DEVELOPMENT OF ROBUST MEMORY-TYPE CHARTS UNDER REPETITIVE SAMPLING AND TRIPLE SAMPLING CHARTS FOR THE GAMMA PROCESS

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LIST OF SYMBOLS

3σ	3-sigma
$\bar{X}$	Sample mean
δ	Size of a standardised mean shift
δ_{min}	Lower bound of the size of the standardised mean shift
δ_{max}	Upper bound of the size of the standardised mean shift
X_i	Process variable/observation (quantitative); or plotting/charting statistics of the Shewhart *X* chart and TCC
μ_0	In-control process (population) mean
μ_1	Out-of-control process (population) mean
σ	In-control process (population) standard deviation
n	Sample size; or sample size of individual observation-based charts
n_1	Fixed sample size of the Shewhart $\bar{X}$ chart; or sample size at the first sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
L_1	Control limit coefficient of the Shewhart $\bar{X}$ chart; or control limit coefficient at the first sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
α	Type-I error probability; or probability of a false alarm
β	Type-II error probability
$f(x)$	Probability density function of random variable *X*
a	Shape parameter of the gamma distribution
b	Scale parameter of the gamma distribution
XmR	Individual/moving range chart
Q_1, Q_2, Q_3	First, second and third quartiles of the specific distribution
IQR	Interquartile range
k	Control limit coefficient of different competing charts
g	Number of trials in Monte Carlo simulations

$E(\cdot)$	Expected value
bmc	Benchmark chart
R_i^-, G_i^-, C_i^-	Lower CUSUM statistics or plotting/charting statistics of the CUSUM-TCC, MEC-TCC and RS-SEC-TCC
R_i^+, G_i^+, C_i^+	Upper CUSUM statistics or plotting/charting statistics of the CUSUM-TCC, MEC-TCC and RS-SEC-TCC
Q_1^*, Q_3^*	Adjusted first and third quartiles
t, r, d	Slack value of the CUSUM-TCC, MEC-TCC and RS-SEC-TCC
H	Decision interval of the CUSUM-TCC and MEC-TCC for symmetric distributions
h	Decision interval constant of the CUSUM-TCC and MEC-TCC for symmetric distributions
H_l, H_u	Lower and upper decision intervals of the CUSUM-TCC and MEC-TCC for skewed distributions
h_l, h_u	Lower and upper decision interval constants of the CUSUM-TCC and MEC-TCC for skewed distributions
λ	Smoothing constant/parameter; or weighting parameter
D_i, F_i, M_i	Plotting/charting statistics of the EWMA-TCC, DEWMA-TCC and TEWMA chart
k_l, k_u	Control limit coefficients of the EWMA-TCC for skewed distributions
v	Degrees of freedom in Student's *t* distribution
W	Wilcoxon rank-sum test statistic
$LCL_{2\text{RS}}$, $UCL_{2\text{RS}}$	Inner control limits of the RS-TEWMA-TCC
$LCL_{1\text{RS}}$, $UCL_{1\text{RS}}$	Outer control limits of the RS-TEWMA-TCC
w_1	Warning limit coefficient at the first sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
L_2	Control limit coefficient at the second sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
LCL_1, LCL_2	Lower control limits of the first and second sampling stages of the DS $\bar{X}$ and TS $\bar{X}$ charts

UCL_1, UCL_2	Upper control limits of the first and second sampling stages of the DS $\bar{X}$ and TS $\bar{X}$ charts
$\bar{X}_1$, $\bar{X}_2^*$	Plotting/charting statistics at the first and second sampling stages of the DS $\bar{X}$ and TS $\bar{X}$ charts
$\bar{X}_2$	Sample mean at the second sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
n_2	Sample size at the second sampling stage of the DS $\bar{X}$ and TS $\bar{X}$ charts
$f(\bar{X}_i \| an_i, b/n_i)$	Probability density function of the gamma distribution with shape parameter an_i and scale parameter b/n_i, for i = 1, 2, 3
Z_i	Plotting/charting statistic of the TEWMA-TCC and RS-TEWMA-TCC
k_s	Control limit coefficient of the TEWMA-TCC for symmetric distributions
k_{l1}, k_{u1}	Control limit coefficients of the TEWMA-TCC for skewed distributions
k_{s1}, k_{s2}	Control limit coefficients of the RS-TEWMA-TCC for symmetric distributions
$k_{l2}, k_{u2}, k_{l3}, k_{u3}$	Control limit coefficients of the RS-TEWMA-TCC for skewed distributions
r_0	Desired value of the in-control average run length
$F^{-1}(0.25)$, $F^{-1}(0.50)$, $F^{-1}(0.75)$	Quantiles for a specific distribution
$\hat{\mu}_0$	Estimator of the in-control process mean
$\hat{\sigma}$	Estimator of the in-control process standard deviation
$\hat{Q}_1$, $\hat{Q}_2$, $\hat{Q}_3$	Estimators of the first, second and third quartiles
$\widehat{IQR}$	Estimator of the interquartile range
H_1, H_2	Pair of decision intervals for upper and lower CUSUM statistics of the RS-SEC-TCC for symmetric distributions
H_{l1}, H_{l2}	Pair of asymmetric decision intervals for lower CUSUM statistic of the RS-SEC-TCC for skewed distributions

H_{u1}, H_{u2}	Pair of asymmetric decision intervals for upper CUSUM statistic of the RS-SEC-TCC for skewed distributions
h_1, h_2	Decision interval constants of the RS-SEC-TCC for symmetric distributions
$h_{l1}, h_{u1}, h_{l2}, h_{u2}$	Decision interval constants of the RS-SEC-TCC for skewed distributions
L	Control limit coefficient of the Shewhart X chart for symmetric distributions
L_l, L_u	Control limit coefficients of the Shewhart X chart for skewed distributions
n_3	Sample size of the third sampling stage of the TS $\bar{X}$ chart
w_2	Warning limit coefficient of the second sampling stage of the TS $\bar{X}$ chart
L_3	Control limit coefficient of the third sampling stage of the TS $\bar{X}$ chart
LWL_1, LWL_2	Lower warning limits of the first and second sampling stages of the TS $\bar{X}$ chart
UWL_1, UWL_2	Upper warning limits of the first and second sampling stages of the TS $\bar{X}$ chart
LCL_3, UCL_3	Lower and upper control limits of the third sampling stage of the TS $\bar{X}$ chart
$\bar{X}_3$	Sample mean of the third sampling stage of the TS $\bar{X}$ chart
$\bar{X}_3^*$	Plotting/charting statistic of the third sampling stage of the TS $\bar{X}$ chart
n_0	Desired in-control average sample size of the TS $\bar{X}$ chart
$f(\delta)$	Probability density function of the size of the standardised mean shift
$m_X(t)$	Moment generating function of X
$m_{\bar{X}}(t)$	Moment generating function of $\bar{X}$
M	Mean of $\bar{X}$
S	Standard deviation of $\bar{X}$
Y	Normally distributed process variable

$\bar{Y}$	Sample mean of the process variable Y
$\bar{Y}_1, \bar{Y}_2, \bar{Y}_3$	Sample means of the first, second and third sampling stages when the process variable Y follows the normal distribution
τ_{01}	Desired in-control value of the average number of observations to signal of the TS $\bar{X}$ chart
τ_{02}	Desired in-control value of the average run length of the TS $\bar{X}$ chart
τ_{03}	Desired in-control value of the expected average number of observations to signal of the TS $\bar{X}$ chart

LIST OF ABBREVIATIONS

ACL	Asymmetric control limits
ADI	Asymmetric decision intervals
ANOS	Average number of observations to signal
ANOS(0)	In-control average number of observations to signal
ANOS(δ)	Out-of-control average number of observations to signal
ARL	Average run length
ARL(0)	In-control average run length
ARL(δ)	Out-of-control average run length
ASS	Average sample size
ASS(0)	In-control average sample size
ASS(δ)	Out-of-control average sample size
AATS	Adjusted average time to signal
ATS	Average time to signal
CL	Centreline
CLC	Control limit coefficient
CUSUM	Cumulative sum
CUSUM-TCC	Cumulative sum Tukey control chart
DEWMA	Double exponentially weighted moving average
DI	Decision interval
DIC	Decision interval constant
DS	Double sampling
DSCN	Drain saturation current of *N*-type
EWMA	Exponentially weighted moving average
EWMA-TCC	Exponentially weighted moving average Tukey control chart
EQL	Extra quadratic loss
EANOS	Expected average number of observations to signal
EANOS(0)	In-control expected average number of observations to signal
EANOS(δ_{min}, δ_{max})	Out-of-control expected average number of observations to signal, based on the shift interval (δ_{min}, δ_{max})
FSS	Fixed sample size
GWMA	Generally weighted moving average
IP	Industrial production

IQR	Interquartile range
K-S	Kolmogorov-Smirnov
LCL	Lower control limit
LWL	Lower warning limit
MATLAB	Matrix laboratory
MCE	Mixed cumulative sum exponentially weighted moving average
MDEWMA-TCC	Mixed double exponentially weighted moving average Tukey control chart
MEC	Mixed exponentially weighted moving average cumulative sum
MEC-TCC	Mixed exponentially weighted moving average cumulative sum Tukey control chart
MGF	Moment generating function
MRL	Median run length
MRL(0)	In-control median run length
MRL(δ)	Out-of-control median run length
MOS	Metal oxide semiconductor
PM	Particulate matter
PCI	Performance comparison index
PCC	Proposed control charts
pdf	Probability density function
RARL	Relative average run length
RL	Run length
RS	Repetitive sampling
RS-SEC-TCC	Repetitive sampling Shewhart exponentially weighted moving average cumulative sum Tukey control chart
RS-TEWMA-TCC	Repetitive sampling triple exponentially weighted moving average Tukey control chart
SAS	Statistical analysis system
SCL	Symmetric control limits
SD	Standard deviation
SDRL	Standard deviation of the run length
SDRL(0)	In-control standard deviation of the run length
SDRL(δ)	Out-of-control standard deviation of the run length
SPC	Statistical process control
SS	Steady-state
SSARL	Steady-state average run length

SSARL(0)	In-control steady-state average run length
SSARL(δ)	Out-of-control steady-state average run length
SSMRL	Steady-state median run length
SSSDRL	Steady-state standard deviation of the run length
TBE	Time between events
TCC	Tukey control chart
TEWMA	Triple exponentially weighted moving average
TEWMA-TCC	Triple exponentially weighted moving average Tukey control chart
TS	Triple sampling
UCL	Upper control limit
UWL	Upper warning limit
VP	Variable parameter
VSI	Variable sampling interval
VSS	Variable sample size
VSSI	Variable sample size and sampling interval
WAT	Wafer acceptable test
WLC	Warning limit coefficient
ZS	Zero-state

LIST OF APPENDICES

PEMBANGUNAN CARTA TEGUH JENIS INGATAN DI BAWAH PENSAMPELAN BERULANG DAN CARTA PENSAMPELAN BERGANDA TIGA BAGI PROSES GAMA

ABSTRAK

Proses pengeluaran dalam industri moden biasanya menghasilkan produk dengan variasi kecil disebabkan oleh kemajuan teknologi. Carta jenis Shewhart adalah tidak peka untuk mengesan anjakan proses yang kecil. Dengan membangunkan carta jenis ingatan dan carta jenis penyesuaian, para penyelidik telah menyelesaikan kelemahan carta jenis Shewhart dalam pengesanan anjakan kecil. Disebabkan oleh kos pensampelan yang tinggi dan ujian pemusnahan, jurutera kualiti menggunakan carta kawalan individu untuk memantau min proses. Terdapat tiga objektif dalam tesis ini. Pertama sekali, skim purata bergerak berpemberat eksponen berganda tiga (TEWMA) dan carta kawalan Tukey (TCC) digabungkan untuk membangunkan TEWMA-TCC dan carta yang berasaskan pensampelan berulang (RS), iaitu RS-TEWMA-TCC untuk memantau min bagi proses yang bertaburan normal dan bukan normal. TEWMA-TCC, RS-TEWMA-TCC dan carta-carta lain dibandingkan berdasarkan metrik panjang larian purata (ARL), sisihan piawai panjang larian (SDRL) dan panjang larian median (MRL) di bawah kedua-dua keadaan sifar (ZS) dan keadaan mantap (SS). TEWMA-TCC dan RS-TEWMA-TCC menunjukkan dominasi dalam pengesanan anjakan min dalam kedua-dua arah. Carta ini juga teguh kepada taburan terpencong kerana tidak mempunyai masalah kepincangan ARL. Kedua, RS untuk statistik jenis hasiltambah kumulatif (CUSUM) yang dibincangkan oleh Riaz et al. (2017) digabungkan dengan carta Shewhart untuk mencadangkan RS Shewhart purata bergerak berpemberat eksponen CUSUM TCC (RS-SEC-TCC). Kekuatan RS-SEC-TCC berbanding

pesaingnya telah diukur dengan menggunakan metrik ARL, SDRL, MRL, kerugian kuadratik tambahan (EQL), ARL relatif (RARL) dan indeks perbandingan prestasi (PCI). Keputusan menunjukkan bahawa RS-SEC-TCC adalah baik dalam mengenal pasti anjakan min yang kecil hingga besar dalam kedua-dua arah untuk taburan simetri dan terpencong. Penggunaan selang keputusan tak simetri (ADI) dan TCC memastikan RS-SEC-TCC adalah teguh kepada taburan proses terpencong dengan menghapuskan masalah kepincangan ARL. TEWMA-TCC, RS-TEWMA-TCC dan RS-SEC-TCC dibangunkan dengan menggunakan n = 1; yakni, carta-carta tersebut ialah carta individu dan boleh digunakan dengan berkesan apabila pengujian adalah merosakkan dan kos pensampelan adalah tinggi. Ketiga, carta pensampelan berganda tiga (TS) $\bar{X}$ untuk memantau anjakan min bagi proses bertaburan gama dicadangkan. Carta TS $\bar{X}$ dibina dengan lapan parameter reka bentuk yang memberikan pengamal fleksibiliti dalam pembinaan carta. Keupayaan pengesanan carta TS $\bar{X}$ dinilai dan dibandingkan dengan carta lain dengan menggunakan metrik purata bilangan cerapan untuk berisyarat (ANOS), ARL dan ANOS jangkaan (EANOS). Apabila carta TS $\bar{X}$ direka bentuk untuk pengesanan anjakan kecil dan besar, perlu dinyatakan bahawa saiz sampel pada peringkat pensampelan pertama, kedua dan ketiga adalah lebih besar bagi pengesanan anjakan kecil. Berbanding dengan carta Shewhart $\bar{X}$ dan pensampelan berganda (DS) $\bar{X}$, carta TS $\bar{X}$ secara puratanya memerlukan bilangan cerapan yang kurang untuk memberikan isyarat berlakunya anjakan, maka, carta ini menggunakan sumber dengan lebih cekap dan menanggung kos pemeriksaan yang lebih rendah. Aplikasi praktikal bagi semua carta yang dicadangkan juga diberikan.

DEVELOPMENT OF ROBUST MEMORY-TYPE CHARTS UNDER REPETITIVE SAMPLING AND TRIPLE SAMPLING CHARTS FOR THE GAMMA PROCESS

ABSTRACT

Production processes in modern industries usually produce products with small variations due to technological advancement. The Shewhart-type charts are insensitive in detecting small process shifts. By developing memory-type and adaptive-type charts, researchers have solved the shortcomings of the Shewhart-type charts in detecting small shifts. Also, due to high sampling costs and destructive testing, quality engineers use individual control charts to monitor the process mean. There are three objectives in this thesis. Firstly, the triple exponentially weighted moving average (TEWMA) scheme and Tukey control chart (TCC) are combined to develop the TEWMA-TCC and repetitive sampling (RS) based RS-TEWMA-TCC, to monitor the mean of normal and non-normal distributed processes. The TEWMA-TCC, RS-TEWMA-TCC and competing charts are compared based on average run length (ARL), standard deviation of the run length (SDRL) and median run length (MRL) metrics under both zero-state (ZS) and steady-state (SS) conditions. The TEWMA-TCC and RS-TEWMA-TCC display dominance in detecting mean shifts in both directions. They are also robust to skewed distributions in that they are devoid of the ARL-biased problem. Secondly, the RS for cumulative sum (CUSUM)-type statistics discussed by Riaz et al. (2017) is coupled with the Shewhart chart to propose the RS Shewhart exponentially weighted moving average CUSUM TCC (RS-SEC-TCC). The supremacy of the RS-SEC-TCC over its competitors has been measured using ARL, SDRL, MRL, extra quadratic loss (EQL), relative ARL (RARL) and performance

comparison index (PCI) metrics. The results indicate that the RS-SEC-TCC excels in identifying small to large mean shifts in both directions for symmetric and skewed distributions. The use of asymmetric decision intervals (ADI) and the TCC ensure that the RS-SEC-TCC is robust to skewed process distributions by eliminating the ARL-biased problem. The TEWMA-TCC, RS-TEWMA-TCC and RS-SEC-TCC are developed using n = 1; hence, they are individual charts and can be used effectively when testing is destructive and sampling cost is high. Thirdly, the triple sampling (TS) $\bar{X}$ chart to monitor shifts in the mean of the gamma distributed process is proposed. The TS $\bar{X}$ chart is constructed with eight design parameters, which provides the practitioner with some flexibility in the chart's construction. The detection ability of the TS $\bar{X}$ chart is evaluated and compared with competing charts via the average number of observations to signal (ANOS), ARL and expected ANOS (EANOS) metrics. When the TS $\bar{X}$ chart is designed for detecting small and large shifts, it is worth noting that the sample sizes at the first, second and third sampling stages are larger for detecting small shifts. Compared to Shewhart $\bar{X}$ and double sampling (DS) $\bar{X}$ charts, the TS $\bar{X}$ chart requires fewer observations on the average to signal a shift, hence, it uses resources more efficiently and incurs lower inspection costs. Practical applications of all the proposed charts are also given.

CHAPTER 1

INTRODUCTION

1.1 Quality and Quality Variation

All businesses in the manufacturing and service industries are aimed squarely toward the customers, who are the ultimate goal because it is they who determine the success or failure of industries. To remain competitive globally, industries must ensure that their goods and services are affordable to consumers of all income levels. As a result, all products and services are purposefully made with varied quality levels, referred to as "quality of design" (Montgomery, 2019). An industry's effort to maintain a high "quality of design" comes at a higher price. Although the definition of quality varies from customer to customer, as different customers have different expectations, every customer wants products that meet their level of satisfaction. The reputation of an industry improves when a customer receives products with uniform quality, that is, having less variation in their intended features. Quality is, thus, defined as being "inversely proportional to variability", among other definitions (Montgomery, 2019). The term "quality improvement" refers to the methods through which industries save time, money, and resources by working to reduce variability in their goods so that fewer repairs, replacements, and warranty claims are necessary (Montgomery, 2019).

The stability in production processes substantially impacts product quality, which is a key aspect in maintaining industrial competitiveness and a point of difference for the customer to choose among the competitive products. Process instability occurs when some attributable reasons, known as assignable causes of variation, are present in the process, which is otherwise stable when only common causes of variation are present. Even when the process is carefully maintained, it is still possible for common or random causes of variation to emerge. A process can continue to function despite the existence of common causes of variation, and the said process is considered as "in-control" (Montgomery, 2019). The transition of the process parameters from an in-control state to an out-of-control state can be attributed to a number of causes, including operator errors, the use of substandard raw materials, improper configuration, and loss of power (Montgomery, 2019). All these assignable causes of variation can be tracked, isolated, and eradicated to bring the process back under control. When a process works only with common causes of variation, i.e., centred to a target level set by the manufacturer, the distribution of the process can be predicted (Alwan, 2000). As long as only common causes of variation are present, industries can count on the process to reliably produce products within predetermined tolerances. Contrary to a stable process, an unstable process leaves the management unpredictable about the future process in that there is no guarantee that the process will produce products according to the specifications (Alwan, 2000). Note that a stable process is a process that operates with only common causes of variation, and this results in a desired situation.

1.2 Statistical Process Control

To achieve the goal of being left with only common causes of variation after identifying and eliminating the special causes of variation in the production processes, statistical tools in Statistical Process Control (SPC), like histogram, check sheet, Pareto chart, cause-and-effect diagram, defect concentration diagram, scatter diagram, and control chart, play a vital role. SPC is a sophisticated set of problem-solving methods that can be used to stabilise a process which produces products of uniform quality and boost its potential to control the variation within certain limits by lowering the level of variability in the process. When used effectively, SPC can pave the way for an organisation-wide commitment to quality and productivity enhancement (Montgomery, 2019). SPC tools have proven beneficial in control and surveillance to maintain process stability. Reduced waste is a direct result of SPC's ability to facilitate the manufacture of consistently high-quality goods with few non-conformities.

Processes continue to be under control for extended periods of time when common causes of variation are present, but the transition to an out-of-control state occurs when special causes of variation are present (Montgomery, 2019). Then, SPC is adopted to monitor the process and figure out when problems arise and what might be causing them so that management can take action to fix the assignable causes and get the process back under control before too many defective products are produced. SPC has two main benefits: (i) rapidly identifying assignable causes for further inquiry and (ii) reducing process variability (Montgomery, 2019). Industries will benefit by implementing SPC tools, for example, earning more because of less scrap, meeting delivery deadlines because of reduced rework, replacement, and warranty claims, gaining a decent reputation among the customer, and cutting expenses throughout the whole manufacturing process. Adopting SPC as a core component of a company's long-term strategy for improving product quality and process stability is essential (Montgomery, 2019).

1.3 Control Charts and its Applications

For maintaining process stability and producing high-quality products, control charts are potent tools in SPC. Control charts are often preferred over other SPC tools because of their ability to quickly detect process deviation, which in turn leads to process stability. Shewhart pioneered constructing control charts in 1924, which are currently being utilised in nearly every process monitoring situation. It was Shewhart who found setting limits upon the natural variation of any process as probable and necessary. Now, the occurrence of process variation is because of only assignable causes of variation if the process observations lie outside of the set limits, i.e., the process will follow a different distribution than the distribution of common variation. If the variation is actually because of common causes, process observations will lie within the set limits (Alwan, 2000).

A control chart is a graphical representation of a critical quality feature as a function of time, with sampling time or sample number often shown along the horizontal axis and data scale along the vertical axis. There are three horizontal lines on a standard control chart, one of which is the centreline (CL), representing the average plotting statistic value when the process is in-control. The other two lines, the lower control limit (LCL) and upper control limit (UCL), are used to sentence the current state of the process; they are called 3σ limits and located 3σ below and above the CL, respectively, so that if the process is in-control or "in statistical control", most points will be contained within these two lines. Therefore, proof of assignable cause(s) will be present if these control limits are exceeded. In this way, control charts can identify when a process deviates from its intended in-control state. Not only do control charts flag points that go outside of control limits, but they also graphically display any discernible trends in the points which appear due to some special cause(s). To determine how a process functions at any given time, a quality or industrial engineer takes a representative sample of products, measures or observes the quality characteristic, and then plots the results (statistics) on the control chart. A control chart thus tests the state of the process, i.e., either in-control or out-of-control, repeatedly at different points in time. While interpreting the control charts, an observer of a control chart should consider the distribution of the process (the population) and not the individual data points because the distribution of the process tells the probable cause of the pattern of the data points (Ishikawa, 1990).

The use of control charts is widespread across all industries because of their ability to decrease variation and improve the quality of products and services to give consumers higher satisfaction. An additional use of the control chart is to estimate process parameters, such as the mean, standard deviation (SD), process fallout, etc., when the process works according to nominal conditions (i.e., in-control) (Montgomery, 2019). The quality or industrial engineer can use these estimates to understand better whether the process is capable of producing marketable goods. Control charts are employed in the surveillance of the stability of past data and check the state of the process in the future. Using control charts in industries can increase productivity since they help ensure consistent product quality, reducing waste and the need for rework (Montgomery, 2019). A control chart isolates the common and assignable causes of variation. If the operator adjusts the machine(s) without getting information from the control chart, the adjustments can be counterproductive since the operator can incorrectly attribute the process variation to special cause(s). A control chart saves the operator from tinkering with the process needlessly (Montgomery, 2019). Hence, control charts are indispensable and necessary as an efficient process monitoring tool.

Control charts help keep tabs on both quantitative and qualitative measures of quality. The measurable quality characteristic-based control charts are called variable control charts, while in cases where the quality feature of interest cannot be measured, attribute control charts are constructed. Continuous data are measured and recorded on a continuous scale of measurement (Montgomery, 2019). Reducing process variation is one of the primary goals of variable control charts, guaranteeing practitioners a steady improvement in product quality. Some quality characteristics with continuous data include the internal diameter (in millimetres) of oil seals, time (in minutes) required to admit a patient, compressive strengths (in pounds per square inch), viscosity (in Pascal second) of a polymer, etc. The nature of these quality characteristics is continuous and, hence, can be measured numerically and monitored by the standard variable control charts, such as $\bar{X} - R$ and $\bar{X} - S$ charts (Montgomery, 2019).

Sometimes, the customer is interested in the number of non-conformities in rolls of cotton cloth (measured in yards), the number of defective light bulbs, the number or fraction of faulty parts in the production of computers, the number of imperfections in rolls of paper etc., which form a discrete count or fraction. Products are classified as imperfect, non-defective, conforming, or non-conforming based on whether or not they meet specific criteria. Such attribute or count data can be monitored with the help of attribute control charts, such as p, np, c and u charts, which help the process of producing non-defective products. The data in variable control charts are presented on a continuous scale, allowing for a better understanding of measurements of central tendency and variability in the process. As a result, variable control charts are generally preferred over attribute control charts (Montgomery, 2019). Despite this, attribute control charts might be useful in service and non-manufacturing sectors when it is not feasible to quantify the quality characteristic.

As stated earlier, control charts are applied to the historical or past data (called retrospective samples) to look for the assignable causes, and if any, they are removed or eliminated to reduce the variability in the process. In the Phase-I process, standard control charts are used to establish trial control limits to see if the process observations are stable throughout the collection period. Once the process has been deemed under control, the trial control limits (now the control limits) can be utilised to continue monitoring future processes. Suppose that a quality or industrial engineer finds any unwanted variation in the historical data, which results in the exceedance of the process observation(s); then, he will look for the root cause and remove the issue(s). In Phase-I, the processes usually have large deviations from their nominal dimensions, and the quality engineer aims to reduce them to an acceptable level. This is the phase when reliable estimates of the process parameters are obtained after getting the clean data, which show the stability of the Phase-I process (Montgomery, 2019). Shewhart-type charts are handy in Phase-I, as they are superior for large shifts.

Quality engineers use the same control limits (calculated and finalised during Phase-I) to track future samples from the same process once they have established process stability in Phase-I. In Phase-II application (prospective stage) of control charts, process monitoring takes precedence over control implementation, i.e., in Phase-II, quality engineers mainly focus on process monitoring and not on regaining control of an out-of-control process. Shewhart-type charts are less effective in Phase-II because this phase typically has small deviations (shifts) in the process parameters compared to Phase-I. The control limits are periodically revised by taking additional samples (process observations) in Phase-I to establish reliable control limits for online process monitoring in Phase-II. A control chart's performance can be evaluated in any of the two conditions, i.e., when either the process parameters shift (i) immediately at the beginning of process monitoring or (ii) at a random time after the process has been under statistical control for some time. Conditions (i) and (ii) are known as the zero-state (ZS) and steady-state (SS) conditions, respectively.

1.4 Problem Statement

Process variations drive practitioners to investigate the vulnerabilities of existing control charts, resulting in the development of new and improved designs. Industries certainly prefer control charts that have quick detection ability of any assignable cause(s). Although widely implemented, the lack of sensitivity in detecting small shifts in the process parameters remains a significant shortcoming of the Shewhart-type charts. The cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts (popularly known as memory charts), proposed by Page (1954) and Roberts (1959), respectively, overcome the problem of the Shewhart-type charts by spotting small and moderate shifts early. Recently, Alevizakos et al. (2021d) constructed the triple exponentially weighted moving average (TEWMA) chart and hypothesised that the TEWMA chart outperforms the EWMA and double EWMA (DEWMA) charts. The TEWMA chart with time-varying control limits has proven to be more effective in detecting small shifts quicker than its competitors. We are driven to benefit from the TEWMA chart because its prominent features, such as its superior inertial properties and its ability to beat EWMA and DEWMA charts, are especially noticeable for small shifts.

In cases where there is insufficient information to make an assumption about the distribution of the quality characteristic being monitored, distribution-free control charts could be the best option for online monitoring to attain a robust in-control performance of the proposed schemes (Celano & Chakraborti, 2021). Thus, distribution-free control charts can be suitable substitutes for parametric charts. Moreover, manufacturers of electronics use the destructive testing approach with only a single observation per time period to monitor the process. To this end, destructive testing procedures use individual control charts to monitor the process mean (Raza et al., 2019; Torng, Lee, & Tseng, 2009). However, a powerful alternative to parametric control charts has recently gained widespread use in monitoring the process mean of a single observation, and it is the Tukey control chart (TCC) developed by Alemi (2004). Using the box plot principle as its foundation, the TCC allows for the monitoring of a process with fewer data points. When the normality assumption of the process is significantly violated, the TCC is not sensitive in detecting shifts (Torng & Lee, 2008). To monitor the skewed process distributions outlined in objectives (i) and (ii) of Section 1.5, we propose TCC-type charts based on individual observations. This is inspired by the remarkable characteristics of TCC, which are robustness in the face of skewed process distributions and utility when the number of observations is limited.

Even though single sampling is the most used decision-making scheme in control charts, its efficacy has been questioned. To improve the process monitoring using control charts, repetitive sampling (RS) schemes were initiated (Adeoti & Olaomi, 2018). Many researchers have utilised the RS approach in enhancing the performance of the control charts. Chen et al. (2022) used the RS to produce a non-parametric generally weighted moving average (GWMA) sign chart and demonstrated its superiority when large design and adjustment parameters are used. Some references for the latest research on control charts using RS include Alevizakos, Chatterjee, Koukouvinos, et al. (2023) and Malela-Majika (2023). Based on the literature, it is clear that the RS procedure outperforms both the single and double sampling (DS) procedures in minimising the average sample size (ASS). These recent publications highlight the importance of the RS procedure in process monitoring and the widespread recognition it has among researchers has actually motivated us to benefit from the RS.

After a thorough literature search, it is found that the utilisation of the RS strategy on TCC designs has yet to be considered in the current literature. Moreover, TEWMA charts have been used to monitor the process mean and variance. However, the memory structure of the TEWMA chart has not been integrated into the TCC, where the latter is resistant to skewed distributions. Furthermore, no research has been undertaken in deriving asymmetric control limits (ACL) for skewed distributions under RS. All the TCCs and their mixture designs that are available in the literature were developed under the ZS condition. It is unclear whether the TCCs designed for the ZS performance will also operate effectively under the SS condition. Therefore, due to the advantages of RS and the salient features of different TCC designs, such as the EWMA-TCC and mixed DEWMA-TCC (MDEWMA-TCC), as well as the outstanding performance of TEWMA charts compared with EWMA and DEWMA charts, we propose two new control charts, i.e., the TEWMA-TCC and RS-TEWMA-TCC for monitoring the process mean under both the ZS and SS conditions. In Section 1.5, this is the first objective of the thesis.

While it is true that Sherman developed the RS technique in 1965 and Page proposed the CUSUM control chart in 1954, we are unable to locate single research where the CUSUM-type charts were designed using the RS technique. Therefore, this thesis pioneers using the RS technique on CUSUM-type statistics discussed by Riaz et al. (2017). In addition, the mixed EWMA-CUSUM TCC (MEC-TCC) has also fared less well than its competitors in detecting large shifts quickly. Hence, an overarching aim of this thesis is to develop a control chart that is robust to non-normality (using the TCC), as well as efficient in detecting small to large shifts (coupling Shewhart chart with the MEC-TCC) in the process mean, while further boosting the proposed scheme's performance with the use of the RS technique (called the RS-SEC-TCC). This is the second objective of the thesis given in Section 1.5.

Shewhart $\bar{X}$ chart is widely employed to identify exceedances in the process mean, even as the sample size gets larger with no corresponding increase in the false alarm rate. However, increase in the sampling and measurement costs are implications of increasing the sample size (Torng, Lee, & Liao, 2009). By developing adaptive-type charts, researchers have solved the shortcomings of the Shewhart-type charts. Adaptive-type charts offer the flexibility of varying the charts' parameters, for example, the sample size and sampling interval. Multiple sampling charts, such as DS and triple sampling (TS), are also adaptive-type charts that utilise two and three sampling stages, respectively, to reach the in-control and out-of-control decisions. In addition to retaining the benefits of the Shewhart-type charts, the DS and TS charts can quickly detect small and moderate shifts in the process parameters using reduced sample sizes (Iziy et al., 2017; Torng, Lee, & Liao, 2009; Tuh et al., 2022). Except for a handful of studies, such as those by Torng et al. (2010) and Torng and Lee (2009), most of the existing DS and TS charts assume the normal underlying distribution. Given the outstanding performance of the TS charts and their superiority over competing charts, we extend the work of Torng and Lee (2009) for the DS $\bar{X}$ chart by proposing the TS $\bar{X}$ chart for the gamma underlying process distribution. This relates to the third objective of the thesis, which is stated in Section 1.5. Furthermore, Torng and Lee (2009) evaluated the DS $\bar{X}$ chart only for a specific shift size. The efficiency and behaviour of the DS $\bar{X}$ and TS $\bar{X}$ charts for the gamma distribution should also be compared across the whole range of shift sizes. Torng and Lee's (2009) DS $\bar{X}$ chart also requires further analysis in finding its optimal parameters and performance measures and this issue is also addressed in this thesis.

1.5 Objectives of the Thesis

This thesis has the following primary objectives:

(i) To propose TEWMA-TCC and RS-TEWMA-TCC utilising single sampling and RS schemes, respectively, for normal and non-normal processes to monitor the process mean under ZS and SS conditions.

(ii) To pioneer the use of RS for CUSUM-type statistics and couple it with the Shewhart chart to propose RS-SEC-TCC by targeting two purposes, i.e., robustness to non-normality and detection of small to large mean shifts quickly.

(iii) To derive formulae for the calculation of control regions' probabilities and statistical properties and develop TS $\bar{X}$ chart for the process mean when the quality characteristic follows a gamma distribution.

1.6 Layout of the Thesis

There are six chapters in this thesis, which are structured as follows: Chapter 1 introduces the thesis's core concepts, including quality, SPC, control charts, the impetus for the thesis, its objectives, and its layout.

Chapter 2 provides a synopsis of the performance measures used in evaluating the performance of the proposed and existing control charts. The literature review and design structures of some non-adaptive charts, such as Shewhart $\bar{X}$ chart, TCC, EWMA-TCC, CUSUM-TCC, MEC-TCC, MDEWMA-TCC, TEWMA chart and adaptive chart such as DS $\bar{X}$ chart are also part of Chapter 2.

The designs of the proposed TEWMA-TCC and RS-TEWMA-TCC are presented in Chapter 3. The optimisation criteria and the algorithm used to find the optimal parameters are also provided. The results, comparison and performance evaluation of the proposed control charts (PCC) and competing charts are given in Chapter 3. To facilitate the PCC's implementation, an example of an application is provided at the end of Chapter 3. A summary of the whole of Chapter 3 is also provided.

Chapter 4 details the design and construction of the RS-SEC-TCC. The asymmetric decision intervals (ADI) for the skewed distributions under RS are presented. The algorithm for optimising the RS-SEC-TCC's parameters is also provided. The application of RS to CUSUM-type statistics is laid out in detail in this chapter. This chapter also examines the performance evaluation of the RS-SEC-TCC compared to the competing charts. In addition, an illustration of how to implement the RS-SEC-TCC in real life is also provided.

Chapter 5 gives the details, structure, control charting procedure, mathematical equations, and probabilities of the charting statistics falling in each of the control charting regions under the gamma distribution for the proposed TS $\bar{X}$ chart. The derivations of the probabilities of the control charting statistic falling into each of the regions of the chart and other statistical properties, along with optimal designs of the proposed TS $\bar{X}$ chart are also provided in Chapter 5. The proposed TS $\bar{X}$ chart is assessed and contrasted with other competing charts, in terms of their performance evaluation. In Chapter 5, we demonstrate how the proposed TS $\bar{X}$ chart is applied in a real-life scenario.

The conclusions, findings and contributions of this thesis are summed up in Chapter 6. This chapter also includes practical contributions, potential real-life applications, recommendations for further research on relevant areas and limitations of the thesis.

The design parameters of all the charts that are proposed in Chapters 3 – 5 of this thesis and the performance metrics that are used to evaluate the ability of the proposed and the competing charts have been determined and computed using R language software, matrix laboratory (MATLAB) software and statistical analysis system (SAS) software. Appendices A, B and C present all the programs that are used to compute all the results and performance metrics.

CHAPTER 2

A REVIEW ON PERFORMANCE METRICS, EXISTING CONTROL CHARTS AND DESIGNS OF RELATED CHARTS

2.1 Introduction

A thorough literature search has been conducted to get insight into the prior research, the current gap in the research, and the prospective future directions for exploration. The performance metrics, algorithms and optimisation criteria used in finding the optimal design parameters of the charts, evaluation and comparison of the performances of the existing and the proposed charts are determined after investigating existing research that is related to the objectives in Section 1.5 of this thesis. An investigation of related research lays the groundwork for the PCC developed in Chapters 3 – 5. The literature has facilitated discussions, comparisons and interpretations of the results, which helps in the development of the PCC in Chapters 3 – 5. To sum up, all the referenced works are crucial in the completion of this thesis.

Since its development in the 1920s by W. A. Shewhart of the Bell Telephone Laboratories, statistical control charts have been irreplaceable for keeping tabs on production and service quality. It wasn't until 1931 that Shewhart presented control charts, the first data-driven approach to monitoring. In reality, Shewhart's approach towards the process control techniques constituted the foundations of the control charts that quality practitioners from all over the world have utilised up till now. Even now, Shewhart's charts are the primary and foremost choice for quality practitioners to keep monitoring their processes. Among all of Shewhart's charts, $\bar{X}$ chart is the most widely used to monitor the process mean, but it has the shortcoming of being unable to detect small shifts early. There has been significant progress in the field of control charts, leading to the development of many different kinds of control charts that are more effective than the Shewhart $\bar{X}$ chart.

In order to assess and compare the effectiveness of the PCC with competing charts, Section 2.2 of this chapter covers the individual (or specific) and overall shift size-based performance metrics that are employed in the thesis. Following is the structure for the rest of Chapter 2: The design structure of the Shewhart $\bar{X}$ chart is given in Section 2.3. Section 2.4 is divided into multiple subsections to give the literature and design structures of the TCC and its mixture charts, such as CUSUM-TCC, EWMA-TCC, MEC-TCC and DEWMA-TCC. Section 2.5 presents the literature on memory-type charts. The design structure of the TEWMA chart and its related literature are provided in Sections 2.5.2 and 2.5.1(b), respectively, of Section 2.5. The literature on some RS-based charts is provided in Section 2.6. The DS $\bar{X}$ chart and some associated literature on adaptive-type charts are presented in Section 2.7. Sections 2.4 – 2.6 cover the literature and design structures of existing charts related to objectives (i) and (ii), while Sections 2.3 and 2.7 include the existing charts and related literature that support objective (iii) of this thesis. Section 2.8 recaps the chapter with a summary. Section 2.8 also presents a summary table for the strengths and shortcomings of the competing charts.

2.2 Performance Metrics

A chart's effectiveness is generally measured by its ability to detect a shift or range of shifts more quickly than competing charts. To calculate this detecting power, practitioners use a variety of statistical performance metrics/indices/measures/indicators. Various performance metrics are employed in the control chart literature to assess the effectiveness of the charts. The performance metrics taken into account when analysing the performance of the proposed and competing charts in this thesis are the average run length (ARL), median run length (MRL), SD of the run length (SDRL), extra quadratic loss (EQL), relative ARL (RARL), performance comparison index (PCI), average number of observations to signal (ANOS) and expected ANOS (EANOS). The effectiveness of the proposed and existing charts in identifying shifts in the process mean is compared by employing ARL, MRL, SDRL, EQL, RARL and PCI that have been adopted by numerous researchers for the Tukey-type, memory-type and RS-based charts (Alevizakos, Chatterjee, Koukouvinos, et al., 2023; Chen et al., 2022; Khaliq et al., 2016; Malela-Majika, 2023; Riaz et al., 2017). ANOS and EANOS are the recommended performance metrics for adaptive-type charts such as DS $\bar{X}$ and TS $\bar{X}$ charts (Khoo et al., 2013; Mim et al., 2022; Saha et al., 2018).

2.2.1 Specific Shift Size-Based Performance Metrics

This section is divided into multiple subsections to discuss the specific shift size-based performance metrics such as ARL, MRL, SDRL and ANOS, established on the run length (RL) distribution characteristics, except for the ANOS.

2.2.1(a) Average Run Length

The speed at which a control chart detects a process shift determines the efficiency of the chart. Many researchers utilise the ARL as an individual performance criterion since it accurately evaluates a chart's detection speed. ARL gives the expected value of the RLs, where a RL is defined as the count of samples from the start of the process to the time when the chart first alerts an out-of-control signal (Montgomery, 2019). A tolerable false alarm rate is the basis of setting the in-control ARL (ARL(0)) value. Besides, interest lies in having a small out-of-control ARL value, denoted as ARL(δ), so that a specific shift can be detected as early as possible before the condition of a process deteriorates (Alevizakos et al., 2022b; Malela-Majika, 2023). When two charts have an equal false alarm rate, the chart with a smaller ARL(δ) value triggers the specific shift earlier and is considered as more efficient than the other chart. Here, δ refers to the standardised shift size in the process mean in multiples of SD.

2.2.1(b) Median Run Length

MRL is defined as the median of the RLs, i.e., it represents a certain RL value below which 50% of all the RLs lie. Researchers have criticised the sole use of ARL due to the skewed nature of the in-control RL distribution and proposed using the MRL criterion in evaluating a chart's performance (Khoo, 2004; Qiao et al., 2022; Tang et al., 2019). As the form of the RL distribution varies in response to the size of the shift, interpretations based on ARL become increasingly complex (Gan, 1994). However, this problem is eliminated when the MRL is utilised because it is less affected by the skewness of the RL distribution. This can lead to a more straightforward interpretation and a core central tendency measure that surpasses the ARL. Chin and Khoo (2012) have elaborated very clearly on the benefit of using MRL: "When using a $\pm$3 SD wide limits on Shewhart $\bar{X}$ chart, the ARL(0) is 370, although, in reality, the majority (60 – 70%) of the RLs will be below 370. There will be half (50%) as many RLs below 257 that represent the in-control MRL (MRL(0)) value. Practitioners using the ARL as a performance indicator can incorrectly conclude that an out-of-control signal will be generated by the 370th sample 50% of the time, even if the process is stable. When applied to a real-world scenario, however, the 370th sample predicts an out-of-control result 60 – 70% of the time. Actually, the MRL(0) (= 257) shows that an out-of-control signal is generated by the 257th sample in 50% of cases." With MRL, we can calculate the likelihood of a signal once a specific number of samples have been taken. On the other hand, the ARL just gives the average number of samples to signal, which is not a probabilistic metric at all.

2.2.1(c) Standard Deviation of the Run Length

A control chart triggers multiple out-of-control signals, which are represented through its RLs. The deviation of all the RLs from the ARL, i.e., the spread in the RL distribution, can be calculated with the help of SDRL. If SDRL is large, then the RL distribution is highly spread out; if SDRL is small, then the RL distribution is less spread out (Khaliq et al., 2016; Riaz et al., 2017). When all the charts are set with the same ARL(0) value, the chart with the lowest SDRL is preferred over the other chart(s) because the performance of the preferred chart is more consistent in detecting a specific shift (Khaliq et al., 2016; Riaz et al., 2017). For example, charts A and B both trigger five out-of-control signals when there is a shift of size δ = 1.00 in the mean of the process with the RLs as 10, 139, 469, 201 and 181 for chart A and 200, 196, 209, 188 and 207 for chart B (setting ARL(0) = 370.40 for both the charts). A greater variability in chart A's RL distribution is abundantly apparent by the fact that a shift of size δ = 1.00 is triggered for the first time at the 10^{th} sample and for the third time at the 469^{th} sample. Consequently, this allows the process to produce many non-conforming products because chart A's performance is inconsistent in the detection of a shift. On the other hand, chart B has a relatively less spread-out RL distribution; hence, chart B is preferred over chart A.

2.2.1(d) Average Number of Observations to Signal

ANOS is the average count of all observations recorded from the onset of the process shift to the detection of an out-of-control signal on the chart (Mim et al., 2022). In non-adaptive charts with a fixed sample size (FSS), multiplying the ARL and FSS values together gives the ANOS. The total number of observations required to signal an out-of-control situation by adaptive charts is not a constant multiple of the ARL since the number of observations adopted at each sampling time is allowed to fluctuate with the quality of the process (Mim et al., 2022). In this case, it is recommended to use the ANOS criterion for adaptive charts rather than to rely entirely on the ARL criterion, which only measures the average number of samples required to detect a particular shift and ignores the change in the sample size. For practical purposes, practitioners prefer a lower value of our-of-control ANOS (ANOS(δ)), which means that the chart initiates a shift early (i.e., after gathering information from a small number of observations). On the other hand, ANOS(0) represents an in-control ANOS value and is anticipated to be large enough to circumvent false alarms. A chart with a smaller ANOS(δ) value generally incurs a lower sampling cost (Lin & Chou, 2007; Motsepa et al., 2022).

2.2.2 Overall Shift Size-Based Performance Metrics

The sole use of ARL, MRL, SDRL and ANOS in evaluating the performance of a control chart is not adequate as the actual shift size differs significantly from the specified shift size and when a practitioner wants to assess the efficacy of a control chart over an extensive range of shifts. This highlights the need for a control chart to be designed to have better overall performance (Wu et al., 2008). Some of the overall performance metrics used in this thesis are discussed below in the subsequent sections.

2.2.2(a) Extra Quadratic Loss

When comparing the efficacy of various control charts, the ARL does not depict the comprehensive situation even when a special cause endures until it is spotted. The loss function, according to Spiring and Yeung (1998), is frequently employed in production to quantify the cost of poor quality. Since EQL takes into account all of the factors that affect the quality cost, such as the time to signal and the magnitude of δ, EQL, based on loss function, is able to provide an exhaustive assessment of the complete performance of the charting scheme. EQL is a metric which measures and compares the performance of the charts over an extensive range of shifts (Reynolds &

Stoumbos, 2004; Wu et al., 2008). EQL can be calculated by using the formula in Equation (2.11) with the help of numerical integration method. When EQL is minimised, the loss in quality (as well as the associated cost and damage) related to out-of-control scenarios is mitigated (Wu et al., 2008). The EQL can be used to compare the impact of various shifts in addition to contrasting the performance of different charts. For instance, it offers a way to determine if a large shift that is spotted quickly is more or less expensive than a small shift that takes a while to trigger (Reynolds & Stoumbos, 2004). The best chart, given by the chart with the lowest EQL among all the competing charts, is used as the benchmark chart. The competing charts then offer EQL > 1.

2.2.2(b) Relative Average Run Length

An alternative metric to show how well a chart performs over an entire range of shift sizes is the RARL. RARL is based on the ratio of ARL(δ) of the competing and benchmark charts. RARL is, therefore, a relative measure that evaluates the overall effectiveness of the competing charts, in terms of its ARL(δ) performance relative to the performance of the benchmark chart. RARL examines the degree to which a chart performs close to a benchmark chart (Khaliq et al., 2016; Riaz et al., 2017). The rest of the charts (other than the best chart, i.e., the chart with lowest EQL) will have RARL $>$ 1, as they produce larger ARL(δ)s than the best chart over the entire range of shifts. For a given interval of shifts ($\delta_{min} \leq \delta \leq \delta_{max}$), a RARL < 1 implies that the competing chart is superior to the benchmark chart, while a RARL = 1 indicates that the two charts are equivalent in performance (Khaliq et al., 2016; Riaz et al., 2017). Here δ_{min} and δ_{max} denote the predefined lower and upper bounds, respectively, of the shift interval for the standardised mean shift.

2.2.2(c) Performance Comparison Index

Many researchers advocated for the use of PCI as an additional metric, alongside EQL and RARL, to assess the relative efficacy of the two charting schemes under the same conditions (Malela-Majika et al., 2021; Motsepa et al., 2022; Wu et al., 2008). PCI is the ratio of the EQL of the competing chart(s) to the EQL of the benchmark chart; hence, it examines the degree to which a chart performs close to a benchmark chart over the range of shifts. As it is based on EQL, in some way or the other, it compares the loss in quality of the competing chart(s) to the benchmark chart incurred in the out-of-control cases. While the rest of the charts have PCIs greater than 1, the best chart has PCI = 1 (Motsepa et al., 2022; Wu et al., 2008).

2.2.2(d) Expected Average Number of Observations to Signal

For the adaptive charts, practitioners prefer EANOS over ANOS because ANOS is designed to detect a specific shift, which is often not the interest of practitioners as they are interested in such a metric, which gives an early alert for a range of shifts. Secondly, even if the practitioners monitor their production processes to detect a specific shift with a chart based on ANOS criteria, the chart performs poorly if the actual shift size differs from the specified shift size. The quality engineers may be unnecessarily re-adjusting the production process, having alarm(s) of a specific shift size that does not persist in that magnitude. This will waste time, delay the shipment to the market and incur extra monitoring cost. The performance of a chart is therefore measured across a range of shift sizes using the EANOS. It is common practice to set the value of the in-control EANOS (EANOS(0)) to be comparable to the ANOS(0) (Mim et al., 2022; Saha et al., 2018). For a given shift interval ($\delta_{\min}$, $\delta_{\max}$), the expected average number of observations needed by a chart to issue a signal is calculated using EANOS($\delta_{\min}$, $\delta_{\max}$). The shift sizes $\delta_{\min}$ and $\delta_{\max}$ are defined in Section 2.2.2(b).

2.3 Shewhart $\bar{X}$ Chart

Let X be a quantitative feature of a process whose properties are described by a specific distribution with mean μ_0 and SD σ. In a univariate setup, let $X_1, X_2, X_3, \ldots, X_{n_1}$ be a sequence of independent and identically distributed observations measured across time. Then, the sample mean $\bar{X}$ is calculated as

$$\bar{X} = \frac{\sum_{i=1}^{n_1} X_i}{n_1}, \tag{2.1}$$

At each sampling time, we have a FSS of n_1 observations from the process. The Shewhart $\bar{X}$ chart uses the sample mean, $\bar{X}$, which is computed at various sampling times as the plotting statistic. The LCL, CL and UCL of the Shewhart $\bar{X}$ chart are given below (Montgomery, 2019).

$$LCL = \mu_0 - L_1\sigma/\sqrt{n_1}\,, \tag{2.2a}$$

$$CL = \mu_0 \tag{2.2b}$$

and

$$UCL = \mu_0 + L_1\sigma/\sqrt{n_1}\,, \tag{2.2c}$$

where L_1 signifies the control limit coefficient (CLC) that can be chosen to attain a pre-specified ARL(0) value. The Shewhart $\bar{X}$ chart triggers an out-of-control signal in the process if $\bar{X} < LCL$ or $\bar{X} > UCL$. ARL is the most common measure to evaluate the performance of the Shewhart $\bar{X}$ chart, which is, for the in-control process, defined as

$$\text{ARL(0)} = 1/\alpha, \tag{2.3a}$$

where

$$\alpha = 1 - P(LCL \leq \bar{X} \leq UCL | \mu = \mu_0). \quad (2.3b)$$

The probability of a false alarm on the Shewhart $\bar{X}$ chart is denoted by α. In cases when an assignable cause occurs, the process shifts to an out-of-control mean $\mu_1 = \mu_0 + \delta\sigma$, and the probability that the Shewhart $\bar{X}$ chart fails to detect the shift (δ) is represented by β. Then ARL(δ) given in Equation (2.4a) measures the out-of-control detection speed of the Shewhart $\bar{X}$ chart.

$$\text{ARL}(\delta) = 1/(1 - \beta), \quad (2.4a)$$

where

$$\beta = P(LCL \leq \bar{X} \leq UCL | \mu_1 = \mu_0 + \delta\sigma). \quad (2.4b)$$

Let X follows a gamma distribution with probability density function (pdf) given in Equation (2.5), where a and b are two parameters that determine the shape and scale of random variable X, respectively.

$$f(x) = \frac{x^{a-1} e^{-\frac{x}{b}}}{b^a \Gamma(a)}, \qquad x > 0, \ a, b > 0, \quad (2.5)$$

The gamma distribution has the mean and variance equal to ab and ab^2, respectively. The sample mean, $\bar{X}$, for a sample of size n will also have a gamma distribution with shape and scale parameters na and b/n, respectively, if the underlying distribution of X is gamma with the shape and scale parameters a and b, respectively (Lin & Chou, 2007; Torng & Lee, 2009). Equations (5.8a) – (5.8c) in Section 5.3.2 can be consulted for the proof that $\bar{X}$ follows the gamma distribution. Let $\bar{X}$ be the sample mean at a certain sampling time. To determine α and β probabilities in Equations (2.3b) and (2.4b), the following probability is computed: