate of g and $\psi_{a,b}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right)$, which is a continuous function with respect to both the time and frequency domains.

In the preceding equation, the function $\psi(t)$, usually belonging to the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, is called the *mother wavelet* and satisfies the admissibility condition,

$$C_\psi = \int_0^\infty \frac{\left|\hat{\psi}(\xi)\right|^2}{\xi}\, d\xi < \infty, \tag{2.5}$$

which implies that wavelets have zero mean, that is, $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$. To see this, note that since $\psi \in L^1(\mathbb{R})$, it follows by Proposition 2.2.1 that $\hat{\psi}$ is continuous. Thus for (2.5) to hold, necessarily $\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)\, dt = 0$. The converse is almost true. We need only to require that ψ additionally satisfies $\int_{-\infty}^{\infty} (1 + |t|)|\psi(t)|\, dt < \infty$ [50, 124].

We now reveal a theorem for recovering f from its wavelet transform and alongside it, the importance of the admissibility condition (2.5).

Theorem 2.3.1 ([50, 124]). *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfy (2.5). Then for any $f \in L^2(\mathbb{R})$,*

$$f(t) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} (\mathcal{W} f)(a, b)\psi_{a,b}(t)\, db\, \frac{da}{a^2}. \tag{2.6}$$

It is clear why we require the wavelets to fulfill the admissibility condition: the reconstruction formula in Theorem 2.3.1 would not hold had C_ψ been infinite!

2.4 Multiresolution Analysis

We venture forth into another colossal contribution of wavelet theory – MRA[1] developed by Mallat [125] and Meyer [129]. We start with a definition.

[1]The notion of MRA has many names; typically, any combination of the words 'multiresolution' or 'multiscale', and 'analysis' or 'appproximation' refers to an MRA.

Definition 2.4.1 (Multiresolution analysis [129]). *A sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces in $L^2(\mathbb{R})$ is a multiresolution analysis if it satisfies*

1. $V_j \subseteq V_{j+1}$ *for all* $j \in \mathbb{Z}$
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
4. $f(\cdot) \in V_j \iff f(2\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$
5. $f(\cdot) \in V_0 \iff f(\cdot - k) \in V_0$ *for all* $k \in \mathbb{Z}$
6. *there exists* $\phi \in V_0$ *(called the scaling function) such that* $\{\phi(\cdot - k) \colon k \in \mathbb{Z}\}$ *forms an orthonormal basis of* V_0.

Consequently, due to Conditions 4 and 6, the function $\phi_{j,k} := 2^{j/2}\phi(2^j t - k)$ span each of the V_j. We also note that the V_j are simply V_0 scaled by a factor of 2^j. This scaling of the V_j's, alongside Condition 5, reveal an interesting and useful characteristic of MRA – translation invariance of the spaces, that is, $f \in V_j \implies f(\cdot - 2^{-j}k) \in V_j$ for all $k \in \mathbb{Z}$. The keen reader might have observed that there is no mention of wavelets in the definition of a MRA. While this is indeed true, wavelets do play a big role in the construction of a MRA, serving as a complementary component to the scaling functions.

We claim that, due to the nature of the MRA, we are able to expand functions $f \in L^2(\mathbb{R})$ in terms of wavelet bases, that is,

$$P_{j+1}(f) = P_j(f) + \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \tag{2.7}$$

where $P_j(f)$ denotes the orthogonal projection of f onto V_j and $\psi_{j,k} := 2^{j/2}\psi(2^j t - k)$. To show this, we define the wavelet spaces W_j that are orthogonal complements to the V_j's in V_{j+1}, so that

$$V_{j+1} = V_j \oplus W_j$$

and that $W_i \perp W_j$ for $i \neq j$. Note that equation (2.7) is equivalent to saying that given j, the collection of translated and dilated wavelets, $\{\psi_{j,k} \colon k \in \mathbb{Z}\}$ constitutes an orthonormal basis for W_j.

