# IMPROVEMENT OF CURVE CONSTRUCTION USING BI-QT BÉZIER CURVES AND APPROXIMATION TO TWO TYPES OF BÉZIER CURVES

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**by**

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| | |
|---|---|
| **CAGD** | Computer Aided Geometric Design |
| **CNC** | Computer Numerical Control |
| **PSO** | Particle Swarm Optimization |
| **QT** | Quadratic Trigonometric |
| **PIA** | Progressive Iterative Approximation |

# LIST OF SYMBOLS

| | |
|---|---|
| $\alpha$ | side length of the first control triangle |
| $\beta$ | side length of the second control triangle |
| $r$ | ratio between the length of the first isosceles control triangle, $\alpha$ and the length of the second isosceles control triangle, $\beta$ |
| $\bar{\alpha}$ | value of $\alpha$ obtained by PSO |
| $\theta$ | angle in degree |
| $m$ | shape parameter |
| $t$ | time parameter |
| $d$ | a positive integer representing the number of subdivisions or intervals between the lower and upper ratios, with the center serving as a reference point |
| $d_{new}$ | a new value of $d$ |
| $\rho$ | a real value serving as a component to determine the new value of d used for the next iteration |
| **P** | control polygon |
| **B** | biarc curve |

**PENAMBAHBAIKAN DALAM PEMBINAAN LENGKUK MENGGUNAKAN LENGKUK BI-QT BÉZIER DAN PENGHAMPIRAN KEPADA DUA JENIS LENGKUK BÉZIER**

## ABSTRAK

Pendekatan baharu, iaitu lengkung dwi-QT-Bézier yang dioptimumkan, untuk menyuaikan lengkung pada 2D, telah dicadangkan. Pendekatan konvensional mempunyai kekangan tambahan bagi menentukan dwi-lengkuk yang unik. Kaedah yang dicadangkan menyepadukan rumus dwi-lengkuk tunggal yang berasaskan lengkung Trigonometri Kuadratik (QT)-Bézier dengan Pengoptimuman Kawanan Zarah (PSO). Dwi-QT-Bézier yang dicadangkan adalah bermanfaat dalam penyuaian lengkung kerana ia mempunyai nilai $\alpha$ yang optimum daripada kaedah PSO. Selain itu, skim yang dicadangkan juga menyediakan kelonggaran untuk membina lengkung yang dikehendaki iaitu dengan menghasilkan lengkung yang lebih dekat dengan poligon kawalannya berbanding kaedah-kaedah sebelum ini. Keputusan eksperimen disediakan bagi menunjukkan kegunaan dan kecekapan kaedah yang dicadangkan. Dwi-QT-Bézier yang dioptimumkan digunakan untuk menyuai bulatan dan bentuk komposit, dan hasilnya turut dianalisa. Keputusan menunjukkan bahawa kaedah yang dicadangkan merupakan alat yang sangat baik dalam penyuaian lengkung. Kerja ini juga memperkenalkan Penganggaran Lelaran Progresif (PIA) yang telah diubahsuai iaitu PIA Jenis 1 dan Jenis 2, yang masing-masing bertujuan untuk menukarkan pembinaan lengkung kepada Bézier dan QT-Bézier. PIA Jenis 1 meningkatkan darjah lengkung, manakala PIA Jenis 2 menyesuaikan parameter bentuk $m$. Keputusan berangka bagi beberapa contoh untuk kedua-dua kaedah ini turut dibincangkan.

**IMPROVEMENT OF CURVE CONSTRUCTION USING BI-QT BÉZIER CURVES AND APPROXIMATION TO TWO TYPES OF BÉZIER CURVES**

## ABSTRACT

A new approach, namely an optimized bi-QT-Bézier, for fitting curves to given 2D, is proposed. The conventional approach includes additional constraints to uniquely determine the biarc. The proposed method integrates the formulation of a single biarc based on the Quadratic Trigonometric (QT)-Bézier curve with Particle Swarm Optimization (PSO). The proposed bi-QT-Bézier curve is advantageous in curve fitting as it provides an optimized value of $\alpha$ from the PSO method. Besides, the proposed scheme also provides the flexibility to construct the desired curve with a smaller distance between the curve and its control polygon compared to the previous methods. An experimental result is provided to demonstrate the usefulness and efficiency of the proposed method. The implementation of the optimized bi-QT-Bézier curve to fit circular and composite shapes will also be analyzed. The result shows that the proposed method is an excellent tool in curve fitting. This work also presents the modified Progressive Iterative Approximation (PIA), namely PIA Type 1 and Type 2, to convert the curve construction into Bézier and QT-Bézier respectively. PIA Type 1 deals with the increment of degree while PIA Type 2 deals with the adjustment of shape parameter $m$. Numerical results of the application of both approaches to several examples are discussed.

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

A crucial operation in engineering design is the approximation of data, points, lines, or arbitrary curves of various types. Because of the simplicity of these curves and the ability of milling machines to run along straight lines and circular pathways, Parkinson and Moreton (1991) mentioned that the approximation of curves by straight lines and circular arcs is really important. A commonly employed method for approximating arc splines is the biarc. A biarc is a curve formed by connecting two circular arcs with $G^1$ continuity, ensuring a smooth transition with a matching tangent (Bolton, 1975; Meek and Walton, 1992; Park, 2004b; Piegl and Tiller, 2002a).

Biarc exhibits $G^1$ continuity, which means that the tangents at the connection point of the two circular arcs are collinear. This ensures a smooth and continuous transition between the arcs without any abrupt changes in direction or slope. To understand $G^1$ continuity in a biarc, imagine two circular arcs that make up a curve. At the point where the arcs meet, the tangents of both arcs are extended to form a line. In a $G^1$ continuous biarc, these extended tangents lie on the same straight line. This alignment ensures that there is no visible break or corner at the connection point. It is worth noting that the length of each tangent at the connecting point of a biarc is different. This is because a biarc is a curve formed by joining two circular arcs with different radii and centers. The difference in radii and centers results in varying lengths of the tangents at the connecting point.

According to Bolton (1975), biarc is used to replace the conventional approach of curve fitting, which is cubic spline. The conventional biarc approach comprises an additional constraint to determine the biarc uniquely. When considering the specific data points $p_0$ and $p_4$, as well as their corresponding tangents $t_1$ and $t_2$, it is possible to create a biarc that connects these endpoints. This concept, described by Sakai (1994) and Ong et al. (1996), involves representing the biarc using two circular arcs, which are labeled as $D_1$ and $D_2$. These arcs form the components of the biarc and satisfy both conditions as follows:

1. $D_i$ (where $i = 1, 2$) passes through endpoints and are tangential to end tangents at respective endpoints. This ensures that each arc starts and ends at the correct point and smoothly transitions into the direction indicated by the tangent.

2. $D_i$ (where $i = 1, 2$) are tangential to each other at their joint. This guarantees that the transition between the two arcs is seamless and that they form a continuous curve without any abrupt changes in direction or curvature.

Figure 1.1 illustrates the biarc construction to demonstrate the conditions. The first condition emphasizes that the arcs $D_1$ (brown arc) and $D_2$ (purple arc) pass through their respective endpoints. This means that the curve starts at $p_0$ and ends at $p_4$, ensuring that the biarc spans the desired interval. Additionally, at each endpoint $p_0$ and $p_4$, the tangent of the arc at endpoints which are indicated by black arrows, are aligned with the end tangents that are represented by the blue and orange dashed lines, respectively. This ensures a smooth transition between the tangent direction and the curve, creating a visually pleasing and continuous curve.

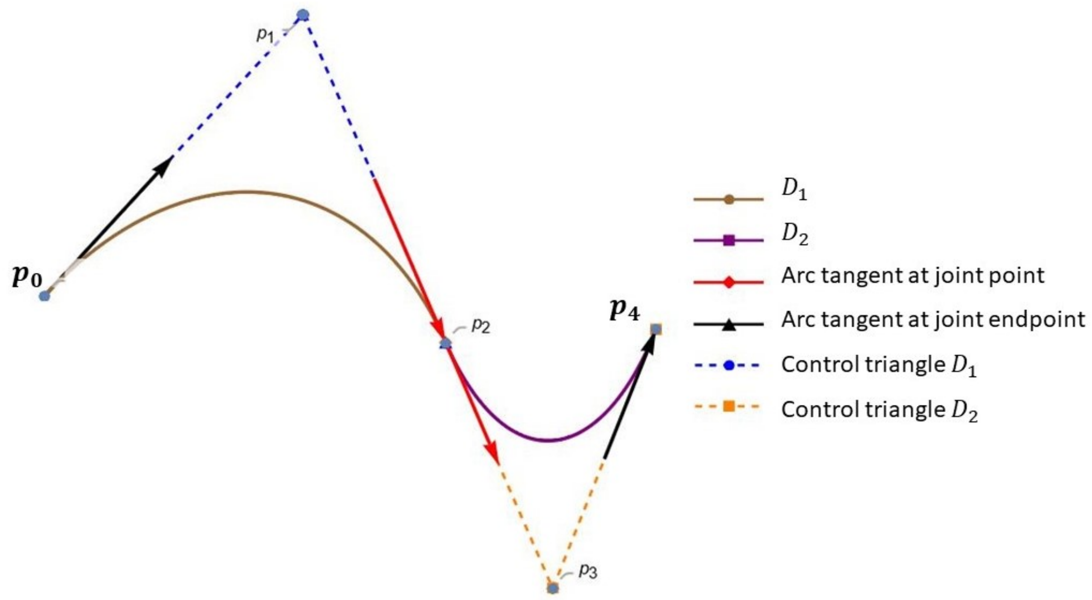


Figure 1.1: Biarc construction

Furthermore, the second condition emphasizes the tangential relationship between the two arcs $D_1$ (Brown arc) and $D_2$ (Purple arc) at their connecting point, $p_2$. When the arcs meet, they are tangential to each other, meaning they share the same direction at the connecting point $p_2$, as indicated by the red arrows. This ensures a seamless transition between the two arcs, without any abrupt changes in direction or curvature. The tangential connection guarantees that the curve flows smoothly and maintains a consistent direction throughout, enhancing its visual continuity and aesthetic appeal.

In summary, the first condition focuses on the behavior of the arcs at the endpoints, ensuring that they pass through the respective endpoints and align with the end tangents. The second condition, on the other hand, emphasizes the tangential connection between the two arcs at their joint, guaranteeing a seamless transition and a continuous curve.

The biarc curve possesses six degrees of freedom, which correspond to six independent parameters that define its shape. These parameters include the start point (2 coordinates), the endpoint (2 coordinates), the start tangent (1 direction), and the end tangent (1 direction). However, when considering conditions (1) and (2), only five of these degrees of freedom are utilized. This is due to the fact that the endpoint of the first arc coincides with the start point of the second arc, resulting in one shared degree of freedom between the two arcs. In the context of a biarc curve, the missing degree of freedom would result in an incomplete specification of the curve. This could lead to ambiguity or multiple possible curve configurations, making it challenging to achieve the desired curve.

To address this issue and distinguish the biarc, additional constraints are introduced. One common constraint is that the tangent at the connecting point remains parallel to the line connecting the two endpoints (Sabin, 1977). This constraint provides an additional condition that aids in uniquely defining the biarc. By considering conditions (1) and (2) along with the additional constraints, a well-defined and uniquely determined biarc curve can be obtained.

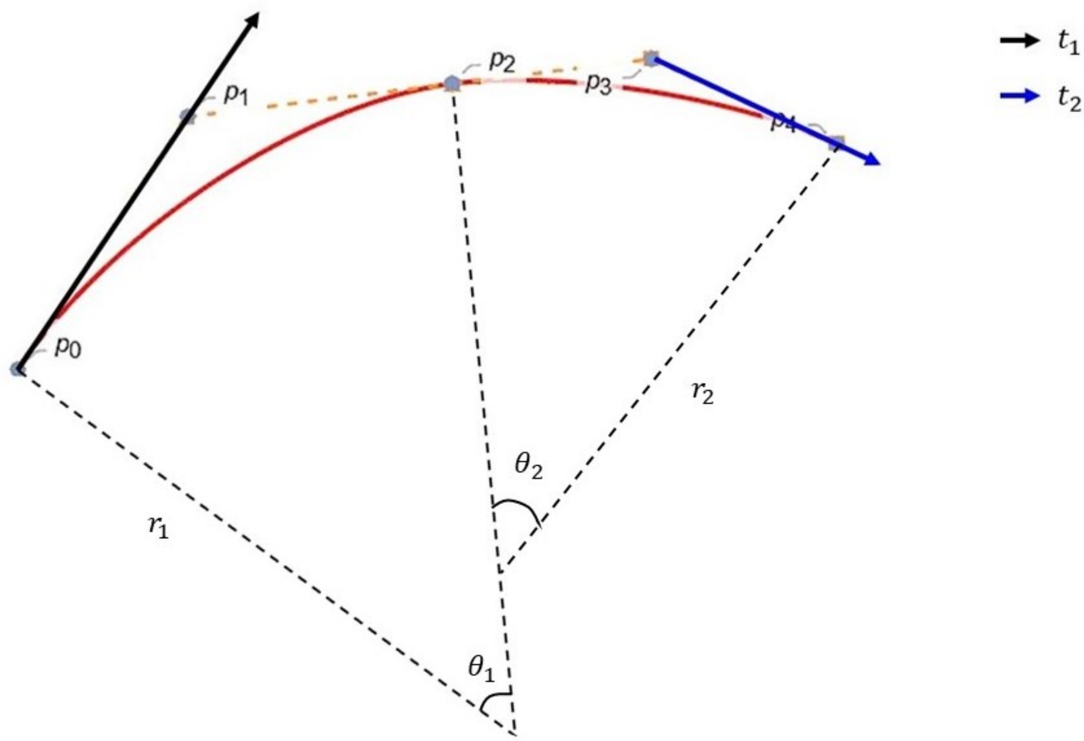


Figure 1.2: C-shaped biarc

Figures 1.2 and 1.3 provide visual representations of biarc interpolation. In Figure 1.2, a C-shaped biarc is illustrated, which is formed when the end tangents, represented as $t_1$ and $t_2$, exhibit different directions. Specifically, this occurs when $t_1$ is positive and $t_2$ is negative, or vice versa. The C-shaped biarcs are characterized by having the data points $p_1$ and $p_3$ located on the same sides of the curve, as shown in Figure 1.2. On the other hand, Figure 1.3 illustrates an S-shaped biarc, which arises when the end tangents have the same direction. In this case, both $t_1$ and $t_2$ are either positive (both greater than zero) or negative (both less than zero). The S-shaped biarcs exhibit the characteristic of having the data points $p_1$ and $p_3$ located on the opposite side of the curve, as shown in Figure 1.3.

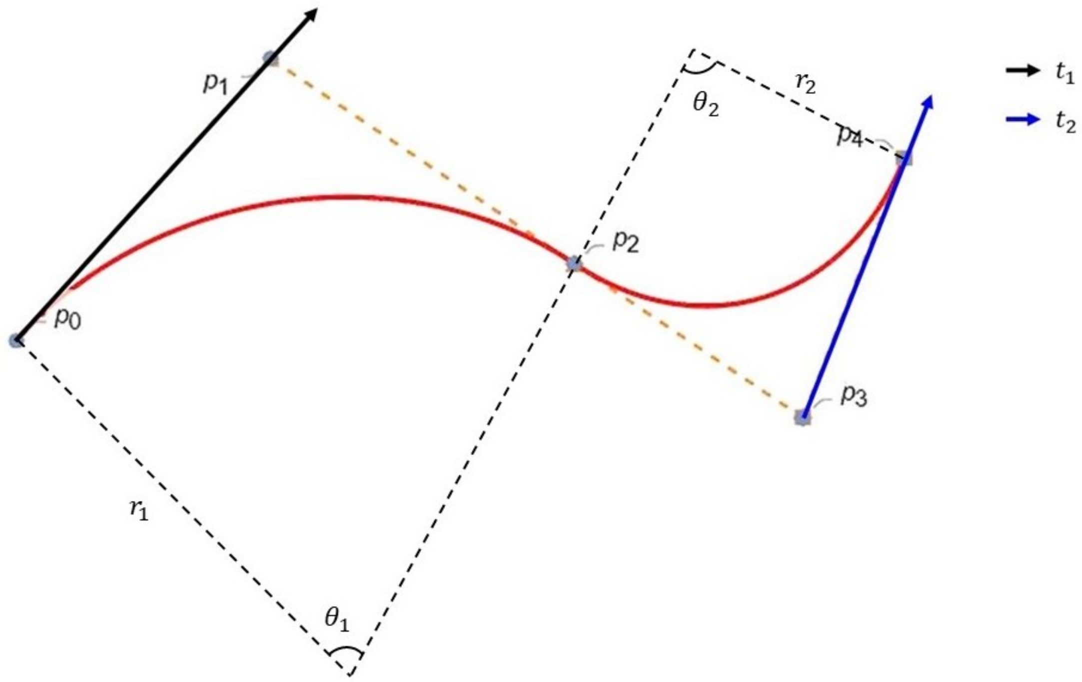


Figure 1.3: S-shaped biarc

This thesis focuses on enhancing the arc spline approximation. The goal is to construct a curve that closely approximates the control polygon. Instead of using a biarc, this study proposed an optimized bi-QT-Bézier curve. The optimized bi-QT-Bézier is a curve formed by connecting two QT-Bézier curves in a $G^1$ continuity manner, ensuring a minimal distance between the curve and the control polygon using an optimization method. The optimization method aims to find the optimal value of the length of the control triangle which is denoted as $\alpha$. The value of $\alpha$ is crucial in shaping the curves. Therefore, by optimizing the value of $\alpha$, the resulting optimized bi-QT-Bézier curve can fit the given data points and reduce the distance between the curve and the control polygon.

Additionally, the thesis explores the conversion of curves into two different basis functions using a modified iterative approach. This conversion process aims to transform the curve representation into alternative basis functions, which can offer advantages in terms of computational efficiency. Overall, this thesis investigates the improvement of curve construction by incorporating optimization methods and exploring alternative basis functions. The objective is to enhance the accuracy and effectiveness of curve fitting while minimizing the distance between the curve and the control polygon and reducing the number of segments needed to achieve a good fit.

## 1.2 Motivation

The work by Park (2004b) proposed an interesting biarc formulation namely optimal single biarc. This study highlights the key properties of the proposed biarc, which does not impose any additional constraints. The approach leverages the flexibility in determining the biarc to achieve a more accurate fit by minimizing the distance between the curve and the control polygon. Furthermore, the proposed biarc results in a reduced number of segments in spline approximation. The study is also simple in concept since the approach does not require any additional constraints to uniquely determine the biarc. This makes the approach simpler to understand and implement. Additionally, the approach is computationally inexpensive and can be applied efficiently. The utilization of the optimal single biarc is extensive and has been employed to address issues related to the arc spline approximation of 2D data points (Park, 2004a).

In addition, the study conducted by Uzma et al. (2012) emphasized the characteristics of the Quadratic Trigonometric (QT) Bézier curve with a single shape parameter. The QT-Bézier curve plays a significant role in minimizing the distance between the curve and the control polygon. It is noteworthy that the properties of the QT-Bézier curve are similar to those of the ordinary quadratic Bézier curve. However, the QT-Bézier curve offers added utility due to the inclusion of the shape parameter. Moreover, under specific conditions, the QT-Bézier curve can effectively represent arcs of circles and ellipses.

Furthermore, the Particle Swarm Optimization (PSO) algorithm is a powerful optimization technique that has been widely employed for solving complex optimization problems. Its ease of implementation and fast convergence rate make it a popular choice among practitioners. In the study conducted by Liu and Li (2016), PSO was utilized to determine the optimal shape parameters for curves. The results demonstrated its effectiveness in enhancing the fairness and smoothness of the curves, thereby validating its usefulness in curve optimization.

Lastly, the work by Chantakamo and Dejdumrong (2013) points out the advantage of using the Progressive Iterative Approximation (PIA) algorithm, where it can construct a non-rational Bézier curve that fits the sampling points that are obtained from the input rational Bézier curve. This allows for the conversion of a rational Bézier curve into a non-rational Bézier curve, which can be helpful in certain applications. Additionally, the PIA algorithm can be used to adjust the control points of the non-rational Bézier curve to improve the approximation.

## 1.3 Problem Statements

Existing approaches in biarc fitting often lack critical components, such as Trigonometric Bézier with shape parameters and optimization algorithms, which can enhance curve accuracy and minimize the distance between the curve and the control polygon.

The basis transformation in Chantakamo and Dejdumrong's (2013) work is limited to transforming rational Bézier curves into non-rational Bézier curves, which restricts its application in basis transformation.

## 1.4 Objectives

This thesis deals with the improvement of the biarcs construction by minimizing the distance between the curve and the control polygon. The main focus of this study is to observe the usefulness of the proposed approach called optimized bi-QT-Bézier and also to convert the input curve using Progressive Iterative Approximation. The objectives of this study are as follows:

1. To improve a curve fitting method namely optimized bi-QT-Bézier via Particle Swarm Optimization.
2. To enhance the efficiency and accuracy of generating curves by minimizing the number of segments and reducing the distance between curves and control polygons.
3. To achieve higher precision in curve approximation using modified Progressive Iterative Approximation methods.

## 1.5 Scope and Limitations of Research

This research employed the QT-Bézier with a single shape parameter as the fundamental basis function in the optimized bi-QT-Bézier. The utilization of QT-Bézier is advantageous as it represents the lowest degree of basis function incorporating a shape parameter. Furthermore, for the purpose of this research and to answer the objectives, this research is focused on fitting two-dimensional data points by improving the curve construction and minimizing the distance between the curve and its control polygon.

### 1.6 Outline of Thesis

The description of the chapters included in this thesis is as follows:

Chapter 2 presents an overview of the usefulness of the biarc curve fitting. This chapter will review the journey and the process of the optimized bi-QT-Bézier as well as the advantage of the Particle Swarm Optimization algorithm as an optimization algorithm in solving a problem. Furthermore, this chapter also discusses the importance of the Progressive Iterative Approximation method. All domains are supported by previous works of literature and related work.

In Chapter 3, the methodology is discussed in detail. All the formulations and algorithms are presented. Flowcharts of the optimized bi-QT-Bézier using Particle Swarm Optimization and the modified Progressive Iterative Approximation of Type 1 and Type 2 are also provided.

In Chapter 4, the usefulness of the proposed approach, optimized bi-QT-Bézier will be demonstrated. The optimized bi-QT-Bézier will be applied to fit a circle and a composite shape composed of arcs and straight lines. A comparison between the proposed approach and the previous approaches will also be presented to verify its effectiveness.

Chapter 5 deals with the conversion of curves into different basis functions using the modified Progressive Iterative Approximation of Type 1 and Type 2. Both methods use different approaches to minimize the error in converting the curve. Examples are provided to demonstrate the algorithm of both PIA Type 1 and Type 2.

Lastly, Chapter 6 concludes the research work with a summary of the findings and some suggestions for future work.

# CHAPTER 2

# BACKGROUND AND LITERATURE REVIEW

## 2.1 Biarc Curve Fitting

B-spline curves are used in applied geometry and Computer Aided Geometric Design (CAGD) to describe objects. However, in many applications, such as Computer Numerical Control (CNC) machines, B-splines are less practical and paths made from simple geometric forms are more advantageous. This leads to the need for quality approximations of B-spline curves by simpler forms to get an acceptable transfer between design and implementation. Previously, the cubic spline is the most common form of curve fitting. The segments of the curve between subsequent data points are represented by cubic equations, and the smoothness of the curve is derived by matching the first and second derivatives of each segment over all intermediate data points.

However, higher-order equations can be used to describe each curve segment, but they are more difficult to solve and are more likely to result in undesirable inflections. Furthermore, the intersection of a cubic spline with a line, circle, or another cubic spline is non-analytic and must be found through iteration (Bolton, 1975). The easiest way to approximate curves is to use straight line segments, but this method has disadvantages such as a lack of smoothness and a lot of line segments needed. Arc splines are tangent-continuous, piecewise curves made of circular arcs, which are smooth, easy to calculate arc length, and easy to find offset curves. These properties make them suitable candidates for curve fitting. According to Bolton (1975), biarc is used to replace the conventional approach of curve fitting, which is a cubic spline. A biarc is widely used for arc spline approximation (Park, 2004a; Piegl and Tiller, 2002a).

A biarc is a curve formed by smoothly connecting two circular arcs that interpolate two endpoints and two end tangents (Bolton, 1975; Meek and Walton, 1992; Park, 2004a; Piegl and Tiller, 2002b). In order for the biarc to have $G^1$ continuity, the two arcs must have the same tangent at their intersection point. It is important to note that the conventional method of biarc fitting requires additional constraints to uniquely determine the biarc. The work by Bolton (1975) proposed a biarc with equal angles for the two arcs. The aim was to minimize the difference in radii between the circular arcs. This study also highlighted the properties of biarcs and demonstrated their usefulness in the shipbuilding industry. According to Bolton (1975), there are often situations in ship forms where a curve needs to smoothly transition to a predefined line. Biarc curves have been found to be highly adaptable in such cases, avoiding the tendency of certain analytic curves to overshoot. Bolton (1975) provided valuable insights into the potential applications of biarcs, inspiring further extensive research in this area by scholars.

Furthermore, Sabin (1977) proposed another variant of the biarc, incorporating an additional constraint, where the tangent at the connecting point remains parallel to the line connecting the two endpoints. This constraint allows the ratio of the two radii to approach $r = 1$, indicating the robustness of the biarc (Park, 2004a; Piegl and Tiller, 2002b). Additionally, Meek and Walton (1992) proposed a biarc with an additional constraint, where the intersection point lies on the bisector of the line segment connecting the two endpoints. The paper also introduces modifications to two existing algorithms used for approximating discrete data with a polygon, transforming them to approximate the data with an arc spline instead. The first algorithm focuses on finding locally optimal approximations, while the second algorithm utilizes a bisection approach. These algorithmic modifications generated approximations composed of both biarcs and straight-line segments, which serve as the fundamental building blocks of an arc spline.

The research conducted by Schönherr (1993) introduces a method for constructing biarcs with two important goals. Firstly, the method ensures that the biarcs have equal tangents at the point where they connect, creating a smooth transition with consistent direction. Secondly, it aims to achieve continuous curvature as the curve transitions from one biarc to the other, maintaining a consistent curve shape. To achieve smoothness, the proposed biarc fitting method uses a linearized approach, which minimizes curvature changes and promotes local smoothness. This approach is efficient and works well even for curves with significant changes in tangent direction. To determine the tangents at the data points, the method solves nonlinear equations to minimize strain energy. Practical examples are provided to demonstrate the effectiveness of the method in generating smooth and accurate curves for various curve-fitting problems.

However, these claims can be contended by Ong et al. (1996) and Park (2004b), who have not imposed any additional constraints on the proposed biarcs. According to Ong et al. (1996), the additional constraints imposed in biarc construction need more segments for a good fit. They presented an approach to the optimal fitting of a biarc-spline to a given B-spline curve to achieve a better fit and improve surface quality. The approach described in the paper introduces two main advancements in biarc curve-fitting techniques. Firstly, it offers more flexibility in selecting the biarc that best fits a given pair of endpoints and their tangents, allowing for a more customized fitting process. Secondly, unlike previous methods, the approach allows the endpoints to be off-curve, enabling better control over tolerance and enhancing the precision of the fitting. To evaluate the approach, the paper utilized numerical integration and discretization techniques to simulate curves based on discrete data points. It also presented numerical results from applying the approach to various examples. The results demonstrate that the proposed method achieves a close fit between the biarc curve and the original curve using fewer segments. This leads to improved surface finish, facilitates CNC code verification, and reduces memory requirements.

The research by Piegl and Tiller (2002a) proposes an algorithm for approximating arbitrary NURBS curves using biarcs. The algorithm employed a geometric design formulation that incorporates curves to represent the biarcs, building upon the well-established NURBS framework. One notable advantage is its ability to ensure $G^1$ continuity, ensuring smooth transitions between the biarcs and the original NURBS curve. The algorithm follows a two-step process. First, it constructs a polygon approximation of the NURBS curve, simplifying the subsequent biarc approximation. Then, it refines the polygon by replacing straight segments with biarcs until the desired tolerance level is achieved. This iterative approach enables efficient and accurate approximation of NURBS curves with biarcs. By combining the flexibility of biarcs with the power of NURBS curves, the algorithm offers an effective solution for curve approximation. The two-step process of polygon approximation followed by biarc refinement provides a systematic and efficient framework for accurate curve representation.

Besides, the study by Park (2004b) proposed a new approach for fitting an optimal single biarc to a given 2D polygon and its two end tangents. The conventional method of biarc construction imposes additional constraints to ensure uniqueness, but this new approach leverages the variability in choosing the biarc to achieve a better fit. The proposed approach was extensively tested on various 2D polygons, and its performance was compared to the conventional method. The experimental results clearly demonstrate that the proposed approach outperforms the conventional method by achieving a better fit with fewer segments. This reduction in segments has practical implications as it improves efficiency and accuracy in applications that rely on biarc curves. The ability to optimize the fit of biarc curves has several benefits. It enables the generation of smoother and more precise paths for robots and machine tools, ultimately improving their performance and efficiency. By minimizing the number of segments, the proposed approach streamlines design processes and reduces computational complexity. This has significant implications for computer-aided design, path planning for robots, and machining operations.

The study by Mu et al. (2019) presented a biarc method tailored to address the complexities of determining the end position, end direction, and task configuration for concentric wire-driven manipulators, particularly in the context of minimally invasive surgery. Unlike traditional methods, this approach employed biarc parameters to parameterize the manipulator's configuration and optimized task configuration by considering basic module parameters, end direction, and bending angle simultaneously. Through simulations and experiments conducted on a concentric wire-driven manipulator, the method showcased its ability to precisely track desired trajectories within confined spaces, underscoring its relevance for real-time control in minimally invasive surgical scenarios. The results emphasized the efficiency and effectiveness of the proposed biarc method, providing a robust solution for kinematics and configuration planning.

Additionally, the study by Han et al. (2019) introduced an efficient algorithm for computing the Minkowski sum of two planar geometric models with B-spline curve boundaries. Initially, $G^1$-biarc splines approximated the boundary curves within a specified error bound. The biarc approximation facilitated the creation of a superset of the Minkowski sum boundary, which might have contained redundant arcs for non-convex models. The main challenge lies in the effective and robust elimination of these redundancies. To address this, the study employed the Minkowski sum of interior disks of the input models, leveraging the biarc approximation for enhanced efficiency. By testing each redundant arc against a selected set of interior disks, the majority of redundancies were successfully eliminated. The remaining arcs were then used to construct the Minkowski sum boundary with the correct topology. The study underscored the significant role of biarc splines in achieving real-time performance and stability in circle-based Minkowski sum computation.

Moreover, the study by Bertolazzi et al. (2020) introduced an algorithm for computing a spline composed of biarcs to interpolate a given set of ordered planar points. Biarcs were optimized at each point to minimize three criteria which are the overall spline length, the integral of absolute curvature, and the integral of squared curvature. Conditions for spline existence, based on admissible point sequences, were outlined. The proposed method's effectiveness was demonstrated through numerical experiments. Additionally, the study highlighted the use of equations and identities for simplification, with a specific formula playing a crucial role in initializing the nonlinear programming process.

## 2.2 Quadratic Trigonometric Bézier

Han (2002) presented quadratic trigonometric polynomial curves with a shape parameter. These curves are constructed with three consecutive control points for each curve segment and can be closer to the given control polygon than the quadratic B-spline curves. This is because the shape parameter can be adjusted to yield tight envelopes for the control polygon. Note that, tight envelopes refer to curves that closely follow the shape of a given control polygon or set of points. They minimize the distance between the curve and the control points, ensuring a close fit. The study also demonstrated the ability of the trigonometric polynomial curves to be decreased to linear trigonometric polynomial curves, which can represent ellipses. Furthermore, the study emphasized the utilization of non-uniform knot vectors to enhance the flexibility of curve representation. By strategically adjusting the distribution of knots, it becomes possible to increase the number of knots in areas that demand more intricate detail, while simultaneously reducing the number of knots in smoother regions. This approach ultimately results in a more efficient representation of the curve

Wu et al. (2007) introduced quadratic trigonometric spline curves with multiple shape parameters. These curves, constructed from four consecutive control points, offer global or local adjustment through shape parameter manipulation. The paper also explored quadratic trigonometric Bézier curves as a special case. The study also reported that by using non-uniform knot vectors, $C^1$ continuity is achieved, while a uniform knot vector ensures $C^3$ continuity when all shape parameters are set to 1.

These trigonometric spline curves exhibit the ability to represent ellipses and generate families of ellipses sharing control points. Additionally, Misro et al. (2017) presented a technique involving five templates of transition curves constructed using cubic trigonometric Bézier spirals. This technique utilizes shape parameters in trigonometric Bézier curves, allowing precise control over curvature without modifying control points or employing weightage in rational functions. The study defined planar parametric curves, tangent vectors, and curvature, further enhancing understanding and control of the curves. Both studies emphasized the importance of shape parameters in achieving flexibility and control over the resulting curves.

The studies by Xu et al. (2011) and Uzma et al. (2012) introduced methods for constructing quadratic curves with shape parameters. In Xu et al. (2011), a novel approach is presented for quadratic TC-Bézier curves, utilizing a shape parameter to precisely adjust the curve's shape and accurately represent circles and ellipses. The study compares this method with previous approaches and concludes that a single shape parameter effectively controls the curve's shape. Similarly, Uzma et al. (2012) proposed a method for constructing quadratic trigonometric Bézier curves using a single shape parameter. The basis functions of the curve are defined, and the shape parameter is used to modify the curve's shape. The properties of the curve and its basis functions are analyzed and compared to those of the quadratic Bézier curve. The research demonstrated that the quadratic trigonometric Bézier curve accurately represents a wide range of shapes without altering the control polygon. By adjusting the shape parameter, the curve can be made closer to the quadratic Bézier curve or approximate the control polygon more closely. Both studies highlighted the effectiveness of shape parameters in controlling the shape of quadratic curves and their ability to accurately represent various geometric shapes. The proposed methods offer flexibility and ease of manipulation without requiring modifications of control points or increasing curve complexity.

The study by Eshan and Tey (2017) investigates the effect of the shape parameter on the equation formed via the Moving Least Square (MLS) Method, a technique for creating smooth functions from scattered points. The researchers applied it to heat transfer simulation data with different shape parameter values to observe their influence on the curve's shape. The study reported that the shape parameter plays a large factor in the curve fitting. Moreover, the study by Ismail and Misro (2020) evaluated and compared continuous curves with varying levels of continuity as the degree of the Bézier curve increased. The study also mentioned that the higher degree of Bézier polynomials resulted in smoother curves due to the involvement of more control points, enhancing precision. The introduction of shape parameters provided users with flexibility in constructing the curve. Additionally, Adnan et al. (2020) employed multiple degrees of trigonometric Bézier curves to fit complex shapes, emphasizing the flexibility and adaptability of these curves to data. The method ensured smooth curve fitting by achieving various levels of continuity between segments. The study also reported that the piecewise approach simplified shape design and to optimize the fitting process, the Harmony Search (HS) algorithm was utilized, proving more effective than other methods in the comparison. Trigonometric Bézier curves with shape parameters emerged as a suitable choice for curve fitting in the study, offering flexibility, smoothness, and optimization possibilities.

## 2.3 Particle Swarm Optimization

Optimization is a mathematical technique used to find the best solution within given constraints. It is applied across industries to maximize profits, minimize costs, and improve efficiency. Optimization involves finding optimal values for variables that satisfy constraints and optimize an objective function. Genetic Algorithm and Particle Swarm Optimization are two commonly used algorithms in optimization. In the study conducted by Gulsen et al. (1995), two methods for curve fitting in the presence of noisy data are compared, which are a genetic algorithm approach and a least-squares approach. The authors evaluated the performance of these methods on three test problems and determined that the genetic algorithm approach outperforms the least-squares approach in terms of accuracy and robustness to noise. They demonstrated that the genetic algorithm approach can converge to the true solution even when initial estimates are significantly different from the true values. However, it is significant to highlight that the genetic algorithm approach is computationally more expensive. Overall, the study provides valuable insights into the strengths and limitations of these methods for curve fitting with noisy data.

The study conducted by Kumar et al. (2003) explores the application of B-spline curves and surfaces for CAD data representation and investigates the emerging technique of freeform shape synthesis from point cloud data. The study focuses on the challenge of achieving precise curves and surface fitting from point clouds, which requires the development of a robust parameterization model. To tackle this challenge, the authors proposed an innovative approach that utilizes genetic algorithms for parameter optimization. They introduced a novel population initialization scheme that ensures global optimization while minimizing the time needed to converge. The study specifically concentrates on Non-Uniform B-spline curve fitting, a widely used technique in computer-aided design. The effectiveness of the optimization procedure is evaluated using the Root Mean Square (RMS) error, a common metric for assessing curve fitting quality. The results demonstrated that, despite the complexity of the optimization process, convergence can still be achieved with additional generations. By presenting the proposed approach, the study makes a valuable contribution to the field by providing insights into achieving accurate curve and surface fitting from point cloud data. This knowledge is crucial in various applications, including computer graphics, virtual reality, and shape modeling.

On the other hand, Gálvez and Iglesias (2011) presented an approach for data fitting using B-splines, which are commonly employed mathematical functions for approximating curves. The method employs Particle Swarm Optimization (PSO) to automatically determine knot placements, which mark points of direction change in the B-spline. This utilization of PSO yields improved accuracy in data fitting. PSO is a metaheuristic optimization algorithm inspired by the collective behavior of bird flocks or fish schools, enabling it to handle complex, multimodal, and nonlinear optimization problems. The proposed approach effectively handles curves with singularities or cusps, thereby surpassing the limitations of prior methods. In addition, Liu and Li (2016) studied a method for obtaining optimal shape parameters in CAGD. While shape parameters are typically given as intervals, it is often necessary to determine the optimal values to achieve the desired fairness and smoothness in curves. The proposed method tackles this problem by utilizing a fairing criterion and applying the PSO algorithm to find the optimal model. The study established a general automatic mathematical model for solving optimal parameters and demonstrates its effectiveness through examples of three curve classes. The method can be applied to various curves with shape parameters, ensuring fairness and smoothness in the resulting curves

The study by BiBi et al. (2022) presents a method for optimizing the assembly of GHT-Bézier developable surfaces using the Particle Swarm Optimization (PSO) technique. The primary objective is to improve the efficiency of complex engineering products by constructing highly accurate developable surfaces. In this approach, the control points of the GHT-Bézier surface are considered as design variables, and the degree of developability of the ruled surface is defined as the objective function. The shape parameters are treated as optimization variables, and PSO is employed to search for the optimal shape control parameters within a specified value range. The effectiveness of the proposed method is demonstrated through modeling examples, showcasing its ability to generate fair surfaces that closely approximate the original surface. By utilizing PSO to search for the optimal shape control parameters, the method enables the creation of developable surfaces with exceptional accuracy in their developability. This advancement has significant implications for the manufacturing process of complex free-form surfaces, as it addresses various challenges and enhances overall performance. PSO is selected for its simplicity, ease of implementation, and its ability to effectively explore the search space as a population-based algorithm. Furthermore, PSO's capability to handle non-differentiable or noisy objective functions makes it well-suited for a wide range of optimization problems.

### 2.4 Progressive Iterative Approximation

Lin et al. (2005) focused on investigating the Progressive Iteration Approximation (PIA) property of curves and tensor product surfaces that are generated by blending a given set of data points and a set of basis functions. The study highlighted the significance of the PIA property, which ensures that a sequence of curves or surfaces interpolates the given data points. This property plays a crucial role in guaranteeing that the iterative adjustments of control points converge to a unique solution. The PIA property holds great relevance in computer-aided design and manufacturing, where curves and surfaces are employed to represent intricate shapes and surfaces. The authors established that B-spline, NURBS, and Bezier curves and surfaces possess the progressive iteration approximation property if the corresponding collocation matrix is nonsingular. This finding contributes to the understanding of the behavior and properties of these widely used curve and surface representations in the context of iterative approximation.

Huang et al. (2008) introduced a method for approximating a rational Bézier curve by utilizing a sequence of Bézier curves. These Bézier curves have control points obtained from degree-elevated rational Bézier curves. The method is extendable to other cases involving rational functions, such as rational B-splines and rational surfaces. Degree-elevation of a given rational Bézier curve involves increasing its degree and the number of control points required to represent it. This elevation can impact the curve's shape and properties, potentially introducing undesired oscillations. However, by carefully selecting the control points of the elevated curve, it is possible to manage and optimize these changes, achieving the desired level of accuracy and smoothness.