POISSON TRANSMUTED EXPONENTIAL DISTRIBUTION FOR COUNT DATA WITH SKEWED, DISPERSED AND EXCESS ZERO

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by

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Ademola Abiodun ADETUNJI

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LIST OF ABBREVIATIONS

AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
CDF	Cumulative Probability Function
CRT	Cubic Rank Transmutation
CRTED	Cubic Rant Transmuted Exponential Distribution
CRTWED	Cubic Rant Transmuted Weighted Exponential Distribution
CRTNWED	Cubic Rant Transmuted New Weighted Exponential Distribution
ED	Exponential Distribution
HRF	Hazard Rate Function
KURT	Kurtosis
LL	Log-Likelihood
MGF	Moment Generating Function
MLE	Maximum Likelihood Estimation
NB	Negative Binomial
NWED	New Weighted Exponential Distribution
PDF	Probability Distribution Function
PED	Poisson Exponential Distribution
PGF	Probability Generating Function
PMF	Probability Mass Function
PTED	Poisson Transmuted Exponential Distribution
PTWED	Poisson Transmuted Weighted Exponential Distribution
PTNWED	Poisson Transmuted New Weighted Exponential Distribution
QT	Quadratic Transmutation
QTED	Quadratic Transmuted Exponential Distribution
QTWED	Quadratic Transmuted Weighted Exponential Distribution
QTNWED	Quadratic Transmuted New Weighted Exponential Distribution
SKEW	Skewness
TED	Transmuted Exponential Distribution
TNWED	Transmuted New Weighted Exponential Distribution
TWED	Transmuted Weighted Exponential Distribution
WED	Weighted Exponential Distribution
ZI	Zero-Inflated
ZIP	Zero-Inflated Poisson

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TABURAN POISSON TRANSMUTASI EKSPONEN BAGI DATA BILANGAN YANG TERPENCONG, TERSERAK DAN TERLEBIH SIFAR

ABSTRAK

Dengan mengandaikan taburan Poisson klasik untuk data bilangan, ia berkemungkinan salah dalam kebanyakan kes kerana ia mengandaikan serakan yang sepadan, sedangkan data bilangan biasanya terlebih terserak. Dalam sains aktuari, kekerapan tuntutan biasanya merupakan unimodal, terpencong, terlebih terserak, dan dengan kekerapan bilangan sifar yang lebih tinggi; oleh itu, dengan andaian taburan Poisson boleh menyebabkan model tersalah dinyatakan. Kajian ini memperluaskan teori dan skop taburan diskret untuk pemerhatian bilangan yang terpencong dan terlebih terserak dengan lebihan bilangan sifar dalam proses Poisson tercampur dengan mengubahsuaikan sifat unimodal dan terpencong taburan eksponen secara lanjut. Transmutasi kuadratik dan transmutasi dua-kubik digunakan untuk menerbitkan taburan eksponen, taburan eksponen berwajaran, dan taburan eksponen berwajaran yang baru. Taburan selanjar yang diperoleh diandaikan sebagai taburan tercampur untuk parameter taburan Poisson dalam prosess Poisson tercampur. Sembilan taburan Poisson tercampur yang baru dan bentuk terlebih sifar masing-masing dicadangkan dari taburan tercampur. Sifat bermatematik yang berasaskan momen yang berbeza yang baru dicadangkan telah diperoleh. Algoritma yang berbeza digunakan untuk menilai anggaran kebolehjadian maksimum bagi menganggar parameter. *Newton-Raphson* dan *Nelder-Mead*, dengan lelaran minimum bagi nilai penumpuan terhadap log kebolehjadian, memberikan nilai anggaran yang terbaik. Cadangan baru tersebut telah dinilai dengan taburan diskrit yang lain terhadap pelbagai cerapan bilangan terlebih serak dalam kehidupan sebenar dengan terlebih sifar. Cadangan baru tersebut

menunjukkan prestasi yang baik dalam pelbagai senario dan boleh menjadi model alternatif untuk menganalisa cerapan bilangan yang terlebih serak dengan terlebih sifar. Hasil kajian juga menunjukkan bahawa cadangan baru tersebut mengatasi bentuk model yang terlebih sifar ketika dinilai pada data bilangan yang disifatkan dengan terlebih sifar.

POISSON TRANSMUTED EXPONENTIAL DISTRIBUTION FOR COUNT DATA WITH SKEWED, DISPERSED AND EXCESS ZERO

ABSTRACT

Assuming Poisson distribution for count data may be misleading in most cases because it assumes equidispersion, whereas count data is usually overdispersed. In actuarial science, claim frequencies are usually unimodal, skewed, overdispersed, and with a higher frequency of zero counts; hence, assuming the Poisson distribution may lead to model misspecification. This study expands theories and scopes of discrete distributions for skewed and overdispersed count observations with excess zero frequency in the mixed Poisson process by leveraging extended exponential distributions' unimodality and skewness properties. Three transmutation maps are used to extend the exponential distributions, the weighted exponential distribution, and the new weighted exponential distribution. The obtained distributions are assumed as mixing distributions for the parameter of the Poisson distribution in the mixed Poisson process. Nine new mixed Poisson distributions and their respective zero-inflated forms are proposed from these mixing distributions. Different moment-based mathematical properties of the new proposed distributions are obtained. Different algorithms are used to assess the maximum likelihood estimates for the parameters of the proposed distributions.The Newton-Raphson and the Nelder-Mead, with minimum iterations for convergence and log-likelihood values, provide optimum estimates. The new proposed distributions are assessed with other discrete distributions on various real-life dispersed count observations with excess zero. The new proposed distributions perform well in diverse scenarios and can be better alternatives to analyzing overdispersed count observations with excess zero. Results also show that the new proposed distributions

outperform their zero-inflated forms when assessed on count data plagued with excess zero counts.