We now attempt to extend this result, so that (2.7) can be written in a form that does not rely on the projection onto V_j; to be precise, we wish to be able to express a function $f \in L^2(\mathbb{R})$ in terms of purely wavelet bases. We continue from the definition of the spaces W_j. First, recall that $V_{-\infty} = \{0\}$ by Properties 1 and 3 of Definition 2.4.1. Now,

$$\overline{V_{j+1}} = \overline{\bigcup_{n=-\infty}^{j+1} V_n} = \bigoplus_{n=-\infty}^{j} W_n$$

as the W_n's are closed. Then as $j \to \infty$, Property 3 from Definition 2.4.1 gives

$$\bigoplus_{j \in \mathbb{Z}} W_j = \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

This allows us to rewrite (2.7) as

$$\begin{aligned}
P_{J+1}(f) &= P_J(f) + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J,k} \rangle \psi_{J,k} \\
&= P_{J-1}(f) + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J-1,k} \rangle \psi_{J-1,k} + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J,k} \rangle \psi_{J,k} \\
&= P_{J-2}(f) + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J-2,k} \rangle \psi_{J-2,k} + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J-1,k} \rangle \psi_{J-1,k} \\
&\quad + \sum_{k=-\infty}^{\infty} \langle f, \psi_{J,k} \rangle \psi_{J,k} \\
&= P_{J-2}(f) + \sum_{j=J-2}^{J} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k} \\
&\;\;\vdots \\
&= \sum_{j=-\infty}^{J} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}, \qquad (2.8)
\end{aligned}$$

or, in more generality,

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \qquad (2.9)$$

We remark that projecting f onto V_j with respect to the scaling functions is also a useful technique (and so too is the combination of the two); we cast the spotlight onto the projection relative to the wavelet bases as, contextually, this expansion of a function f in orthonormal wavelet bases serves more importance to us. We will, however, give the expression of the orthogonal projection of f onto V_j in terms of the scaling functions:

$$P_j(f) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{j,k} \rangle \phi_{j,k}.$$

2.5 The Integrals and Derivatives of Fractional Calculus

It is perhaps easier to approach fractional calculus by giving the reader a geometric intuition of fractional derivatives and thus, we shall do so. It will be important to mention before proceeding any further that we shall discuss only those theories in fractional calculus which will be entailed in future chapters[2].

2.5.1 Half Derivatives and a Primer for Fractional Order Calculus

Consider the function $f(x) = x^2$ and so, $Df(x) = 2x$, where D represents the standard differential operator d/dx. We pose the question: what is $D^{1/2}f$ and what does it represent? Intuition tells us that the half derivative should be of the form $Cx^{3/2}$ with $C \in \mathbb{R}$. In addition, for this fractional order differential operator to obey the standard result of the power rule for monomials,

$$D^n x^m = \frac{m!}{(m-n)!} x^{m-n}, \tag{2.10}$$

the coefficient C should consist of some factorial term. This is, in fact, true and we can actually modify (2.10) slightly to give us an expression for the α-th order derivative of a monomial x^m (the proof of this will come later):

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{m-\alpha}, \tag{2.11}$$

[2]For the reader who is interested in a more textbook-like approach, as well as a more encyclopedic account on the rudiments of fractional calculus, the book [98] is much recommended.

where Γ is the Euler gamma function given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx. \tag{2.12}$$

We shall stray from our discussion of fractional derivatives briefly to discuss some useful properties of this gamma function Γ that will prove crucial to our work in subsequent parts of this thesis. The first of such properties is in order.

Proposition 2.5.1. *Let* $\alpha > 0$. *Then,* $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Proof. This can be seen by a straightforward application of integration by parts:

$$\begin{aligned}
\Gamma(\alpha + 1) &= \int_0^\infty x^\alpha e^{-x}\, dx \\
&= -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx \\
&= \lim_{x\to\infty} -x^\alpha e^{-x} - 0 + \alpha \int_0^\infty x^{\alpha-1} e^{-x}\, dx \\
&= \alpha\Gamma(\alpha).
\end{aligned}$$

That $\lim_{x\to\infty} -x^\alpha e^{-x} = 0$ follows from L'Hôpital's Rule. □

We derive the next property by mathematical induction. Set the base case to be $\Gamma(1) = 0!$; this can be verified easily:

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1 = 0!.$$

Now, assume that $\Gamma(n) = (n - 1)!$ for $n \in \mathbb{N}$ and observe that by Proposition 2.5.1,

$$\Gamma(n + 1) = n\Gamma(n) = n(n - 1)! = n!.$$

This proves the following proposition.

Proposition 2.5.2. *Let* $n \in \mathbb{N}$. *Then,* $\Gamma(n) = (n - 1)!$.

We give the last two formulas without proof; they can both be found in [172].

Proposition 2.5.3 ([172]). *The functional equation*

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}, \quad 0 < \alpha < 1$$

holds. In particular,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proposition 2.5.4 ([172]). *Let $z \in \mathbb{C}$ with $z \neq 0$, and let $\alpha > 0$ be such that $0 < \alpha < 1$ when $\text{Re}(z) = 0$ and that the principal value of z^α is chosen such that z^α is positive for $z > 0$. Then,*

$$\int_0^\infty x^{\alpha-1} e^{-zx}\, dx = \frac{\Gamma(\alpha)}{z^\alpha}.$$

Thus to answer the first half of our question at the start of this subsection, we have

$$D^{1/2} f(x) = \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2} = \frac{2!}{(3/2)(1/2)\Gamma(1/2)} x^{3/2} = \frac{8}{3\sqrt{\pi}} x^{3/2}.$$

Unfortunately, the notion of half derivatives have yet to have a definitive mathematical interpretation established; it does, however, have a significant geometric meaning. To see this, we shall consider the second derivative of f, $f''(x) = 2$ and compute

$$D^{3/2} f(x) = \frac{\Gamma(3)}{\Gamma(3/2)} x^{1/2} = \frac{2 \cdot 2!}{\sqrt{\pi}} x^{1/2} = \frac{4}{\sqrt{\pi}} x^{1/2} = DD^{1/2} f(x).$$

We shall defer the exploration of the interesting relation $D^{3/2} f = DD^{1/2} f$ as it will become apparent when we expose ourselves to the formal definition of this fractional derivative. For now, we illustrate the derivatives of $f(x) = x^2$ to give a clearer description on half derivatives.

One could deduce, from Figure 2.1 that the derivatives $D^{1/2} f$ and $D^{3/2} f$ behave like "intermediary forms" as $f \to f'$ and $f' \to f''$, respectively. Indeed, such is true for fractional order derivatives: given $m, n \in \mathbb{N}$ with $m < n$ and a function f, the fractional derivatives $D^\alpha f$, $m < \alpha < n$, show the change as the function goes from its m-th derivative to its n-th derivative.

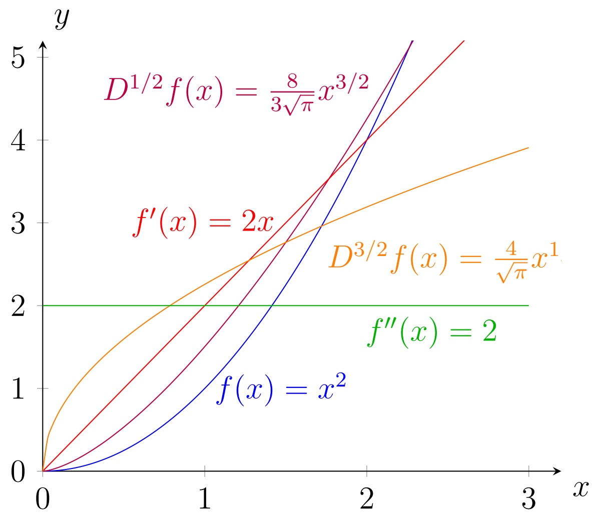

Figure 2.1: Derivatives of $f(x) = x^2$ of order $\alpha = 0, 1/2, 1, 3/2, 2$

The discussion on fractional calculus hitherto had been informal in nature, so as to present a more intuitive overview. Hereon, we shall transition to more technical descriptions of fractional integrals and derivatives as such is necessary for our work henceforth. In this process, a prominent notoriety of fractional calculus will come to surface: the lack of a universal definition for integrals and derivatives. Since the fractional derivatives of our interest are defined based on fractional integrals, we shall first examine the latter.

2.5.2 The Riemann-Liouville Fractional Integral and Derivative

The historical genesis of the countless work which led to what is called the Riemann-Liouville fractional integral is Cauchy's integral formula, first studied in a paper by Sonin [161, 179]. However, to delve into this development of the fractional integral formula would mean to indulge ourselves with many technicalities from complex analysis that will not prove relevant beyond this point, unnecessarily lengthening this thesis. Thus, we will present naught but the product of this process[3].

Definition 2.5.1 (Riemann-Liouville fractional integral [98])**.** *Let* $-\infty \leq a < b \leq \infty$ *and* $\alpha > 0$*. The left- and right-sided Riemann-Liouville fractional integrals of order* α

[3]The curious reader is directed to [161] for a chronological discussion on the development of fractional calculus.

are given by

$$\left(I_{a+}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)\,dx, \quad t>a$$

and

$$\left(I_{b-}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}f(x)\,dx, \quad t<b,$$

respectively.

Remark. *While we will consider only $\alpha > 0$, the definitions of fractional integrals and derivatives given herein can be generalized to any $\alpha \in \mathbb{C}$ with $\mathrm{Re}[\alpha] > 0$. It is also important to mention that there is no consensus for the notations of fractional integral and differential operators – it simply depends on the author's preference. For instance, another widely used notation for the Riemann-Liouville fractional integral is J^{α} (or some variation of it).*

In most practical situations, only the left-sided Riemann-Liouville fractional integral is of interest; this is also true here. Hence, we shall limit whatever discussions of fractional integrals and derivatives to only the left-sided operator, with the above definition serving as a reminder that there also exists a right-sided formula for each of these definitions. For our cause, we are primarily interested in the case where $a = 0$ and $b = \infty$, so that $I_{a+}^{\alpha}f$ is defined on $[0,\infty)$. For this particular case, the following holds [98].

Proposition 2.5.5 ([98])**.** *Let $h \in \mathbb{R}$, and denote by τ_h the translation operator such that $(\tau_h f)(t) = f(t-h)$. For $\alpha > 0$,*

$$\tau_h I_{0+}^{\alpha} f = I_{h+}^{\alpha}\tau_h f.$$

What naturally follows in our current discussion is the Riemann-Liouville fractional derivative. As previously mentioned, its definition is based upon that of the fractional integral, as we shall now see.

Definition 2.5.2 (Riemann-Liouville fractional derivative [98])**.** *Let $-\infty \leq a < b \leq \infty$ and $\alpha > 0$. The left-sided Riemann-Liouville fractional derivative of order α is defined,* *for $t > a$, as*

$$\left(D^{\alpha}_{a+}f\right)(t) = \left(\frac{d}{dt}\right)^{n} \left(I^{n-\alpha}_{a+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} (t-x)^{n-\alpha-1} f(x)\,dx,$$

where

$$n = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N} \end{cases}, \tag{2.13}$$

and $[\alpha]$ denotes the integral part of α.

Similarly, our attention is focused on when $0 < t < \infty$. Hereafter, we will denote by $f^{(n)}$ the classical integer order derivative to avoid confusion. We shall now carry out the promise we made earlier to prove the equation (2.11).

Proof of equation (2.11). Consider Definition 2.5.2 with $f(x) = x^m$ and $a = 0$. Then,

$$\begin{aligned} \left(D^{\alpha}_{0+}x^m\right)(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-x)^{n-\alpha-1} x^m \,dx \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} t^{n-\alpha-1} \left(1-\frac{x}{t}\right)^{n-\alpha-1} t^m \left(\frac{x}{t}\right)^{m} dx \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha-1} \int_{0}^{t} \left(1-\frac{x}{t}\right)^{n-\alpha-1} \left(\frac{x}{t}\right)^{m} dx \end{aligned}$$

It follows by the substitution $u = x/t$ that $dx = t\,du$ and so,

$$\begin{aligned} \left(D^{\alpha}_{0+}x^m\right)(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha-1} \int_{0}^{1} (1-u)^{n-\alpha-1} u^m t\,du \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha} \int_{0}^{1} (1-u)^{n-\alpha-1} u^m \,du. \end{aligned}$$

Now, the beta function B given by

$$B(\alpha,\beta) = \int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1}\,dx \tag{2.14}$$

has the popular identity [17]

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Thus,

$$\begin{aligned}\left(D_{0+}^{\alpha}x^{m}\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha}B(m+1,n-\alpha)\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(m+1)\Gamma(n-\alpha)}{\Gamma(m+1+n-\alpha)}\left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha}\\ &= \frac{\Gamma(m+1)}{\Gamma(m+1+n-\alpha)}\left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha}.\end{aligned}$$

Observe that

$$\begin{aligned}\left(\frac{d}{dt}\right)^{n} t^{m+n-\alpha} &= (m+n-\alpha)(m+n-\alpha-1)\cdots(m-\alpha+1)t^{m-\alpha}\\ &= \frac{\Gamma(m+n-\alpha+1)}{\Gamma(m-\alpha+1)}t^{m-\alpha},\end{aligned}$$

and hence,

$$\begin{aligned}\left(D_{0+}^{\alpha}x^{m}\right)(t) &= \frac{\Gamma(m+1)}{\Gamma(m+1+n-\alpha)}\frac{\Gamma(m+n-\alpha+1)}{\Gamma(m-\alpha+1)}t^{m-\alpha}\\ &= \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}t^{m-\alpha}.\end{aligned}$$ □

2.5.3 Caputo Fractional Derivative

We follow the standard approach endorsed by fractional calculus textbooks, that is, following the introduction of the Riemann-Liouville operators, we acquaint ourselves with the Caputo fractional derivative[4].

Definition 2.5.3 (Caputo fractional derivative [98]). *Let* $-\infty \leq a < b \leq \infty$ *and* $\alpha > 0$. *The left-sided Caputo fractional derivative is defined in terms of the Riemann-Liouville fractional derivative by*

$$\left({}^{C}D_{a+}^{\alpha}f\right)(t) = \left(D_{a+}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k}\right]\right)(t),$$

where n *is given by* (2.13).

[4] Again, only the left-sided derivative will be given here.

While this definition makes the Caputo fractional derivative look intimidating, we assure the reader that in most practical cases, we instead adopt one that is tamer; the presentation of the above definition is merely for the sake of completeneness and formalism. To give this more elegant description of the Caputo fractional derivative, however, we require the following function spaces.

Definition 2.5.4 ([98])**.** *Let $n \in \mathbb{N} \cup \{0\}$ and $-\infty \leq a < b \leq \infty$, and let $C[a,b]$ be the space of continuous functions on $[a,b]$. We denote by $C^n[a,b]$ the space of functions that are n-times continuously differentiable on $[a,b]$. Mathematically,*

$$C^n[a,b] = \left\{ f\colon [a,b] \to \mathbb{C} \mid f', f'', \ldots, f^{(n-1)} \in C[a,b] \right\}.$$

In particular, $C^1[a,b] = C[a,b]$.

Definition 2.5.5 ([98])**.** *Let $n \in \mathbb{N}$ and $-\infty < a < b < \infty$, and let $AC[a,b]$ be the space of functions that are absolutely continuous on $[a,b]$. We denote by $AC^n[a,b]$ the space consisting of functions with continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in AC[a,b]$, that is,*

$$AC^n[a,b] = \left\{ f\colon [a,b] \to \mathbb{C} \mid f^{(n-1)} \in AC[a,b] \right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

We now give the "nicer" expression for ${}^C D^{\alpha}_{a+}$ [98]; this is the main representation that will be adopted in the sequel.

Theorem 2.5.1 ([98])**.** *Let $-\infty \leq a < b \leq \infty$ and $\alpha > 0$, and let n be as in (2.13). If $f \in AC^n[a,b]$, then ${}^C D^{\alpha}_{a+} f$ exists almost everywhere on $[a,b]$ and has the representation*

$$\left({}^C D^{\alpha}_{a+} f\right)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)\, dx, & \alpha \notin \mathbb{N}, \\ f^{(n)}(t), & \alpha \in \mathbb{N}. \end{cases}$$

If, instead, $f \in C^n[a,b]$, then ${}^C D^{\alpha}_{a+} f \in C[a,b]$.