STRUCTURAL PROPERTIES OF EXTREMAL TREES, BALANCED SPIDERS, AND PATH FORESTS WITH RESPECT TO BURNING NUMBER

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by

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etc.	et cetera

LIST OF SYMBOLS

x^a	x to the power of a
$x!$	Factorial
$f(x)$	Function of x
$\lfloor x \rfloor$	Floor function
$\lceil x \rceil$	Ceiling function
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\{\}$	set
$A \cap B$	Intersection
$A \cup B$	Union
$A \subset B$	Subset
$A \subseteq B$	Proper subset/strict subset
$A \not\subset B$ or $A \not\subseteq B$	Not subset
$A \setminus B$	Relative complement
$a \in A$	Element of
$a \notin A$	Not element of
$\varnothing$	Empty set

$\mathbb{Z}$	Set of integers
$\mathbb{Z}^+$	Set of positive integers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\sum$	Summation
$\lvert G \rvert$	Order of graph G
$\exists$	Existential quantifier
$\forall$	Universal quantifier
deg	Degree
$\deg(v)$	Degree of vertex v
$\mathrm{root}(T)$	Root of a tree T
$E(G)$	Set of edges
$V(G)$	Set of vertices
$V_{br}(G)$	Set of branch vertices
$T[v]$	Rooted tree with root v
$T_{\varnothing}$	Empty tree

SIFAT STRUKTUR POKOK EKSTREMUM, GRAF LABAH-LABAH SEIMBANG, DAN HUTAN LINTASAN DARI SEGI NOMBOR PEMBAKARAN

ABSTRAK

Pembakaran graf ialah satu proses graf diskrit yang interpretasinya sebagai model penyebaran pengaruh dalam rangkaian sosial. Pada 2016, Bonato et al. mengandaikan bahawa untuk mana-mana graf berkait dengan berperingkat N^2, nombor pembakaran adalah tidak melebihi N. Konjektur ini masih terbuka walaupun kemajuan luar biasa telah dicapai akhir-akhir ini. Dengan memperhatikan bahawa nombor pembakaran sesuatu graf berkait ialah nombor pembakaran pepohon rentang minimumnya, kajian kami menumpu pada pengenalpastian pokok ekstremum dalam erti kata bahawa setiap pokok mencapai peringkat terbesar antara pokok homeomorfisma dengan sesuatu nombor pembakaran yang diberi. Kajian ini bermula dengan menemui peringkat yang ketat pada hutan lintasan, hutan lintasan seimbang, graf labah-labah, dan graf labah-labah seimbang apabila nombor pembakaran adalah tetap. Peringkat yang ketat untuk kelas graf yang diberikan menghasilkan julat yang mungkin bagi nombor pembakaran untuk sebarang graf yang diberi dalam kelas tersebut. Dengan mengamalkan generalisasi, kami memperoleh beberapa sifat am tentang linkungan bersektu yang sepadan dengan sebarang jujukan pembakaran optimum bagi sebarang pokok ekstremum. Berdasarkan ini, kami juga mencadangkan satu rangka kerja baru yang terdiri daripada jujukan teraku atas mana-mana pokok tak terturunkan homeomorfisma sedemikian hingga mana-mana pokok ekstremum dengan diberi suatu nombor pembakaran dapat dihasilkan oleh suatu jujukan teraku dalam beberapa kata tertentu. Dengan memanfaatkan sifat-sifat jujukan ini, kami memperolehi pokok ekstremum dengan diberi suatu

nombor pembakaran untuk kes empat bucu cabang. Akhirnya, kami menyajikan hasil pada diameter terkecil pada n labah-labah ekstremum dengan diberi suatu nombor pembakaran.

STRUCTURAL PROPERTIES OF EXTREMAL TREES, BALANCED SPIDERS, AND PATH FORESTS WITH RESPECT TO BURNING NUMBER

ABSTRACT

Graph burning is a discrete-time deterministic graph process that can be interpreted as a model for spread of influence in social networks. Bonato et al. conjectured in 2016 that for any connected graph of order N^2, the burning number is at most N. This conjecture remains open, although remarkable progress has been achieved lately. By noting that the burning number of any connected graph is the minimum burning number of its spanning trees, our work focuses on identifying extremal trees in the sense that each tree attains the largest possible order among homeomorphic trees with a given burning number. The study initiates with finding the tight bounds on the orders of path forests, balanced path forests, spiders, and balanced spiders when the burning number is fixed. The tight bounds for a given class of graphs render the possible range of burning numbers for any given graph in the class. Upon generalizing, we obtain some general properties on the associated neighbourhoods corresponding to any optimal burning sequence of any extremal tree. Based on that, we propose a new framework consisting of admissible sequences over any homeomorphically irreducible tree such that any extremal tree with a given burning number can be induced by some admissible sequence in some sense. Utilising the properties of admissible sequences corresponding to extremal trees, we obtain the extremal trees with any given burning number for the case of four branch vertices. Finally, we present some results on the smallest diameter attainable by extremal n-spiders with a given burning number.

CHAPTER 1

INTRODUCTION

1.1 General Introduction

In 2016, the paper titled "How to Burn a Graph" by Bonato et al.[8] introduced graph burning as a graph parameter called burning number associated with the spreading of social contagion in social networks. Graph burning is inspired by contact processes on graphs, such as the firefighter problem [12]. The concept of graph burning is similar to the firefighter problem, in which firefighters are placed at chosen unburned vertices. The main objective of the firefighter problem is to prevent the "fire" from spreading. On the other hand, graph burning aims to turn each vertex of a graph from unburned to burned in discrete time steps or rounds as fast as possible.

In social network analysis, the spread of social influence is an increasing focus, particularly in light of the recently widespread COVID-19 pandemic [1, 32]. Within the realm of social networking, the graph burning problem aims to transmit information efficiently throughout the graph while minimising the initial number of "burned" vertices. In order to illustrate this concept, consider a network whereby each vertex corresponds to a person, and the act of "burning" represents the spread of information, influence, or a specific message. The objective is to determine the most effective strategy for distributing this information with a minimum number of initial persons (vertices).

The burning process begins with all vertices in an unburned state, and at the beginning of each round $t \geq 1$, a vertex is chosen as the new burning source. Subsequently, any unburned vertex adjacent to a burned vertex in the t-th round will be burned in the $(t+1)$-th round. A burned vertex remains burned throughout the process until all vertices are burned, at which point the graph is considered "burned". The burning source that chosen in the i-th round denoted by x_i. The sequence $(x_1, x_2, x_3, \ldots, x_k)$ is called a burning sequence of a graph.

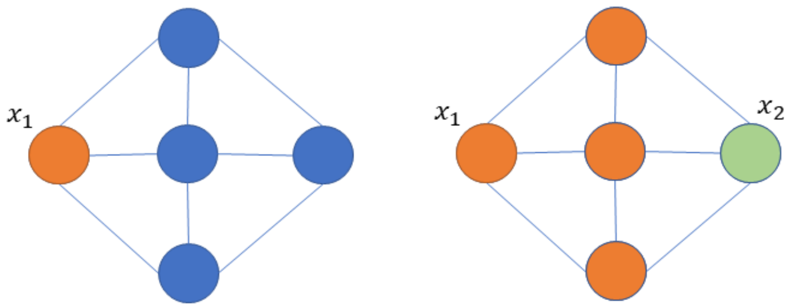

Figure 1.1: The burning process of a graph

The burning number of a graph, denoted as $b(G)$, is defined as the minimum number of rounds required to completely burn the graph. The burning process is conducted using a burning sequence, which is a sequence of chosen burning sources that effectively burns the graph. Thus, the burning number provides a measure of how efficiently the graph can be burned.

The graph burning problem, despite its relevance and applicability in social network research, but it also presents computing difficulties. It has been proven to be NP-complete for several graph classes, including planar graphs, bipartite graphs, spider graphs, path forests, and general graphs [3], as well as directed trees [19]. In the fields of mathematics and computer science, a problem is classified as "NP-complete" when it belongs to the NP class (nondeterministic polynomial time) and has a unique degree of computational complexity. NP-complete problems have two important features. First, NP-complete problems can be verified in polynomial time when given a solution, which makes it relatively quicker to verify if the proposed solutions work. Second, these problems are thought to be as hard as the hardest problems in the NP complexity class. Identifying the polynomial-time approach to solve any NP-complete problem would be important because it would give the possibility of solving all the problems in the class NP. For more information regarding NP-completeness can refer to [14].

The researchers have continuously explored various approaches to improve the upper bounds on the burning number. Initially, Bessy et al. [4] established an upper bound of $b(G) \leq rt\frac{12n}{7} + 3$ for any connected graph G with order n. This bound was subsequently enhanced by Land and Lu [24] to $\left\lceil \frac{-3+\sqrt{24n+33}}{4} \right\rceil$. Landmark breakthroughs have been achieved in recent years, Bastide et al. [2] tighter the upper bound to $\left\lceil \frac{8+\sqrt{12n+64}}{3} \right\rceil$ and Norin and Turcotte [33] proving the asymptotic results supporting the burning number conjecture.

The burning number conjecture, proposed by Bonato et al. [5, 8], suggests that for any connected graph G of order n, its burning number is at most $\lceil \sqrt{n} \rceil$. Although the conjecture has not been proven in general, researchers have extensively studied its validity for specific families of graphs, yielding affirmative results for a wide range of cases, including circulant graphs [13], grid and interval graphs [7, 17], point in plane [21], t-unicyclic graphs [40], generalised Petersen graphs [35], fence graphs [6], theta graphs [28], caterpillars [18, 26], Cartesian product and the strong product of graphs [30], spiders, and path forests [10, 11, 27]. Furthermore, researchers have developed systematic methodologies to test whether a given graph class satisfies the burning number conjecture or not in [34].

Aside from focusing on the graph burning problem and the burning number conjecture, some researchers work on variations of the burning number. The variation k-burning problem was initially investigated by [31]. This variation allows k vertices to be chosen each round where $k = 1$ is the classical burning problem. Later, the k-burning problem was shown to be NP-complete for spiders and permutation graphs [16]. Additionally, graph burning is also studied from the perspectives of parameterized complexity [20, 23] and randomness [29]. Furthermore, a few graph burning approximation algorithms are also proposed in [9, 15] and three heuristics proposed in [22] based on centrality. Later, [36] presents a novel approach for determining the burning number of a graph, which is attained by the strategic selection of activators. The selection of activators was determined by using the core vertex set obtained from the spanning tree of the original graph, in combination with weighted centrality measurements. From the provided list of nodes, an additional activator is chosen as a potential selection.

In [25], the authors also suggest a new graph parameter which is generalised burning number of graph G denoted as $b_r(G)$. This burning problem asks for the smallest number of steps required to burn every vertex in G by burning vertices only when they are close to at least r burned neighbours. Clearly, $b_1(G) = b(G)$ is our standard burning problem, which only considers a single channel of information spreading.

1.2 Scope of Research

If the burning number of a graph is at most m, then we can say the graph is m-burnable. Bonato and Lidbetter [10] and Das et al. [11] have proved that every spider of order at most m^2 is m-burnable. Later, a tight upper bound on the order of a spider to guarantee that it is m-burnable were obtained by Tan and Teh [37], and surprisingly the tight bound depends simply on the number of arms. These insightful findings served as motivation for our own research, prompting us to investigate tight bounds on the order of an extremal tree if given a fixed burning number.

In this work, our primary objective is to identify tight bounds on the order of trees with a given burning number within various graph homeomorphism classes. To achieve this, we commence by examining the tree with one branch vertex, namely spiders, and the closely related path forests. The balanced spider has been shown to have unusual characteristics in [37], hence providing encouragement for further investigation into the spider graph. The balanced path forest plays an important part in the burning of a balanced spider. In [38], the researchers are investigating the burnability of a path forest that is sufficiently long. Our study, on the other hand, focuses on examining the length of the path in relation to a relatively smaller burning number. Subsequently, we first extend our investigation to trees with proceed to a higher number of branch vertices, seeking to determine their largest attainable order while maintaining a fixed burning number. However, some problems have arisen due to the order of an extremal trees are depending on the distribution of the degree of branch vertices and the structural of trees. Thus, to overcome this, we are compelled to introduce the concept of admissible sequences, which plays a fundamental role in our analysis. We demonstrate that any tree with a given burning number and in the largest order within its homeomorphic class is induced by some admissible sequence. By comparing and analysing suitable admissible sequences, we can effectively identify extremal trees with specific burning numbers. Our exploration of extremal trees in the context of admissible sequences allows us to make significant contributions to the understanding of the burning number for various tree structures.

In addition, we explore the smallest attainable diameter of an n-spider by considering trees in the context of the largest order with a given burning number. The study of diameter enhances the investigation of the burning number, supporting an overall understanding of the structural attributes of trees show distinct burning characteristics. The balanced spider and path forest also play a crucial role in determining the smallest diameter of a spider. Therefore, our research will investigate the characteristics of the balanced spider and balanced path forest.

This study proposed a list of five research objectives and the corresponding research questions in order to provide a clear direction for our study. The five research questions are:

- What are the tight bounds on the order of trees within different graph homeomorphism classes while maintaining a fixed burning number?
- How do the properties of spiders and path forests relate to their burning characteristics?
- How can we extend our analysis to extremal trees with multiple branch vertices and determine their maximum attainable order with a specified burning number?

- Does there exist any structural characteristics that could be derived from an extremal tree for a given burning number?
- What is the smallest attainable diameter of n-spiders in the largest order with fixed burning numbers?

The five research objectives are:

- To determine tight bounds on the order of trees within various graph homeomorphism classes while maintaining a fixed burning number.
- To investigate the properties of spiders graph and path forests, to understand their relationship with burning characteristics.
- To extend the analysis to trees with multiple branch vertices and identify their maximum attainable order for a given burning number.
- To investigate the structural diversity of extremal tree for a given burning number.
- To explore the smallest attainable diameter of n-spiders in the largest order with fixed burning numbers, with a special focus on balanced spiders and path forests, to gain insights into distinct burning behaviours and structural properties.

1.3 Preliminaries

This section will begin with an introduction to basic notation and terminology, as well as some theorems related to the graph burning problem. For background of graph theory, reader are suggested to read [39].

A graph G consists of a set of *vertices* (also known as nodes) denoted by $V(G)$, and a set of *edges* denoted by $E(G)$. Every edge has either one or two vertices associated with it, called its *endpoints*. If two vertices in a graph are connected by an edge, we say the vertices are *adjacent*. An edge is also said to connect its endpoints. The sets $V(G)$ and $E(G)$, which represent the vertices and edges of graph G, respectively, have the potential to be infinite. The distance between two vertices u and v in G denoted as $d(u,v)$ is defined as the shortest path connecting u to v in the graph G.

A graph with finite vertex and edges graph will be *finite graph*. A graph is classified as simple if it satisfies two conditions: firstly, no two edges link the same pair of vertices, and secondly, every edge connects two distinct vertices. A *multigraph*, also known as a many edges graph, is a kind of graph in which there is the presence of two or more edges connecting the same pair of vertices.

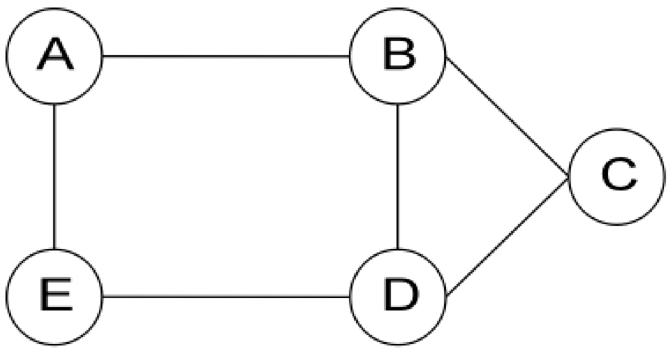

Figure 1.2: An example of a simple graph

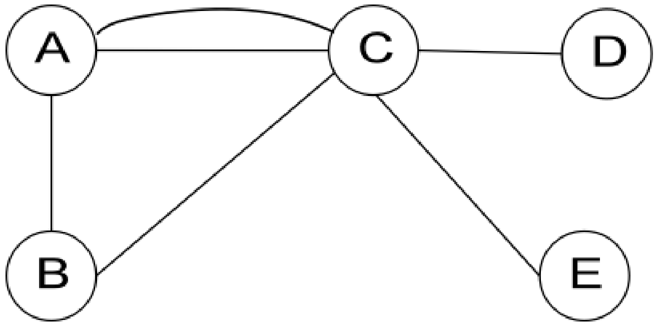

Figure 1.3: An example of a multiple graph

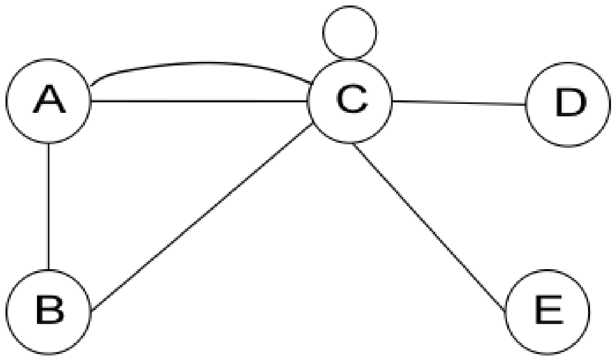

Figure 1.4: An example of a graph with loop on vertex C for diagnostics

The graphs we consider in our work are simple, finite, and undirected. The order of a graph G is the number of vertices in the graph G and denoted by $|G|$. Let $(x_1, x_2, \ldots, x_m)$ be a burning sequence of a graph G that burns in m rounds. Each x_i represents the i-th burning source chosen in the i-th round. The burning sequence is said to be *optimal* if $b(G) = m$. Note that in a burning process of m rounds, the burning source x_i will burn all the vertices within distance $m - i$ from x_i. Therefore, if there are m rounds, the i-th burning source can be used to burn a path of order (at most) $2(m - i) + 1$, by placing this burning source at the centre of the path. This last simple observation will be useful throughout this work.

For any nonnegative integer k, the k-th closed neighbourhood of a vertex v, denoted by $N_k[v]$, is defined to be the set $\{u \in V(G) \mid d(u,v) \leq k\}$. By [8], we know that $(x_1, x_2, \ldots, x_m)$ forms a burning sequence if and only if for every pair i and j, with $1 \leq i < j \leq m$, $d(x_i, x_j) \geq j - i$ and $N_{m-1}[x_1] \cup N_{m-2}[x_2] \cup \ldots \cup N_0[x_m] = V(G)$. These neighbourhoods are the *associated neighhourhoods* of the burning sequence and $N_{m-i}[x_i]$ is the *neighbourhood associated to x_i*. Hence, the burning problem is essentially solving for a cover of the set of vertices with the minimum number of closed neighbourhoods with decreasing radii. The subsequent definitions and theorems will describe some of the properties of burning numbers introduced in [8].

Theorem 1.3.1. *[8] If a given graph G, there exists a collection of connected subgraphs $\{C_1, C_2, \cdots, C_t\}$ and each of their radius is at most k which covers all the vertices of G, then $b(G) \leq t + k$.*

Theorem 1.3.2. *[8] If $\{C_1, C_2, \cdots, C_t\}$ is a covering of the vertices of a graph G, where each C_i is a connected subgraph of radius at most $k - i$, and $t \leq k$, then $b(G) \leq k$.*

Theorem 1.3.3. *[8] If $(x_1, x_2, x_3, \cdots, x_k)$ is a sequence of vertex in a graph G such that $N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \cdots \cup N_0[x_k] = V(G)$, then $b(G) \leq k$.*

The theorem above focused solely on connected graphs. However, for the disconnected graph, we cannot assume that a disconnected graph G with components of $G_1, G_2, G_3, \cdots, G_t$ has a burning number of $b(G) = b(G_1) + b(G_2) + b(G_3) + \cdots + b(G_t)$. As an example, consider two distinct paths, each consisting of two vertices. This graph can be burned in three rounds. However, each path requires two rounds to burn completely. Graph that consists of multiple disconnected path is also known as *path forest*.

The next theorem introduced by [8] provides an alternate approach for observing the burning number. A rooted tree T is a tree in which a branch vertex is designated as the *root*, denoted as root(T). The root serves as the beginning point for traversal. The *predecessor* of v in T refers to the unique vertex adjacent to v from the root leading towards v. The empty rooted tree is denoted as $T_\varnothing$. For any vertex $v \in V(T)$, the rooted tree $T[v]$ is the subtree of T with root v and all its descendants in T.

The *depth* of a vertex in a rooted tree T is defined as the minimum number of edges along the path from the vertex to the root of the tree. The *height* of a rooted tree T is defined as the maximum depth of T. Put simply, the rooted tree may be seen as a partition or subtree of G, with a constraint on the distance between two vertices.

Theorem 1.3.4. *[8] Burning a graph G in k steps is equivalent to finding a rooted tree partition into k trees $T_1, T_2, \cdots, T_k$, with heights at most $(k-1), (k-2), \cdots, 0$, respectively, such that for every $1 \le i, j \le k$, the distance between the roots of T_i and T_j is at least $|i-j|$.*

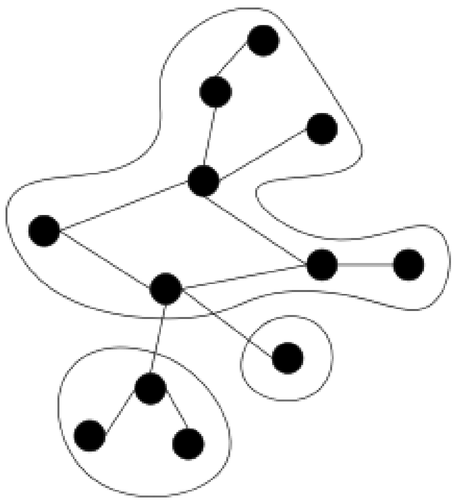

Figure 1.5: The partition of rooted trees

In [8], the authors also showed that the burning number of a connected graph is the minimum burning number of its spanning trees. Let T be a tree. A *leaf* of T is a vertex in T with degree one. A *branch vertex* is a vertex with degree greater than two. The set of branch vertices of T is denoted as $V_{br}(T)$. An *internal path* refers to the path connecting any two branch vertices, while the path joining a leaf to a branch vertex is called an *arm*. Two trees are *homeomorphic* if and only if they are isomorphic up to contraction (deleting a vertex) or extension (adding a vertex) of the internal paths or arms. A tree T is *homeomorphically irreducible* if there is no other homeomorphic tree with a smaller order; equivalently, T has no vertices of degree two. An n-spider is a tree with exactly one vertex of degree $n \geq 3$, called the *head* of the spider, and every other vertex has degree at most two. T is said to be *balanced* if each of the arm of T are in equal length. Also, T is said to be *maximal* with burning number m if adding a vertex to one of the internal paths or arms of T results in a tree that is not m-burnable.

The study published by Bonato et al. [10] on spiders integrates ideas drawn from their proven theory on the burning number of a path. When the two ends of this path are connected, a cyclic graph is formed, and interestingly, this cyclic graph burning number has similar characteristics to the path.

Theorem 1.3.5. *[8] For a path P_n and n vertices, we have a* $b(P_n) = \lceil n^{1/2} \rceil$.

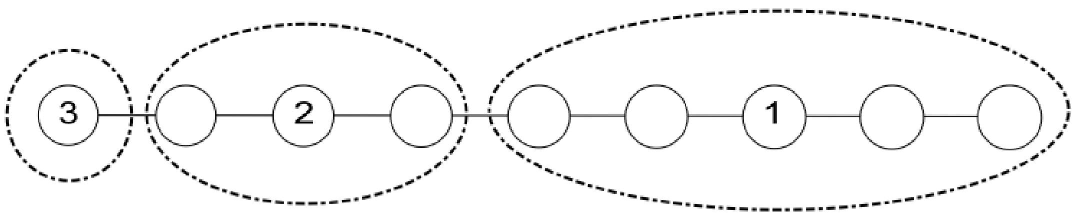

Figure 1.6: Optimum burning sequence of path P_9

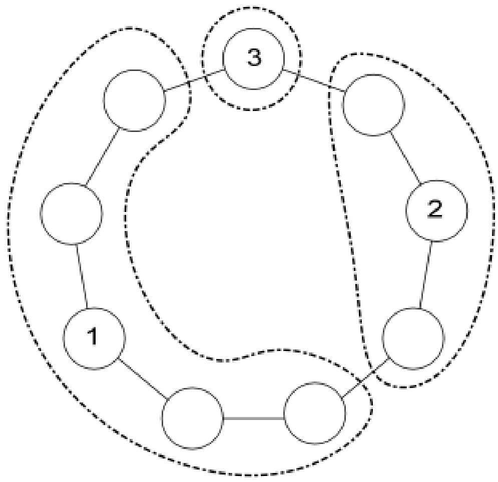

Figure 1.7: Optimum burning sequence of cycle C_9

The Figure 1.6 and 1.7 show an optimum burning sequence of a path and cycle with nine vertices. The following two theorems give a significant breakthrough on the burning number of spiders. They shown that a spider graph does follow the burning conjecture and gives a tight upper bound on the order of a spider to guarantee that it is m-burnable. This upper bound is tied closely to the number of arms. More interestingly, the upper bound works generally, with some exceptional cases that are related to balanced spiders. Therefore, together with the following result, it shows that balanced spiders are worthy of study in the context of graph burning.

Theorem 1.3.6. *[10] Every spider of order at most m^2 is m-burnable.*

Theorem 1.3.7. *[37] Let $n \geq 3$ and $m \geq 3$. If $m > n$, then every n-spider of order at most $m^2 + n - 2$ is m-burnable. On the other hand, if $n \geq m$, then every n-spider of order at most $m^2 + n - 2$ is m-burnable unless it contains the balanced m-spider of order $m^2 + 1$ as a subgraph.*

Theorem 1.3.8. *[3] Let $n \geq 3$. If T is a balanced n-spider with each arm length equals l such that $l \leq n$, then $b(T) = l + 1$.*

The following theorem is stated as a remark in [37]. This theorem presents a concept that helps in determining the smallest order of a spider. Therefore, we will provide a proof of this theorem.

Theorem 1.3.9. *[37] If one of the arm of n-spider (for $n > 2$) has length one, then deleting that arm results in a subspider with the same burning number.*

Proof. Suppose T is an n-spider with at least one arm of length one and subspider H is a subgraph of T with one arm of length one removed (i.e. a leaf e is removed from T). Let the optimum sequence of T be $S = (x_1, x_2, \ldots, x_m)$ and the head of T (and H) be c.

We claim that if $b(T) = m$, then $b(H) \leq m$. This is because if no burning source is positioned at e, then H can also be burned using S. If $e \in S$, then we consider two cases. If $e = x_m$, then in the process of burning H, we follow S to burn H and simply allocate x_m at any unburned vertices at the m-th round or H may be completely burned with burning sequence $(x_1, x_2, \ldots, x_{m-1})$. If $e = x_i$ for $1 \leq i < m$, then we can allocate x_i at c if c is not burned earlier, or else we simply allocate x_i at any other unburned vertices in the i-th round. In all cases, H is burned in at most m rounds and hence $b(H) \leq m$.

To show that $b(H) \geq m = b(T)$ which implies $b(T) \leq b(H) = r$, we may show that T can also be completely burned using at most r burning sources. Let $S' = (x_1, x_2, \ldots, x_r)$ be an optimum burning sequence of H.

If c in H is burned in the i-th round where $i < r$ using S', then T can be burned in at least r rounds using S'. Hence we can assume c in H is burned in r-th round by x_i

where $1 \leq i \leq r$.

If there exists a burning source x_j on an arm l in H that did not burn maximally in l, then we can relocate some burning sources on l to be one step closer to c in T. Then e can be burned in the next round after c is burned and hence $b(T) \leq r$. Now, we assume that all burning sources in S' burn maximally in all arms in H. We consider two cases.

Case 1. c in H is burned by x_r

Without loss of generality, in H, assume x_a be the closest burning source to c at an arm l_1 and x_b be the closest burning source at another arm l_2 where $a < b < r$. First, we relocate x_a in l_1 to be $2(r-b)+2$ steps closer to c and to the direction of x_b. All other burning sources besides x_a, x_b and x_r remain as in S'. Thus, c and first $2(r-b)+1$ vertices nearest to c in l_2 will be burned by the newly positioned x_a. Then we relocate both x_b and x_r to l_1 in order to burn the unburned $2(r-b)+2$ vertices (which were originally burned by x_a) in l_1. Thus, we can completely burn T using the newly relocated x_a, x_b and x_r and remained-position of the rest of burning sources as in H since e in T can be burned after c is burned by x_a where $a < r$.

Case 2. c in H is burned by x_i for some $1 \leq i < r$ in r-th round

Assuming that x_r is initially in H, placed at a vertex in an arm l and each burning source in l burns maximally. Here, we consider two cases by relocating x_i and x_r in H. First, if x_i and x_r are located on different arms, we can relocate x_i to be one step closer to c. Recall that c is burned in r-th round. Consequently, with relocated x_i, c will be burned in $(r-1)$-th round and x_r will be relocated at the last vertex which was originally burned by x_i in l. Second, if x_i and x_r are on the same arm, we can easily relocate x_r to c and x_i to be one step away from c in l. Then, c is now burned by x_r and following the relocation strategy in Case 1, T can then be burned using r burning sources.

This completes the proof. □

Graph burning of spiders is closely related to that of path forests. This is because when we remove the closed neighbourhood associated to some burning sources that contains the head, the subgraph induced by the remaining vertices is a path forest. In fact, the following theorem guarantees that for any spider, there exists an optimal burning sequence such that the head is burned by the fire spread from the first burning source. However, the original proof of Theorem 1.3.10 is very technical. Therefore, here, we provide an alternative and neater proof.

Theorem 1.3.10. *[3] Let $n \geq 3$. Suppose T is an n-spider with spider head c. There exists an optimal burning sequence $(x_1, x_2, \ldots, x_m)$ of T such that $d(x_1, c) \leq m-1$.*

Proof. We argue by contradiction. Suppose T is an n-spider with spider head c. Assume all optimal burning sequences of T are such that $d(x_1, c) > m-1$. Consider an optimal burning sequence of T where x_1 is the closest to the head. Let

$$W = \{v \in V(T) \mid v \notin N_{m-1}[x_1] \text{ and } v \text{ is a vertex along the path from } c \text{ to } x_1\}.$$

Clearly, W is non-empty as $d(x_1,c) > m-1$. Let v^* denote the vertex in W furthest away from head (possibly the head if $d(x_1,c) = m$).

Suppose v^* is included in the neighbourhood $N_{m-i}[x_i]$ for some $2 \le i \le m$. Let us consider the path joining x_1 and x_i. Our strategy is to relocate x_1 closer to x_i such that $N_{m-i}[x_i]$ forms a proper subset of $N_{m-1}[x'_1]$, where x'_1 is the newly relocated x_1. (Our proof works regardless of whether x_i is on the same arm as x_1 or not. Figure 1.8 illustrates the relocation of x_1, using one interesting scenario.)

The maximum possible distance between x_1 and x_i is $(m-1)+(m-i)+1 = 2m-i$, while the minimum possible distance is i for otherwise, $v^* \notin N_{m-i}[x_i]$. Let x'_1 be the vertex along the path joining x_1 and x_i such that $d(x'_1,x_i) = i-1$. This can be done because $d(x_1,x_i) \ge i$. Since $d(x'_1,x_i) = i-1$, the vertex x_i will be burned by the fire spread from the new vertex x'_1 at round i. Therefore, the neighbourhood $N_{m-i}[x_i]$ is included in $N_{m-1}[x'_1]$.

Note that $d(x_1,x'_1) \le 2m-i-(i-1) = 2(m-i)+1$. This implies that upon the relocation of x_1 to x'_1, at most $2(m-i)+1$ vertices originally contained in $N_{m-1}[x_1]$ are now excluded from $N_{m-1}[x'_1]$. Furthermore, these excluded vertices form a path segment on the same arm as x_1. Hence, these vertices can be burned by relocating x_i to a vertex x'_i at the center of that path of order at most $2(m-i)+1$. There is no harm in doing this since the vertices originally burned by the fire from x_i is now taken care by the fire from x'_1. Therefore, letting $x'_j = x_j$ for every other $j \notin \{1,i\}$, it can be seen now that $(x'_1,x'_2,\ldots,x'_m)$ is also an optimal burning sequence of T. However, x'_1 is closer to c compared to x_1, which leads to a contradiction. $\square$

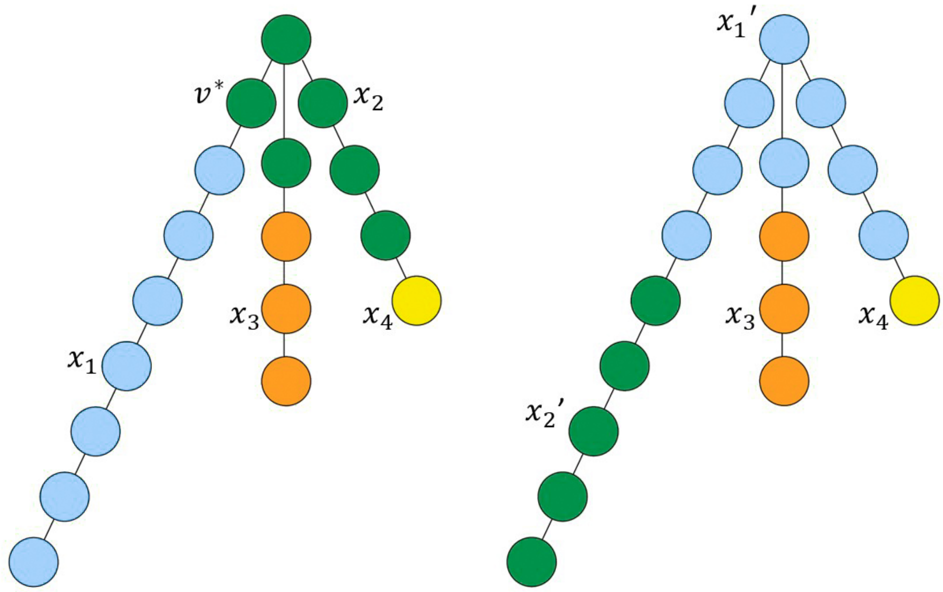

Figure 1.8: An example of the relocation of x_1 to x_1'

Theorem 1.3.11. *[37] Let $m \geq n \geq 2$ and suppose T is a path forest with n paths. If*

$$|T| \leq m^2 - (n-1)^2 + 1,$$

then T is m-burnable unless $|T| = m^2 - (n-1)^2 + 1$ and T is the unique path forest with path orders $m^2 - n^2 + 2, \underbrace{2, 2, \ldots, 2}_{n-1 \text{ times}}$.

Apart from extending Theorem 1.3.7 and Theorem 1.3.11 in [38], the authors also showed that every path forest of order m^2 with sufficiently long paths is m-burnable.

Theorem 1.3.12. *[38] Let $n \geq 2$. There exists $L \in \mathbb{Z}^+$ with the following property: if T is a path forest with n paths of order m^2 such that the order of its shortest path is at least L, then the burning number of T is m.*

Theorem 1.3.12 leads to the following definition.

Definition 1.3.13. [38] For $n \geq 2$, define L_n to be the least integer with the following property: if T is a path forest with n paths of order m^2 where each of its paths has order at least L_n, then $b(T) = m$.

It was claimed that $L_2 = 3$, $L_3 = 18$, and $L_4 = 26$, but the study of the values of L_n generally was left as a future direction.

1.4 Structure of thesis

In this part, the structure of the thesis will be clarified. This thesis contains six chapters. The first chapter of this study centres on providing an introduction to the field of graph theory, specifically addressing the graph burning problem. This includes the discussion on the current research attempt in this field, outlining the fundamental notion of graph burning, and presenting the path of research along with the necessary preliminary information.

The subsequent chapter of this thesis will provide the findings of our research related to a tree with a single branch vertex, namely spiders. In addition, we investigate the tight bound on the order of the balanced spider and path forest, since both graphs have significance for the next chapter. In addition, we investigate the concept of the balanced spider and path forest, since both elements have significance for a subsequent chapter.

The third chapter will provide another option for expanding our research by considering a tree that has multiple branch vertices. Following the first study relate to the spider, we have successfully determined the largest order of tree with two branch vertices. Several significant general observations are also being substantiated in this last subsection. The highlight of this chapter is we propose a general framework namely admissible sequence. The introduction of admissible sequence is to tackle the issue arise when we extend our study to tree that has multiple vertices.

The fourth chapter will be the application of the admissible sequence. We will demonstrate the practicality of admissible sequences by applying this work on tree with three and four branch vertices.

The fifth chapter study the extremal tree from the approach of diameter of a tree.

The last chapter will contain a conclusion and an in-depth discussion. The concluding section of this thesis will provide an in-depth overview and analysis of the significant findings and arguments given throughout the study. Moreover, this chapter will emphasise its limitations and propose future directions for future research.

CHAPTER 2

RESULTS ON THE SPIDERS AND PATH FORESTS

To identify the tight bound on the order for the general tree. We would start by investigating the tree with a single branch vertex, namely the spider, and the related graph path forest.

2.1 Tight bounds on the order of n-spiders

The following result is established by using Theorem 1.3.7.

Theorem 2.1.1. *Let $n \geq 3$ and $m \geq 2$.*

1. *If $m > n+1$, then the smallest order of an n-spider with burning number m is $(m-1)^2 + n - 1$.*
2. *If $n+1 \geq m$, then the smallest order of an n-spider with burning number m is $(m-1)^2 + n - m + 2$.*

Proof. (Proof of Part (1)) Consider the n-spider T of order $(m-1)^2 + n - 1$ with arm lengths $(m-1)^2 - 1, 1, \ldots, 1$. Trivially, T is m-burnable. Since T contains a path of order $(m-1)^2 + 1$ as a subgraph, its burning number is m. As $m - 1 > n$, by Theorem 1.3.7, any n-spider of order at most $(m-1)^2 + n - 2$ is $(m-1)$-burnable. Hence, the proof is complete.

(Proof of Part (2)) Trivially, the smallest order of an n-spider with burning number two is $n+1$. Also, the n-spider of order $n+3$ with arm lengths $2,2,1,1,\ldots,1$ has burning number three and any n-spider of order $n+2$ is 2-burnable. These imply that the result holds for $m \in \{2,3\}$. Hence, we may suppose $m \geq 4$. Consider the n-spider T of order $(m-1)^2+n-m+2$ with arm lengths

$$\underbrace{m-1,m-1,\ldots,m-1}_{m-1 \text{ times}},\ \underbrace{1,1,\ldots,1}_{n-m+1 \text{ times}}.$$

Clearly, T has the smallest order among any n-spider containing the balanced $(m-1)$-spider of order $(m-1)^2+1$ as a subgraph. As $n \geq m-1$, by Theorem 1.3.7, T is not $(m-1)$-burnable. Clearly, T is m-burnable and thus its burning number is m. Furthermore, by Theorem 1.3.7 again, any n-spider of order at most $(m-1)^2+n-m+1$ is $(m-1)$-burnable because it cannot possibly contain the balanced $(m-1)$-spider of order $(m-1)^2+1$ as a subgraph. Hence, the proof is complete. □

Theorem 2.1.2. *For every $n \geq 3$ and $m \geq 2$, the largest order of an n-spider with burning number m is $(m-1)^2+n(m-1)+1$.*

Proof. Consider the n-spider T with arm lengths $(m-1)^2+m-1,\underbrace{m-1,\ldots,m-1}_{n-1 \text{ times}}$. Its order is $(m-1)^2+n(m-1)+1$. By putting the first burning source at the head, all the arms are burned in m-rounds by its fire, except $(m-1)^2$ vertices from the longest arm. Hence, T is clearly m-burnable. Also, it contains a path longer than $(m-1)^2$ as a subgraph and thus it is not $(m-1)$-burnable. Therefore, $b(T)=m$. To complete the proof, it suffices to show that for any n-spider with burning number m, its order is at most $(m-1)^2+n(m-1)+1$.

Suppose T' is an n-spider with burning number m and suppose in an optimal burning sequence, the head c is burned for the first time in the t-th round by the fire that spread from the burning source x_j for some $1 \le j \le m$. It can be verified that

$$|N_{m-j}[x_j]| \le 2(m-j)+1+(n-2)(m-t).$$

By the choice of x_j, it follows that for every other burning source x_i, at most $2(m-i)+1$ vertices from one of the arms that are not included in $N_{m-j}[x_j]$ are burned by the fire that spread from x_i. Therefore,

$$\begin{aligned}
|T| &\le \sum_{i\in[m]\setminus\{j\}} [2(m-i)+1] + 2(m-j)+1+(n-2)(m-t) \\
&= [(2m-1)+(2m-3)+\cdots+1]+(n-2)(m-t) \\
&\le m^2+(n-2)(m-1) \\
&= (m-1)^2+n(m-1)+1.
\end{aligned}$$

The proof is thus complete. □

2.2 Balanced spiders and path forests

Upon studying the tight bounds on the order of spiders, our focus shifts to path forests. By Theorem 1.3.10, after the vertices in the neighbourhood associated with the first burning source are excluded in an optimal burning sequence for a spider, a path forest forms naturally. As a result, finding out whether the remaining path forest can burn using the remaining burning sources will provide information on the burnability of the spider.

Theorem 2.2.1. *For $m > n \geq 2$, the smallest order of a path forest with n paths and burning number m is $(m-1)^2-(n-1)^2+1$. Furthermore, there is a unique such path forest where its longest path has order $(m-1)^2-n^2+2$, and the remaining $n-1$ paths are of order 2.*

Proof. Suppose T is the said path forest of order $(m-1)^2-(n-1)^2+1$ with path orders $(m-1)^2-n^2+2,2,2,\ldots,2$. Then, T is m-burnable but not $(m-1)$-burnable by Theorem 1.3.11. Hence, the burning number of T is m. Since any path forest with n paths of order at most $(m-1)^2-(n-1)^2+1$ is $(m-1)$-burnable except for the path forest T by Theorem 1.3.11 again, it follows that T is the unique path forest with n paths and burning number m of the smallest order. □

The largest attainable order of a path forest with burning number m is m^2. Indeed, for every $n \geq 2$, the path forest with path orders $1,3,\ldots,2n-3,m^2-(n-1)^2$ has burning number m.

After studying general spiders and path forests, we will now obtain analogous results for balanced spiders and balanced path forests. The proof of Theorem 2.1.2 shows that for an n-spider with a given burning number of the largest order, the first burning source of an optimal burning sequence must be placed at the head of the spider. However, this observation is not true for the case of balanced spiders. Let us consider a balanced 3-spider of order 16 as in Figure 2.1. If the first burning source x_1 is placed at the head, the remaining vertices unburned by x_1 in four rounds form a path forest consisting of three paths each of order two. Clearly, this path forest cannot be burned by the remaining three burning sources. However, we could burn this balanced spider