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

In fitting distributions for count observations, a major demerit of the classical Poisson distribution is its inability to efficiently fit dispersed count data. This is rooted in the distribution's assumption of equality of mean and variance for count data which is rare for real-life observations. This has given rise to various modifications to the distribution. The mixed Poisson process, is prominent among these modifications, usually applied for fitting claim frequency in actuarial science (Karlis, 2005). These distributions arise from the variations inherent in the mean fluctuation of the Poisson random variable by assuming a continuous distribution (mixing distribution) for the distribution's parameter.

Several mixed Poisson distributions exist in the literature to provide more efficiency in fitting count observations. In most of these propositions, the mixing distributions are obtained from the exponential-related distributions. The classical exponential distribution is suitable for systems with constant failure rates (not minding the effect of time or accumulated age). This is a significant shortcoming in its general application hence, the introduction of several forms of its extensions. Extending a classical distribution is usually achieved by adding extra shape parameter(s) using different compounding techniques (Karina *et al.*, 2019). Applications of compounded exponential distributions pervade almost all aspects of life, including but not limited to economic, reliability, environmental, industrial, and engineering spaces (Aguilar *et al.*, 2019; Mohammed *et al.*, 2015; Rasekhi *et al.*, 2017).

This research extends the exponential distribution and other two forms of its generalization using three transmutation maps to obtain a new set of mixing distributions to derive new Poisson distributions and their respective zero-inflated forms. The new proposed distributions' various moment-based mathematical and statistical properties are obtained, and different algorithms for maximum likelihood estimation are assessed. Performances of the new proposed distributions are compared with other refereed count distributions using the negative log-likelihood, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the chi-square statistics as selection criteria. Real-life applications are examined with count data, mostly from actuarial science, and more generally with skewed, unimodal, and over-dispersed count observations with excess zero.

Generally, variances of all mixed Poisson distributions exceed their means (Karlis & Xekalaki, 2005); hence, they are more efficient in fitting dispersed observations. Therefore, the new proposed distributions are expected to give more flexibility and general applicability to over-dispersed count observations with excess zeros.

1.2 Problem Statement

Attempts to obtain distributions that provide better explanations for real-life data are continuous search in distribution theories and applications. Distributions that appeared best in the past have been extended to allow for more flexibility in recent studies. Because of its relative simplicity and broad applicability, the Poisson distribution is at the forefront in fitting count observations. The distribution is best assumed for count data when equi-dispersed (equal variance and mean). Model misspecification arises when the distribution is assumed for heterogeneous count data (Asamoah, 2016). Researchers over the years have developed distributions that can provide a good fit for over-dispersed observations using the Poisson-Exponential relationships (Bhati *et al.*, 2015, 2017; Das *et al.*, 2018; Sankaran, 1970; Zakerzadeh & Dolati, 2009). First among many of these propositions is the negative binomial (NB) distribution (Greenwood & Yule, 1920), obtained by assuming the gamma distribution for the parameter of the Poisson distribution.

Assuming continuos distributions for the Poisson parameter in the mixed Poisson process may ensure adequate flexibility in fitting count data. The general applicability of the generalized exponential distributions transcends reliability, lifetime, engineering, and actuarial modellings. Additional shape parameters in these classes of exponential distributions ensure efficient fitting of dispersed observations (Karina *et al.*, 2019). Generally, count observations in actuarial science are often dispersed and plagued with an excess frequency of zero (Adcock *et al.*, 2015; Adetunji & Sabri, 2021; Khan & Khan, 2010; Omari *et al.*, 2018).

Since its variance exceeds its mean, the NB distribution has been used to fit overdispersed count observations (Greenwood & Yule, 1920; Klugman *et al.*, 2012; Sankaran, 1970; Srivastava, 2016). However, its efficiency in fitting unimodal, skewed, and overdispersed count data is yet to be reported to our best knowledge (Altun, 2021; Nikoloulopoulos & Karlis, 2008). Hence, this study is aimed at proposing discrete distributions that can provide a good fit for count observations that are unimodal, skewed and overdispersed. These properties characterize claim frequency in actuarial science. This will be achieved by exploring some of these properties that characterize compounded exponential distributions by utilizing advantages inherent in extending classical exponential distribution and two of its extensions (weighted exponential distribution and new weighted exponential distribution) to obtain a new set of mixing distributions assumed for the Poisson parameter in the mixed Poisson process.

1.3 Research Objectives

Using extended exponential distributions as the mixing distributions assumed for the parameter of the Poisson distribution, this research aims to develop new count distributions that can fit dispersed and skewed count data with excess zero in the mixed Poisson process. Specific objectives are:

i. to obtain a new set of mixed Poisson distributions and their mathematical properties using extended exponential distributions as the mixing distributions.
ii. to obtain zero-inflated forms of the new proposed distributions for fitting count observations with excess zero.
iii. to assess the asymptotic characteristics of the new proposed distributions.
iv. to compare the performance of the new proposed distributions in fitting count observations with different proportions of zero counts.
v. to investigate the new proposed distributions' performances compared to refereed count distribution for skewed and dispersed observations with excess zero counts.

1.4 Motivation for the Study

Developments in distribution theories have shown that data fitting can continuously be improved upon since most of the new proposed distributions have extra parameters that improve flexibility and can better fit real-life observations. This research attempts to improve the fitting of count data using new mixing distributions to obtain a new set of mixed Poisson distributions. The new proposed distributions aim to provide alternatives for fitting skewed and dispersed count observations with excess frequencies of zero.

Extensions of various continuous distributions have been reported to give better performance over the baseline distributions (Alsikeek, 2018; Bhatti *et al.*, 2018; Elgarhy *et al.*, 2017; Kemaloglu & Yilmaz, 2017; Ogunde *et al.*, 2020). Therefore, the new proposed distributions are expected to provide better fits with higher efficiencies than some of the established discrete distributions in actuarial science in particular and dispersed count observations with excess zero in general.

1.5 Significance of the Study

In general data fitting, the aim is to assume a probability distribution with the propensity to provide the best fit for any observations of interest. Several techniques are employed to introduce new probability distributions to achieve this. In most cases, introducing a new probability distribution involves extending the baseline distributions by adding extra parameter(s). The resulting new distribution tends to incorporate dynamism absent in the baseline distribution to better fit the observation of interest. Extending distribution procedures are more pronounced in the continuous distributions process than the discrete distributions.

Recent advancements in computing power and statistical software have made estimating parameters in multi-parameter continuous distributions less strenuous. For continuous distributions, different four-parameter distributions (Alizadeh *et al.*, 2016; Hamed *et al.*, 2020; Handique & Chakraborty, 2021); five-parameter distributions (Al-Babtain *et al.*, 2014; Tharshan & Wijekoon, 2020); and six-parameter distributions (Alshkaki, 2021; Yousof & Afify, 2016) have been introduced. Techniques of obtaining multi-parameter discrete distributions have gained little patronage. This study presents an approach to generalizing discrete distributions that can be used to obtain multi-parameter discrete distributions. Apart from being able to provide efficient fit for skewed and dispersed count observations with excess zero, the new proposed distributions can also be assumed in regression modelling of relationships between skewed and dispersed count observations and other covariates.

1.6 Scope of the Study

The study uses the classical exponential distribution and its two generalized forms, the weighted exponential distribution (Gupta & Kundu, 2009), and the new weighted exponential distribution (Oguntunde *et al.*, 2016), as baseline distributions. Three one-parameter transmutation maps (Al-kadim, 2018; Rahman *et al.*, 2019; Shaw & Buckley, 2007) are used to extend these baseline distributions to obtain nine transmuted exponential distributions. The resulting distributions are assumed for the parameter of the Poisson distribution to serve as mixing distributions. Using these mixing distributions, a new set of mixed Poisson distributions and their respective zero-inflated forms are obtained.

From unlimited choices of baseline distributions and compounding techniques that could be considered, this study compares three forms of exponential distributions and three one-parameter transmutation maps, leveraging the characteristics of count observations in actuarial sciences. A noticeable characteristic of most mixed Poisson distributions is that they are unimodal (Bhati *et al.*, 2017). Therefore, the proposed distributions in this study may be less efficient in fitting multimodal count data. This study's major applications are in claim frequency (although other refereed dispersed and unimodal count data are assessed). Getting primary data on claim frequency from insurance companies is herculean; hence, real-life applications in the study are limited to available secondary data extracted from different sources, including highly refereed journals.

1.7 Zero-Modified Count Distribution

The nature of some count data may cause observations to be plagued with an unusual frequency of zero. Results of count data fitting may therefore become unreliable if there are too few zero frequencies (zero-deflation) or too many zero frequencies (zero-inflation). Zero-deflated features occur when the observed frequency of zero in a count data is much lower and inconsistent with the expected frequency. This scenario is typically less reported in the literature (Angers & Bsiswas, 2003; Dietz & Böhning, 2000) than in instances where zero frequency is substantially larger (Conceição *et al.*, 2017). In the literature, two-part and mixture models have been used to treat count data with zero inflation. There are two processes in a finite mixture model, one with a zero output (resulting in a count of zero) and the other with a standard count distribution (Heilbron, 1994; Mullahy, 1986). The zero-hurdle models combine a binary model with a zero-truncated count data model for positive outcomes. This formulation is based on the assumption of a Bernoulli variable (with a zero or a positive realization). The "*hurdle*" is crossed if the realization is positive (Mullahy, 1986), and the zero-truncated distribution is used to derive the conditional distribution of positives.

Most count data assumed Poisson distribution, but there are perceived reasons not to utilize the distribution in many situations. The usual circumstances are when the data are dispersed, or there are too many (or too few) zero counts. Dietz and Böhning (2000) reported some conditions that may warrant many zeros in count observations.

(i) All members of a study population may not be concerned in responding to the Poisson process; hence, zero inflation develops as a result of the unaffected members.

(ii) During sampling, some conditions can increase (or decrease) the likelihood of having zero counts. This situation creates room for zero inflation (or zero deflation).

(iii) There may be no chance of observing zero counts in the study. Hence, if zeros are found in the observations, they need to be truncated. This is the case with the positive count process.

Generally, any of the zero-modified distributions is assumed when zero counts in a dataset are assumed to be beyond (or below) average. The zero-modified distribution involves special attention to zero frequency in count observations. Among the standard techniques for zero modifications are the zero-truncated (ZT), the zero-hurdle (ZH), and the zero-inflated (ZI) distributions.

1.8 Thesis Organization

Chapter 1 provides a detailed background to the study. Problem statements, objectives, and significance of the study are stated. The scopes covered by the study and the accompanying limitations are also stated. Some basics of zero-modifications in count data are also presented.

In Chapter 2, the procedures of compounding classical distributions are reviewed. Recent advancements and studies in the mixed Poisson process are also revised. Background studies on the rank transmutation maps used in extending the considered exponential distributions are presented.

In the third Chapter, nine mixing distributions are obtained. Some basic properties and shape characteristics of these distributions are also shown. Some background on characterizing discrete distributions and extimating parameters of zero-modifications are also presented. Different algorithms for estimating the parameters of the proposed distributions in the maximum likelihood estimation process are also discussed. Techniques of simulating random variables to assess the behaviour of the new proposed distributions are also discussed. Finally, four selection criteria used in this study are presented.

Chapters 4, 5, and 6 present the Poisson transmuted distributions for the exponential, weighted exponential, and new weighted exponential distributions, respectively. Properties like the PMF, the CDF, the hazard function, the survival function, the Moment Generating Function (MGF), the Probability Generating Function (PGF), the skewness, the kurtosis, and the shapes of the PMFs are obtained for each distribution in each Chapter. Mathematical expressions for the maximum likelihood estimation of these distributions are also presented.

Real-life count datasets used in the study, results and discussions of each proposed distribution with competing distributions are presented in Chapter 7. The general performance of the best three distributions for each dataset are also discussed. The results obtained are ranked according to the baseline distributions and according to the transmutation maps, and the overall performance of each category on the dataset is assessed.

The summary, conclusion, and recommendations for further studies are presented in the eighth Chapter.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter reviews recent trends in discretizing random variables from continuous distributions. Special attention is given to the techniques of extending continuous distributions and some common generalizations in the mixed Poisson process.

2.2 Discretizing Continuous Distributions

The technique of discretizing continuous distributions was developed to fit discrete observations with similar shapes to the respective continuous distributions. Nakagawa and Osaki (1975) first applied the technique to the Weibull distribution. Since then, many techniques of obtaining discrete analogous to continuous distributions have been developed (Chakraborty, 2015; Roy, 2002; Tovissodé *et al.*, 2021; Drezner & Zerom, 2016), but using the survival function of the continuous distribution has received the major attentions in the literature. According to Roy (2004), the probability mass function (PMF) of a new discrete distribution using survival function $S(x)$ of its continuous equivalent is defined in equation (2.1) as:

$$P(X = x) = S(x) - S(x + 1), \qquad x = 0,1,2, \dots \tag{2.1}$$

where

$$S(x) = 1 - F(x|\Theta) = P(X \geq x) \tag{2.2}$$

$F(x|\Theta)$ is the CDF of a continuous distribution. Table 2.1 shows some discrete distributions obtained using the equation (2.1).

Table 2.1 Some discretized continuous distributions

Distribution	Reference
Weibull	Nakagawa and Osaki (1975)
Lindley	Gómez-Déniz and Calderín-Ojeda (2011)
Generalized Exponential	Nekoukhou *et al.*, (2013)
Lomax	Para and Jan (2016)
Gompertz Exponential	Eliwa *et al.*, (2020)
Marshall–Olkin Generalized Exponential	Almetwally *et al.*, (2020)
Weibull Marshall–Olkin Exponential	Gillariose *et al.*, (2021)
Marshall–Olkin Inverted Topp–Leone	Almetwally *et al.* (2022)

2.3 Compounding Distribution

Several techniques exist for extending classical distributions to obtain more efficient and flexible distributions. This section explores some of the developments in compounding baseline continuous probability distributions.

2.3.1 General Compound Distribution

Several extensions of different classical probability distributions have been proposed. Most of these extensions improve the flexibility and general applicability of the classical distributions. Applications of these principles are usually on the continuous probability distribution with very little attention to the discrete distribution. Few among many inexhaustible techniques of extending continuous distributions are the quadratic transmutation map (Shaw & Buckley, 2007); the Kumaraswamy *G* distributions (Cordeiro & de Castro, 2011); the beta extended *G* distributions (Cordeiro *et al.*, 2012); the gamma *G* distributions (Ristić & Balakrishnan, 2012); the generalized beta *G* distributions (Alexander *et al.*, 2012); the beta exponential *G* distributions (Alzaatreh *et al.*, 2013); the exponentiated exponential Poisson *G* distribution (Ristić & Nadarajah, 2013); the exponentiated generalized *G* distributions (Cordeiro *et al.*, 2013); the modified beta *G* distributions (Nadarajah *et al.*, 2014); the exponentiated transmuted G type 2 (Merovci *et al.*, 2016); the alpha-power transformation (Mahdavi & Kundu, 2017); the Weibull G family (Bourguignon *et al*., 2014); and the cubic rank transmutation maps (Al-kadim, 2018; Aslam *et al*., 2018; Granzotto *et al*., 2017; Rahman *et al*., 2019).

Detailed literature on many distributions due to the transmuted families are provided by Alzaatreh *et al*., (2013); Tahir and Cordeiro (2016); and Klakattawi and Aljuhani (2021), while Ali and Athar (2021) explored recent trends in transmutation maps.

2.3.2 Quadratic Transmutation Map

Suppose a baseline continuous distribution has the CDF $\varphi(x)$, the Quadratic Transmuted (QT) family of distributions (Shaw & Buckley, 2007) with the transmutation parameter $\wp$ has the CDF:

$$F(x) = (1+\wp)\varphi(x) - \wp\big(\varphi(x)\big)^2; \quad x \in \mathbb{R}, |\wp| \le 1 \tag{2.3}$$

Due to inherent transformations of the baseline distributions from the continuous to discrete cases, the conditions imposed on the transmutation parameter $(|\wp| \le 1)$ may not always hold in the mixed Poisson process. An extensive work on transmutation parameters was done by Hameldarbandi and Yilmaz (2020).

2.3.3 Cubic Transmutation Maps

In some cases, the QT family may not capture the complexity of observations of interest; hence, the development of the Cubic Rank Transmutation (CRT) with additional shape parameter which improves flexibility and general applicability. Several CRT techniques pervade the literature (Al-kadim, 2018; Aslam *et al*., 2018; Granzotto *et al*., 2017; Rahman *et al*., 2019), with each proposed distribution adding extra shape parameter(s) to baseline distributions. Rahman *et al.*, (2020) provides a list of extended distribution using the CRT technique.

This study focuses on the CRT techniques that add only one shape parameter to baseline distributions (Al-kadim, 2018; Rahman *et al.*, 2019) to avoid the challenges of estimating too many parameters. Also, these techniques have been reported to perform creditably well (Adetunji & Ademuyiwa, 2020), like those with two additional shape parameters (Aslam *et al.*, 2018; Granzotto *et al.*, 2017; Rahman *et al.*, 2018).

Given that a baseline continuous distribution has the CDF of the form $\varphi(x)$, the CRT map due to Al-kadim (2018) is given as:

$$F(x) = (1 + \wp)\varphi(x) - 2\wp\big(\varphi(x)\big)^2 + \wp\big(\varphi(x)\big)^3, \qquad x \geq 0, |\wp| \leq 1 \tag{2.4}$$

The CDF of another CRT map due to Rahman *et al.*, (2019) has the form:

$$F(x) = (1 - \wp)\varphi(x) + 3\wp\big(\varphi(x)\big)^2 - 2\wp\big(\varphi(x)\big)^3, \; x \geq 0, |\wp| \leq 1 \tag{2.5}$$

2.4 Mixed Poisson Distributions

Since the pioneering works on the compound Poisson distributions (Greenwood & Yule, 1920; Holgate, 1970; Maceda, 1948; Willmot, 1986), the mixed Poisson distribution has received attention with applications in count observations in general and claim frequencies fitting in particular. Iyer-Biswas and Jayaprakash (2014); Das *et al.*, (2018); Simeunović *et al.*, (2018), and Ong *et al.*, (2021) provide references, properties, applications, and recent trends in mixed Poisson distribution. Among many mixed Poisson distributions, the most refereed ones are elucidated below.

2.4.1 Poisson-Exponential Distribution

If $N|Y\sim Poisson(Y)$ and $Y\sim Exponential(\alpha)$, N has a Poisson-Exponential distribution with its PMF obtained as:

$$P_n=\frac{\alpha}{(1+\alpha)^{n+1}}; \qquad n=0,1,2,\dots;\alpha>0 \tag{2.6}$$

Equation (2.6) has a geometric distribution.

2.4.2 Poisson-Lindley Distribution

The Poisson-Lindley distribution (Sankaran, 1970) is a mixture of the negative binomial $\left(2,\frac{\alpha}{1+\alpha}\right)$ distribution, and the geometric $\left(\frac{\alpha}{1+\alpha}\right)$ distribution with $\left(\frac{\alpha}{1+\alpha}\right)$ as the mixing proportion. The PMF is defined as:

$$P_n=\frac{\alpha^2(n+\alpha+2)}{(1+\alpha)^{n+3}}, \qquad n=0,1,2,3,\dots;\ \alpha>0 \tag{2.7}$$

This distribution has enjoyed patronage by researchers since its introduction. Ghitany and Al-Mutairi (2009) investigated some of its mathematical properties. Mahmoudi and Zakerzadeh (2010) proposed the two-parameter discrete Lindley distribution with the generalized Lindley distribution (Zakerzadeh & Dolati, 2009) as the mixing distribution. Bhati *et al.*, (2015) used the two-parameter Lindley distribution (Shanker *et al.*, 2013) as the mixing distribution. Another generalization of the Lindley distribution (Das *et al.*, 2018) was used to derive a three-parameter Poisson-Lindley distribution.

2.4.3 Poisson-Gamma Distribution

If $N|Y\sim Poisson(Y)$ and $Y\sim Gamma(\alpha,\beta)$, a discrete random variable N has a Poisson-Gamma distribution if its PMF is defined as:

$$P_n = \binom{n+\alpha-1}{n}\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left(\frac{1}{1+\beta}\right)^{n}; \qquad n = 0,1,2,\ldots; \alpha,\beta > 0 \tag{2.8}$$

The PMF obtained has a negative binomial (NB) distribution with $p = \left(\frac{1}{1+\beta}\right)$. Various generalizations of this distribution can be found in Albrecht (1984); Willmot (1993), and Sastry *et al.*, (2016). The distribution has been used in modelling (i) wind speed data (Çakmakyapan & Özel, 2016); (ii) rainfall data (Dzupire *et al.*, 2018); and (iii) claim premium (Wu, 2020). Various properties of distribution are studied by Cha and Mercier (2021).

Table 2.2 shows some mixing distributions that have been utilized to obtain different mixed Poisson distributions.

Table 2.2 Mixing distributions for some mixed Poisson distributions

Mixing Distributions	**References**
Gamma	Greenwood and Yule (1920)
Uniform	Bhattacharya (1966)
Lindley	Sankaran (1970)
Lognormal	Bulmer (1974)
Truncated Gamma	Willmot (1993)
Pareto	Willmot (1993)
Lindley-Beta	Gómez-Déniz *et al.*, (2012)
Generalized Gamma	Sastry *et al.*, (2016)
Marshall-Olkin-Generalized Exponential	Gómez–Déniz and Calderín–Ojeda (2015)
Three-Parameter Lindley	Das *et al.*, (2018)

From the above, several baseline continuous distributions have been used as mixing distributions in the mixed Poisson process by assuming them for the Poisson parameter. This research explores discrete distribution space by extending the search for the mixing distributions to be used in the mixed Poisson process.

Many studies (Bertoli *et al.*, 2021; Dzupire *et al.*, 2018; Meytrianti *et al.*, 2019; Nikoloulopoulos & Karlis, 2008; Sharker *et al.*, 2020) have assessed over dispersion in count data but these data, especially in actuarial sciences are not only dispersed but also skewed, unimodal, and plagued with excess frequency of zero (Adetunji & Sabri, 2021; Atikankul *et al.*, 2020; Bhaktha, 2018; Emilio Gómez-Déniz & Calderín-Ojeda, 2018; Tüzen *et al.*, 2020). This study presents a technique of obtaining discerete distributions that can efficiently provide adequate fits to count data which are not only overdispersed but also skewed, unimodal and have excess zero frequency.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

If $N|Y \sim Poisson(Y)$ and $Y \sim \pi(\Theta)$ where $\pi(\Theta)$ is the probability distribution function (PDF) of a distribution with positive supports assumed for *Y*, a mixed Poisson distribution with the mixing distribution $\pi(\Theta)$ is obtained by finding the unconditional distribution for N.

$$P_n = \int_0^\infty f(n|y)\, \pi(\theta)\, dy \qquad (3.1)$$

Different choices for the mixing distribution, $\pi(\Theta)$, have been reported in the literature (*see* Table 2.2 for some choices of $\pi(\Theta)$ in the literature). Karlis and Xekalaki (2005) reported that the shape of a mixing distribution resembles the shape of the equivalent mixed Poisson distribution with appropriate adjustment of its parameters. In addition, the tail properties of a mixed Poisson distribution have similarities with the tail properties of the mixing distribution producing it (Rémillard & Theodorescu, 2000; Willmot, 1990).

In this study, we utilize the quadratic transmutation map (Shaw & Buckley, 2007) and two cubic rank transmutation maps (Al-kadim, 2018; Rahman *et al.*, 2019) on the exponential distribution, the weighted exponential distribution (Gupta & Kundu, 2009), and the new weighted exponential distribution (Oguntunde *et al.*, 2016) to obtain a new set of mixing distributions which are assumed for the parameter of the Poisson distribution. Various mathematical properties and characteristics of the new proposed distributions are explored using different properties earlier applied to similar distributions (Feller, 1968; Holgate, 1970; Karlis & Xekalaki, 2005; Maceda, 1948; Shaked, 1980).

With additional shape parameter $(\wp)$ called the transmutation parameter, transmutation maps add flexibility to baseline distributions. If $\varphi(y)$ is the CDF of the baseline distribution, the QT map (Shaw & Buckley, 2007) has the form given in the equation (3.2):

$$F(y) = (1+\wp)\varphi(y) - \wp\big(\varphi(y)\big)^2; \qquad y \in \mathbb{R}, |\wp| \leq 1 \tag{3.2}$$

Among many CRT maps, the one-parameter ones due to Al-kadim (2018) and Rahman *et al.*, (2019) are receiving many considerations largely due to their ease of representation and relatively good performance when compared with those with two extra parameters (Adetunji & Ademuyiwa, 2020). If the CDF of a baseline continuous distribution is given as $\varphi(y)$, the CRT map introduced by Al-kadim (2018) and Rahman *et al.*, (2019) are respectively given as:

$$F(y) = (1+\wp)\varphi(y) - 2\wp\big(\varphi(y)\big)^2 + \wp\big(\varphi(y)\big)^3, \qquad y \geq 0, |\wp| \leq 1 \tag{3.3}$$

$$F(y) = (1-\wp)\varphi(y) + 3\wp\big(\varphi(y)\big)^2 - 2\wp\big(\varphi(y)\big)^3, \; y \geq 0, |\wp| \leq 1 \tag{3.4}$$

Note: Equations (3.2), (3.3), and (3.4) reduce to $\varphi(y)$ when $\wp = 0$.

3.2 Transmuted Exponential Distribution and Properties

3.2.1 Exponential Distribution

If θ is the scale parameter, then CDF of the Exponential Distribution (ED) is given in equation (3.5) as:

$$\varphi(y) = 1 - e^{-\theta y} \tag{3.5}$$

Among numerous parametric distributions in probability theory, equation (3.5) is widely used in diverse fields of study. The distribution has played an important role in probability theory (Yang *et al.*, 2021). Applications of its extended forms span various fields of endeavour, including environmental, economic, reliability, engineering, and industrial (Aguilar *et al.*, 2019; Rasekhi *et al.*, 2017).

3.2.2 Quadratic Transmuted Exponential Distribution

To obtain the Quadratic Transmuted Exponential Distribution (QTED), we insert equation (3.5) into (3.2). Therefore, the CDF and PDF of the QTED are respectively obtained as:

$$F(y) = 1 - e^{-\theta y} - \wp e^{-2\theta y} + \wp e^{-\theta y} \tag{3.6}$$

$$f(y) = \theta e^{-\theta y}\left(1 - \wp + 2\wp e^{-\theta y}\right) \tag{3.7}$$

The corresponding survival and hazard functions are:

$$S(y) = e^{-\theta y}\left(1 - \wp + \wp e^{-\theta y}\right) \tag{3.8}$$

$$h(y) = \frac{\theta\left(1 - \wp + 2\wp e^{-\theta y}\right)}{1 - \wp + \wp e^{-\theta y}} \tag{3.9}$$

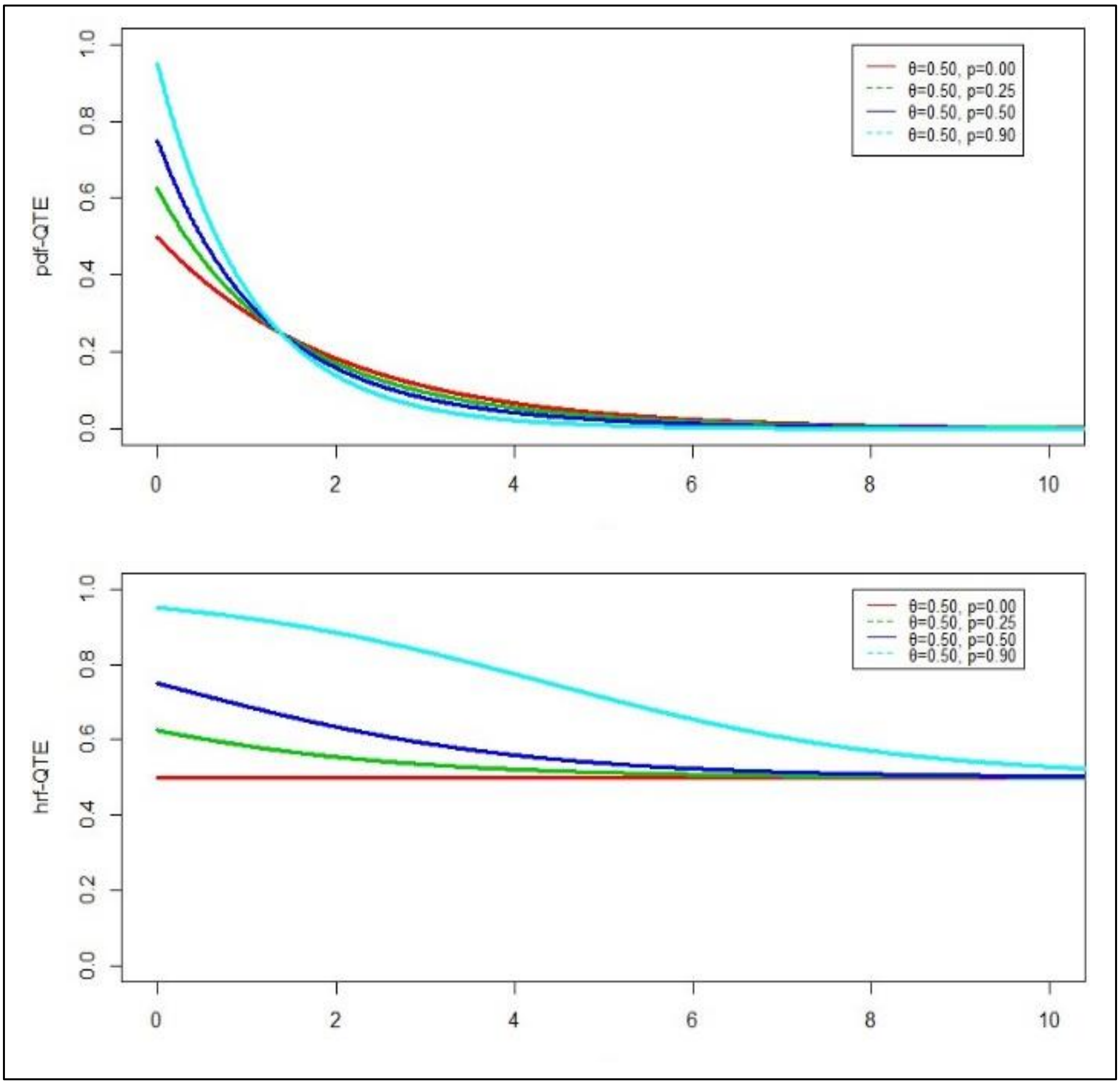

Figure 3.1 Shapes of the PDF and HRF of the QTED

Figure 3.1 shows the shapes of both PDF and HRF of the QTED for constant θ and different values of $\wp$. The PDF of the distribution has an inverted J-decreasing shape. For constant θ, the distribution gives a higher probability to zero as $\wp \rightarrow 1$, except for $\wp = 0$ (constant HRF for the exponential distribution). The HRF is a monotonically decreasing function.

3.2.2.1 r^{th} Moment of the QTED

Proposition 3.2.1: Assume a random variable Y has the QTED as given in equation (3.7), the r^{th} moment is obtained as:

$$E(y^r) = \left(1 - \wp + \frac{\wp}{2^r}\right)\frac{r!}{\theta^r} \tag{3.10}$$

Proof:

$$E(y^r) = \int_0^\infty y^r f(y)\,dy = \int_0^\infty y^r \left(\theta e^{-\theta y} - \wp\theta e^{-\theta y} + 2\wp\theta e^{-2\theta y}\right)dy$$

$$= \theta\int_0^\infty y^r e^{-\theta y}\,dy - \wp\theta\int_0^\infty y^r e^{-\theta y}\,dy + 2\wp\theta\int_0^\infty y^r e^{-2\theta y}\,dy$$

$$= \frac{r!}{\theta^r} - \frac{\wp r!}{\theta^r} + \frac{\wp r!}{(2\theta)^r} = \left(1 - \wp + \frac{\wp}{2^r}\right)\frac{r!}{\theta^r}$$

3.2.2.2 Moment Generating Function of the QTED

Proposition 3.2.2: Given that a random variable Y has the QTED, the MGF is obtained as:

$$E(e^{ty}) = \frac{\theta}{\theta - t} - \frac{\wp\theta}{\theta - t} + \frac{2\wp\theta}{2\theta - t} \tag{3.11}$$

Proof:

$$E(e^{ty}) = \int_0^\infty e^{ty} f(y)dy = \int_0^\infty e^{ty}\left(\theta e^{-\theta y} - \wp\theta e^{-\theta y} + 2\wp\theta e^{-2\theta y}\right)dy$$

$$= \theta\int_0^\infty e^{-(\theta - t)y}\,dy - \wp\theta\int_0^\infty e^{-(\theta - t)y}\,dy + 2\wp\theta\int_0^\infty e^{-(2\theta - t)y}\,dy$$

$$= \frac{\theta}{\theta - t} - \frac{\wp\theta}{\theta - t} + \frac{2\wp\theta}{2\theta - t}$$

Hence, the mean and variance of QTED are:

$$E(Y) = \frac{2 - \wp}{2\theta} \tag{3.12}$$

$$Var(Y) = \frac{4 - 2\wp - \wp^2}{4\theta^2} \tag{3.13}$$

3.2.3 Cubic Rank Transmuted Exponential Distribution I

The Cubic Rank Transmuted Exponential Distribution I (CRTED I) using the CRT map of Al-kadim (2018) is obtained by inserting (3.5) into (3.3). Therefore, the CDF and PDF of the CRTED I are obtained as:

$$F(y) = 1 - e^{-\theta y} + \wp e^{-2\theta y} - \wp e^{-3\theta y} \tag{3.14}$$

$$f(y) = \theta e^{-\theta y}\left(1 - 2\wp e^{-\theta y} + 3\wp e^{-2\theta y}\right) \tag{3.15}$$

The survival and hazard rate functions are given as:

$$S(y) = e^{-\theta y} - \wp e^{-2\theta y} + \wp e^{-3\theta y} \tag{3.16}$$

$$h(y) = \frac{\theta\left(1 - 2\wp e^{-\theta y} + 3\wp e^{-2\theta y}\right)}{1 - \wp e^{-\theta y} + \wp e^{-2\theta y}} \tag{3.17}$$

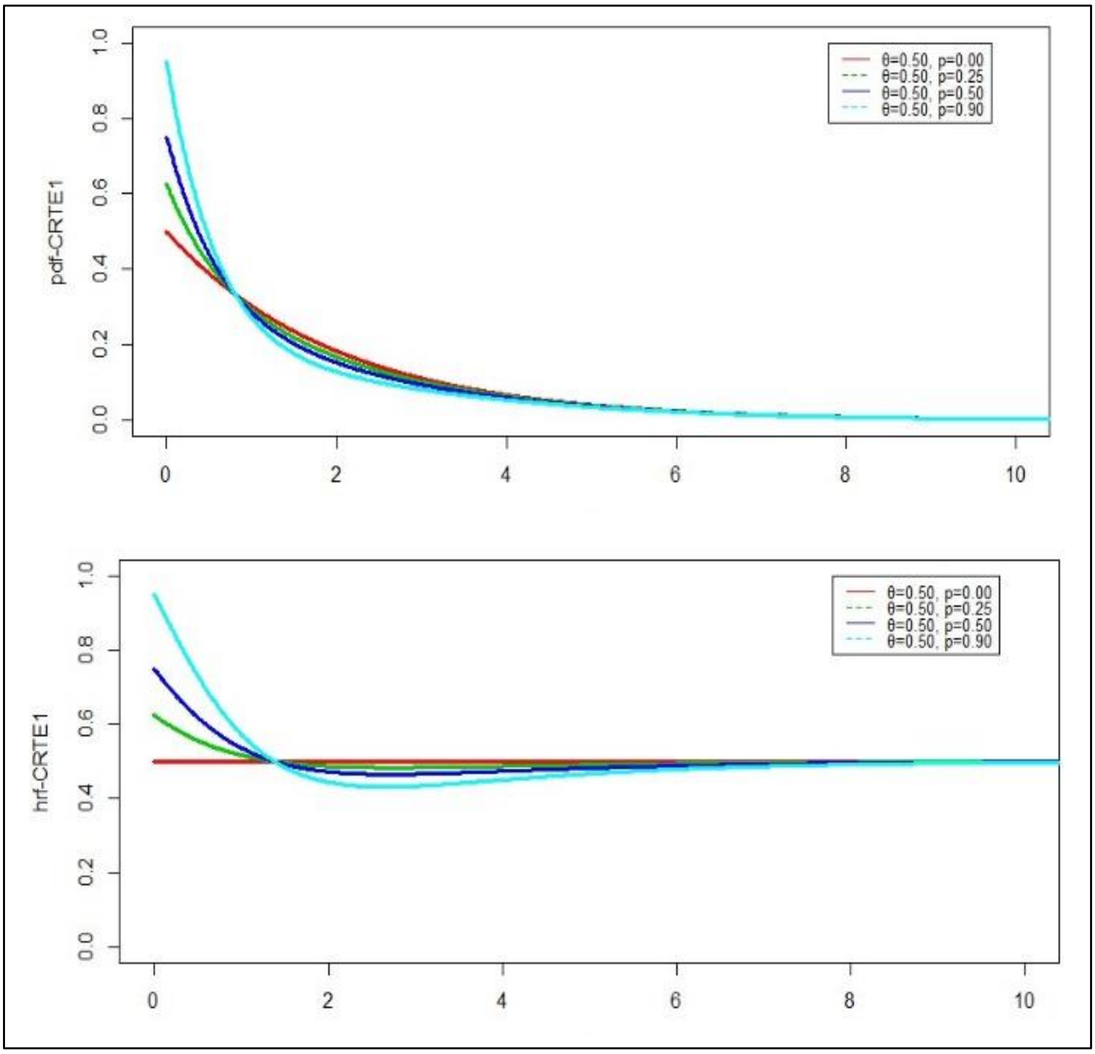

Figure 3.2 Shapes of the PDF and HRF of the CRTED I

For different values of θ and $\wp$, Figure 3.2 shows that the PDF of the CRTED I is a monotonically decreasing function. The HRF is also decreasing except when the exponential distribution ($\wp = 0$) is attained.

3.2.3.1 r^{th} Moment of the CRTED I

Proposition 3.2.3: If a random variable Y has a CRTED I, the r^{th} moment is defined as:

$$E(y^r) = \left(1 - \frac{\wp}{2^r} + \frac{\wp}{3^r}\right)\frac{r!}{\theta^r} \tag{3.18}$$

Proof:

$$E(y^r)=\int_0^\infty y^r f(y)\,dy=\int_0^\infty y^r\left(\theta e^{-\theta y}-2\wp\theta e^{-2\theta y}+3\wp\theta e^{-3\theta y}\right)dy$$

$$=\theta\int_0^\infty y^r e^{-\theta y}\,dy-2\wp\theta\int_0^\infty y^r e^{-2\theta y}\,dy+3\wp\theta\int_0^\infty y^r e^{-3\theta y}\,dy$$

$$=\frac{r!}{\theta^r}-\frac{\wp r!}{(2\theta)^r}+\frac{\wp r!}{(3\theta)^r}=\left(1-\frac{\wp}{2^r}+\frac{\wp}{3^r}\right)\frac{r!}{\theta^r}$$

3.2.3.2 Moment Generating Function of the CRTED I

Proposition 3.2.4: If a random variable *Y* has a CRTED I, the MGF is defined as:

$$E(e^{ty})=\frac{\theta}{\theta-t}-\frac{2\wp\theta}{2\theta-t}+\frac{3\wp\theta}{3\theta-t} \tag{3.19}$$

Proof:

$$E(e^{ty})=\int_0^\infty e^{ty}\left(\theta e^{-\theta y}-2\wp\theta e^{-2\theta y}+3\wp\theta e^{-3\theta y}\right)dy$$

$$=\theta\int_0^\infty e^{-(\theta-t)y}\,dy-2\wp\theta\int_0^\infty e^{-(2\theta-t)y}\,dy\ +3\wp\theta\int_0^\infty e^{-(3\theta-t)y}\,dy$$

$$=\frac{\theta}{\theta-t}-\frac{2\wp\theta}{2\theta-t}+\frac{3\wp\theta}{3\theta-t}$$

Hence, the mean and variance of CRTED I are:

$$E(Y)=\frac{6-\wp}{6\theta} \tag{3.20}$$

$$Var(Y)=\frac{36+2\wp-\wp^2}{36\theta^2} \tag{3.21}$$

3.2.4 Cubic Rank Transmuted Exponential Distribution II

The Cubic Rank Transmuted Exponential Distribution II (CRTED II) using the CRT map of Rahman *et al.*, (2019) is obtained by inserting (3.5) into (3.4). Hence, the CDF and PDF of the CRTED II are respectively obtained as: