MODELING OF CURVES AND SURFACES USING GHT-BERNSTEIN BASIS FUNCTIONS AND USING OPTIMIZATION METHODS TO CONSTRUCT DEVELOPABLE SURFACES

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by

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All praises to ***Almighty Allah***, the creator and sustainer of the universe, Who is the supreme authority, knowing the ultimate realities of universe and source of all knowledge and wisdom. Without His will nothing could be happened. It has been deemed a great favor of ***Allah*** that, I was bestowed upon the vision, initiative, potential and hope to complete my thesis successfully. All regards to the Holy Prophet ***Hazrat Muhammad (SAW)***, Who enable me to recognize my creator and His creations to understand the philosophy of life. Holy Prophet said that, I am the light, whoever follows Me, will never be in the darkness.

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LIST OF ABBREVIATIONS

CAGD	Computer Aided Geometric Design
CAD	Computer Aided Design
CAM	Computer Aided Manufacturing
GHT	Generalized Hybrid Trigonometric
PSO	Particle Swarm Optimization
I-GWO	Improved-Grey Wolf Optimization
USM	Universiti Sains Malaysia
CG	Computer Graphics
CNC	Computer Numerical Control
PGR	Point Geometric Representation
LPGR	Line and Plane Geometric Representation

LIST OF SYMBOLS

$\times$	cross product
$\cdot$	dot product
$\surd$	square root
$\vec{a}$	vector a
$\parallel$	parallel
θ	angle in degree
z	variable
μ	shape parameter
ν	shape parameter
γ	shape parameter
$x(z)$	coordinate axis
$y(z)$	coordinate axis
m	free parameter
k	free parameter
w_k	weight function
h_k	constant functions
$b_{k,m}(z)$	Classical Bernstein basis functions
$B_{k,m}(z)$	Classical Bézier curve
Q^*	control point

P^*	control point
C^0	position parametric continuity
C^1	tangent parametric continuity
C^2	curvature parametric continuity
C^3	rate of change of curvature parametric continuity
G^0	position geometric continuity
G^1	tangent geometric continuity
G^2	curvature geometric continuity
G^3	rate of change of curvature geometric continuity
ϕ	phi
ξ	xi
η	eta
ζ	zeta
ψ	psi
λ	lambda
π	pi
$*$	times
$\kappa(z)$	Curvature
$\kappa'(z)$	Derivative of curvature
ϑ	numerator of curvature derivative
φ	constant parameter

$S(z)$	Parametric curve
$\ell(z)$	directrix of curve
$h(z)$	directrix of curve
$g(z)$	generatrix of curve
$n_\ell(z)$	unit normal vector
$n_h(z)$	unit normal vector
p_{best}	best location where particle has gone through
g_{best}	best location where entire swarm has gone through
$R(\psi)$	Fitness function
d	Number of dimensions
χ_1	Lower bound
χ_2	Upper bound
c_1	Cognitive coefficient
c_2	Social coefficient
v_i	velocity of the PSO particle
ψ_i	position of the PSO particle
$Maxiter$	Maximum number of iterations
$\hat{AL}$	Arc Length
En	Energy
$C\hat{V}En$	Curvature Variation Energy

PEMODELAN LENGKUNG DAN PERMUKAAN MENGGUNAKAN FUNGSI ASAS GHT-BERNSTEIN DAN MENGGUNAKAN KAEDAH PENGOPTIMUMAN UNTUK MEMBINA PERMUKAAN YANG BOLEH DIBANGUNKAN

ABSTRAK

Model Bézier dengan parameter bentuk ialah salah satu topik penyelidikan yang berpengaruh dalam pemodelan geometri dan CAGD. Kerja ini menerangkan pembinaan lengkung trigonometri hibrid umum Bézier menggunakan fungsi asas trigonometri hibrid umum Bernstein dengan tiga parameter bentuk dan aplikasinya dalam pemodelan geometri. Formula rekursif dalam ungkapan eksplisit digunakan untuk menyamaratakan fungsi asas trigonometri hibrid Bernstein darjah 2, dan fungsi asas trigonometri hibrid mempunyai semua sifat geometri fungsi asas bernstein tradisional. Satu kelas permukaan boleh dibangunkan GHT-Bézier dibina dengan menggunakan prinsip dualiti antara satah dan titik. Untuk meningkatkan kecekapan produk kejuruteraan yang kompleks, permukaan boleh dibangunkan dengan tahap kebolehbangunan yang lebih tinggi perlu diperolehi. Teknik pengoptimuman yang diilhamkan oleh Bio, dinamakan sebagai teknik Pengoptimuman Particle Swarm dan teknik digunakan untuk mencari parameter bentuk optimum untuk menentukan tahap kebolehbangunan. Tahap kebolehbangunan permukaan dianggap sebagai fungsi objektif dalam teknik pengoptimuman. Contoh pemodelan menunjukkan keberkesanan kaedah yang dicadangkan dengan kesaksamaan permukaan. Perbandingan antara tahap kebolehbangunan yang diperolehi oleh algoritma PSO dan I-GWO diberikan.

MODELING OF CURVES AND SURFACES USING GHT-BERNSTEIN BASIS FUNCTIONS AND USING OPTIMIZATION METHODS TO CONSTRUCT DEVELOPABLE SURFACES

ABSTRACT

A Bézier model with shape parameters is an influential research topic in geometric modeling and CAGD. This thesis describes the construction of generalized hybrid trigonometric Bézier (GHT-Bézier) curves using generalized hybrid trigonometric Bernstein (GHT-Bernstein) basis functions with three shape parameters and their applications in geometric modeling. The recursive formula in explicit expression is used to generalize the hybrid trigonometric Bernstein basis functions of degree 2, and the new generalized hybrid trigonometric Bernstein basis functions contain all the geometric properties of traditional Bernstein basis functions. A class of GHT-Bézier developable surfaces is constructed by using the principle of duality between the planes and points. To improve the efficiency of complex engineering products, a developable surface with higher developability degree is necessary to be obtained. The optimization techniques named as Particle Swarm Optimization (PSO) technique and Improved-Grey Wolf (I-GWO) technique are used to find the optimal shape parameters for determining developability degree. The developability degree of the surface is the objective function in optimization techniques. The modeling examples demonstrate the effectiveness of the proposed method with fairness of the surfaces. The developability degree obtained by PSO and I-GWO algorithm is given.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Parametric representation of curves and surfaces are widely used in Computer Aided Geometric Design (CAGD) and Computer Graphics (CG) to model various surfaces interactively as described in Han et al. (2008) and a brief survey of the fundamentals of CAGD is given by Schmitt et al. (1986). The preliminary use of CAGD was to express the data as a smooth surface for numerical control which quickly became noticeable that the surfaces could be used for designing purposes. In this era of computer technology, the users demand for a simpler method, for the construction of complex geometric modeling and shape editing which also consumes less computational time.

For the first time, James Ferguson constructed the surface patch system in 1963 and described the concept of parametric surfaces which has become the merit because it provides independence from an arbitrarily fixed coordinate system. Pierre Bézier redefined the Ferguson idea in 1971, thus a draftsman could design a curve/surface without any additional mathematical training. Bézier scheme was used by Renault and became a turning point in the progress of CAGD that epitomized the difference between the surface fitting and surface design given in Farin (2002). The purpose of the design was to provide a draftsman such a computer tool that empowered him/her to use appropriate mathematics of surface/curve representation with a strong intuition about the shape but limited mathematical practice.

Parametric curves/surfaces are not only a research initiative in CAGD but they are also robust tools for shape designing presented in Hu et al. (2017a). Classical Bézier curves are parametric curves defined by using the basis of Bernstein polynomial and control points, which are widely used in CAGD due to their good properties and have become one of the powerful technique for representing free form curves/surfaces. When Bézier curves/surfaces are designed, they usually need to be modified in terms of shape to satisfy our design requirement. Creating more suitable techniques for designing and modifying Bézier curves/surfaces is an important issue as well as a dominant research area in Computer Aided Design (CAD), Computer Aided Manufacturing (CAM), and Computer Graphics (CG) technology.

A Bézier curve is defined by the set of control points P_0 through P_n. It is used a lot in computer graphics, often to produce smooth curves, and yet they are a very simple tool (Farin, 2002). The Bernstein basis functions are used to construct a class of Bézier curves and have many applications in the area of CAGD and CAD. The developable Bézier surfaces are constructed by using optimization techniques which are used in designing of mechanical engineering products.

Rational Bézier curve, B-Spline curve and traditional Bézier curve are frequently used for designing in engineering. The researchers have studied both rational (Bashir et al., 2012), and non-rational forms of Bézier curves (Dejdumrong, 2008), and both of them have some deficiencies. To avoid these deficiencies, trigonometric splines with shape parameters have achieved much attention in CAGD, particularly in curve modeling. Various applications of these vigorous curves in object recognition, fingerprints recognition, modeling objects, CAD/CAM, medical imaging and font designing, are the inspirations in the area of curve designing.

Due to better piecewise representation and shape control on the curve, the B-Spline is more suitable choice for shape designing as compared to Bézier curve. Although, with all these positive characteristics of B-Spline curves, it still have some deficiencies like B-Spline is more complex than Bézier curve (Parekh et al., 2006). Thus for practical applications the Bézier curve is more preferable. Besides these applications, the major inspiration to research this area is the importance of curve/surface designing in robotics, computer visualization, and broadcasting.

Traditional Bézier curves, which are made up of polynomial Bernstein basis functions, cannot be used to create complex designs since doing so takes a lot of time. In order to overcome this cumbersome problem many researchers have constructed various trigonometric Bézier curves with different shape parameters, because trigonometric Bézier curves have great smoothness as compared to polynomial Bézier curves. Uzma et al. (2012) proposed quadratic trigonometric Bézier curves having two different shape parameters together with their geometric properties. The trigonometric Bézier curve of higher degrees with shape parameters are also proposed in Misro et al. (2017c), Dube and Sharma (2013), and Sharma (2016c) with their geometric properties, which are very useful for shape designing and modeling. For the construction of complex shapes and curves, the continuity conditions are necessary to be derived between two adjacent trigonometric Bézier curves of same/different degrees. The continuity conditions between two generalized trigonometric Bézier curves are derived by Maqsood et al. (2020). So, the complex curve modeling can be constructed by using these generalized trigonometric Bézier curves.

In previous research, XiKun (2004), and Yan and Liang (2011) proposed polynomial Bernstein basis functions for the construction of polynomial Bézier curve and various curve modeling with the continuity conditions. By using the recursive approach, some new kind of Bernstein basis functions with adjustable shape parameters are proposed. The shape parameters are used to modify the curves inside the control polygon. The continuity conditions between two adjacent Bézier curves can be derived which are used for the construction of complex curve/surface modeling (Hu et al., 2018a). In this work, a class of GHT-Bernstein basis functions with three adjustable shape parameters is proposed by using the recursive approach. The geometric properties of Bernstein basis functions with graphical representation are given.

A class of GHT-Bézier curves with their properties, and parametric and geometric continuity conditions is given. The duality principle between the points and planes is described which is useful for the construction of a class of GHT-Bézier developable surfaces such as Enveloping GHT-Bézier developable surface and Spine GHT-Bézier developable surface. The conditions of G^3 geometric continuity between two adjacent GHT-Bézier developable surfaces are also described. The developable surfaces with higher developability degree are useful for engineering construction. To find the developability degree of GHT-Bézier developable surfaces, the optimal shape parameters need to be determined. For this, the bio-inspired optimization techniques by Mirjalili et al. (2014) and Bonabeau et al. (1999) are used with the specified fitness functions. The developabilty degree is obtained by using two different optimization techniques. In case of PSO, the single objective function is used for determining optimal shape parameters for developability degree while in case of I-GWO technique the multi-objective functions are used to find optimal shape parameters for the developability degree.

1.2 Motivation

Parametric curves are powerful tools which can represent most geometric entities when properly define. As a valuable tool in CAD/CAM system, Bézier method is one of the parametric representation which is used to generate curves that appear reasonably smooth at all scales, and has received wide acceptance. Bézier curve has been commonly used in CAGD for the construction of various free from curves and surfaces. Several methods like curve manipulation, curve fitting, curve merging and blending have been proposed over the years for better handling and enhancing the uses of Bézier curves with every possible application in CAD domain.

As the researchers deal with both rational/non-rational forms of Bézier curves, the non-rational form never yields complex results while solving (Han et al., 2009b). For this reason, as opposed to the rational Bézier curves, the non-rational Bézier curves are more suitable choice for curve modeling and other designing applications. Scholars constructed various polynomial and trigonometric Bézier curves possess all the characteristics of rational Bézier curve except having weight factors and instead of weight factors these curves carry the shape parameters which is appropriate choice for modification of shape of the curve and has less computational cost. The major inspiration to research this area is the importance of curve designing, which are used in various CAD/CAM fields.

Our proposed GHT-Bézier curves are based on the novel idea that they can control the shape of the curves using shape parameters without changing control polygon.

Inspired from Hu et al. (2017a) and Han et al. (2009a), the GHT-Bézier curve and GHT-Bézier developable surfaces are constructed and some design examples of the surfaces are provided to show the significance of the suggested scheme. From the optimization techniques mentioned in Mirjalili et al. (2014) and Bonabeau et al. (1999), a study about the construction of GHT-Bézier developable surfaces with developability degree is presented by using optimization algorithms. After comparison with previous defined schemes, our suggested scheme of constructing GHT-Bézier curves and GHT-Bézier developable surfaces with developability degree have a more simplified form and clearer geometric meanings.

1.3 Objectives

The dominant purpose of this study is the development of the GHT-Bézier curves with three different shape parameters and a feasible way for the construction of GHT-Bézier developable surfaces. This study also provides a more convenient way for designing of GHT-Bézier developable surfaces with higher developability degree, which can be obtained by the optimal choice of shape parameters determined by the optimization techniques. The objectives and expectations from the proposed research work are:

1. To propose the GHT-Bernstein basis functions of degree m and a family of GHT-Bézier curves with three adjustable shape parameters along with their properties and graphical representations.

2. To derive the parametric (C^0, C^1, C^2, C^3) continuity and geometric continuity (G^0, G^1, G^2, G^3) conditions between two adjacent GHT-Bézier curves with modeling examples.

3. To construct a class of GHT-Bézier developable surfaces by using the duality principle between the points and planes in $3D$ projective space with the geometric continuity $(G^m, m \leq 3)$ constraints between two adjacent GHT-Bézier developable surfaces.

4. To determine the developability degree of GHT-Bézier developable surfaces by using the optimal shape parameters obtained from the two optimization algorithms which are Particle Swarm Optimization (PSO) Algorithm and Improved-Grey Wolf Optimization (I-GWO) Algorithm.

1.4 Problem Statements

Classical Bernstein basis functions have some distinguish properties as compared to the other parametric functions. Various shapes of Bézier curves cannot be obtained by using traditional Bernstein basis functions because they do not possess any shape parameters to modify the curve. To overcome these shortcomings, the GHT-Bernstein basis functions are used to construct GHT-Bézier curves. The GHT-Bernstein basis functions cannot execute any optimal shape parameters from domain $[-1, 1]$ without any optimization technique. So, the Bio-inspired optimization techniques need to be studied, to determine the optimal shape parameters which are used for designing of various surfaces. By using GHT-Bernstein basis functions, two bio-inspired optimization techniques Particle Swarm Optimization (PSO) and Improved-Grey Wolf Optimization (I-GWO) techniques are described to execute the optimal values of shape parameters.

1.5 Scope and Limitations

This research is focused on the modeling of curves in two dimensional space and construction of developable surfaces in $3D$ projective space which are widely used for engineering and CAD/CAM field. It is also focused on the optimization techniques, which are used in this work for determining the optimal shape parameters to construct developable surfaces with higher developability degree. For the purpose of this research and to answer research objectives, the applications of curve are used in engineering designing, sketching and modeling and the developable surfaces are used for the construction of auto mobile bodies, ship hulls and air craft wings. Moreover, the higher developability degree is used for fairness of the surfaces which are helpful during manufacturing of products.

1.6 Outline of Thesis

This thesis is devoted to new techniques for the construction of GHT-Bézier curves and surfaces with three adjustable shape parameters. The GHT-Bézier developable surfaces are also constructed with highly accurate developability degree by using the optimization techniques. The description of each chapter included in this thesis are as follows:

Chapter 2 provides a detailed literature review about Bézier curves/surfaces, rational Bézier curves/surfaces, developable Bézier surfaces and the continuity between these curves/surfaces. The study review about various optimization algorithms such as Genetic Algorithm, Cuckoo Search Algorithm, Particle Swarm Optimization Algorithm and Grey Wolf Optimization Algorithm is also given.

An overview of parametric curves and optimization algorithms, some related definitions and important terminologies are given in **Chapter 3** to understand the proposed research work properly. Moreover, some previous work done in this area, and the issues that will address in the proposed research are also presented in this chapter.

Chapter 4 deals with the construction of GHT-Bernstein basis function and GHT-Bézier curve along their properties and graphical representations. The parametric continuity (C^0, C^1, C^2, C^3) and geometric continuity (G^0, G^1, G^2, G^3) conditions between two adjacent GHT-Bézier curves are derived. Several designing examples of continuous connections are shown to verify the smoothness. The comparison of curvature continuity between GHT-Bézier curve and traditional Bézier curve is also given.

In **Chapter 5**, the duality principle is defined between the points and planes, and a class of GHT-Bézier developable surfaces with adjustable shape parameters is constructed by using the duality principle. The smooth G^3 continuity conditions are derived between two adjacent GHT-Bézier developable surfaces. Various modeling examples are given to verify the smoothness of continuity between GHT-Bézier developable surfaces.

Chapter 6 deals with the developability degree of GHT-Bézier developable surfaces. In this chapter, the optimization technique named as Particle Swarm Optimization (PSO, in short) technique is used, which determine the optimal values of shape control parameters from fitness functions and these optimal parameters are used to construct GHT-Bézier developable surfaces, and to determine developability degree. Few examples of GHT-Bézier developable surfaces with higher developability degree are also given.

In **Chapter 7**, to get more higher developability degree (with less number of iterations), another optimization technique named as Improved Grey Wolf Optimization (I-GWO, in short) is used. Three optimization models based on the minimum values of fitness functions, (i.e arc length, energy and curvature variation energy of the dual curve) are used to determine the optimal values of shape control parameters with fewer iterations. Some examples of GHT-Bézier developable surfaces with higher developability degree are also provided.

Finally, the conclusion followed by some suggestions for future research is provided in **Chapter 8**.

CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

In the early times of manufacturing, products were designed in industry by the manufacturer's viewpoint. The expanding demand for comfort, functionality and aesthetics in products, have forced the designers to handle complex shapes leading to the invention of new schemes to represent complex curves/surfaces. Developments in computer hardware/software technology gradually made it possible to automatically generate these free form curves/surfaces in a digital format with the help of mathematical representations.

The past few decades have witnessed many novel mathematical representations of free form curves/surfaces with the need to make them more computers compatible. Parametric representations of curves/surfaces have availed central importance of being bounded in parameter range, having simple and short programming, which proved it to be the most suitable shape description method.

2.1 Bézier curves and Bézier surfaces

Bézier curves are popular parametric curves that play a key role in CAGD, CG, and many other disciplines (Hoschek and Lasser, 1993; Hu et al., 2015; Mainar et al., 2001). However, they have some flaws such as lack of local control on the curve and by changing the position of one control point affects the entire curve. These imperfections restrict their applications in engineering, animation, automobile industries etc.

Creating curves using Bézier approach ensures that the path starts at the first control point and ends in the last but does not guarantee interpolation of the intermediate control points. Besides, change of one control point affects the shape of the entire curve which makes the approach difficult and not properly applicable.

Bézier surfaces are a species of mathematical spline used in computer graphics, computer-aided design, and finite element modeling. Similar to Bézier curves, a Bézier surface is defined by a set of control points, and first described in 1962 by the French engineer Pierre Bézier who used them to design automobile bodies. It can be of any degree, but bicubic Bézier surfaces generally provide enough degrees of freedom for most applications. Similar to interpolation in many respects, a key difference is that the surface does not, in general, pass through the central control points but they are "stretched" towards them. They are visually intuitive and for many applications they are mathematically convenient. Maqsood et al. (2021a) proposed generalized Bézier trigonometric (GBT-Bézier) surfaces with some geometric and parametric continuity conditions, along with some engineering applications.

2.2 Rational Bézier curves and Trigonometric Bézier curves

Rational Bézier curve is a robust tool for constructing free form curves/surfaces as given by Duan et al. (2005) and Gregory et al. (1994). These curves can be adjusted by introducing weight factors without changing their control points. However, they also have many drawbacks due to their rational form, for instance, repeated differentiation that produces curves of a very high degree. This drawback can be overcome by using classical Bézier curves which possess shape parameters instead of weight factors. A class expansion of Bézier curve having one shape control parameter was introduced by Wu et al. (2006), Houjun (2011) and Yang et al. (2011). Qin et al. (2008) proposed two expansions for the cubic Bézier curves to increase shape adjustability. New expansions for the quartic Bézier curves with multiple shape parameters were defined by Zhang et al. (2009).

Han et al. (2008) presented the designing of various Q-Bézier curves/surfaces with local and global control by using a class of m adjustable shape parameters. An extension of Bézier model with all its features was developed by Hu et al. (2018c). A novel extension of Bézier curves/surfaces of degree m with various shape parameters were described by Qin et al. (2013). These newly constructed curves/surfaces not only shared most features with classical degree m Bézier curves/surfaces but also modified their shapes by fluctuating the values of the corresponding shape parameters. Using GBT-Bézier curves, Maqsood et al. (2021a) constructed a class of engineering surfaces and described their shape adjustment using two different shape parameters. Hu et al. (2017b) derived the continuity conditions between generalized Bézier-like surfaces with modeling examples of smooth continuity. The generalized Bézier trigonometric curves and surfaces with their geometric properties and continuity conditions are proposed by Maqsood et al. (2020).

Approximation of circular arcs using Bézier curves, is a fundamental topic in CAGD, CAD/CAM areas and many other disciplines (Ahn and Kim, 1997; Goldapp, 1991; Kim and Ahn, 2007). However, Bézier curves cannot represent transcendental curves accurately such as cycloid and helix due to their representation in polynomial form (Mainar et al., 2001). In the past few years, many papers have investigated the trigonometric Bézier-like curves, trigonometric splines, and their applications. For the first time, the trigonometric B-Splines were presented by Schoenberg (1964) and the iterative relationship for random order trigonometric B-Splines was developed by Lyche and Winther (1979). In Computer Numerical Control (CNC) technology, Neamtu et al. (1997) proposed some new techniques for designing CAM profiles using trigonometric spline. Applications of trigonometric spline in dynamic systems were discussed by Nikolis and Seimenis (2005).

In recent years, geometric modeling using trigonometric polynomials has attained much attention and a considerable amount of work has been done in this context. Su and Zou (2012) used trigonometric spline to modify a trajectory for robot manipulators. Using single and double shape parameters, quadratic and rational quadratic trigonometric Bézier curves were developed by Uzma et al. (2012) and Bashir et al. (2012) respectively. Han (2004) constructed cubic trigonometric polynomial curves with one shape parameter that can deal precisely with circular arcs, cylinders, cones and many others. Meanwhile, Han (2006) proposed piecewise quartic polynomial curves with a local shape parameter that can approximate an ellipse from both sides. The total positivity property of cubic trigonometric nonuniform spline basis functions have been proved by Yan (2016).

An exact representation of ellipse using cubic trigonometric Bézier (shortly, T-Bézier) curves were presented by Han et al. (2009b). These T-Bézier curves were further extended to construct spiral and transition curves (Misro et al., 2017c). Quartic trigonometric Bézier-like curves with single shape parameter and their corresponding surfaces were defined by Dube and Sharma (2013). Misro et al. (2017a) constructed quintic trigonometric Bézier curves with two shape parameters that later applied in generating five templates of spiral transition curves. They also utilized T-Bézier curves with two shape parameters to develop S-shaped and C-shaped transition curve satisfying G^2 Hermite condition (Misro et al., 2017b).

To justify the work of Hongyi (2005), Yang and Zeng (2009) proposed Bézier curves and triangular Bézier surfaces using m and $3m(m+1) = 2$ shape parameters. Using trigonometric polynomials, Su and Tan (2006) studied quasi-cubic B-Spline curves and surface to show the exact representation of spheres, sine curves, circular arcs and straight lines. Xumin and Weixiang (2008) presented a technique for modeling free form surfaces and provided several geometrical examples to show the effect of adjustment of shape parameters over the surfaces. An algorithm for converting a rectangular patch of a triangular Bézier surface into a tensor product Bézier surface was proposed by Lasser (2002). Sharma (2016c) and Sharma (2016a) suggested quartic trigonometric, quasi-quartic trigonometric and a class of Bézier-type cubic trigonometric curves/surfaces, respectively.

2.3 Developable surfaces

Construction of developable surfaces has received much attention from various industries, including shipbuilding, automotive, architecture, clothing and footwear, computer animation, and image processing. Chu and Chen (2005) presented that developable surfaces have many applications in engineering, manufacturing, and Computer Numerical Control (CNC). Therefore, the developable surfaces have been widely investigated for modeling of industrial products such as architectural free-form surfaces and ship-hulls. Pottmann et al. (2008), Hwang and Yoon (2015), and Tang et al. (2016) make a composition of multiple developable strips to represent free form shapes. Developable shapes were widely used in metal sheet forming (Mancewicz, 1992), shipbuilding (Nolan, 1971), windshield design, and clothing industries (Hinds et al., 1991; Lamb, 1995; Wang et al., 2005). A process of designing shoe uppers by using trianglar Bézier patches was explained by Chung et al. (2008). Geometric design of quadratic and cubic developable Bézier patches from two Bézier boundary curves is studied and the conditions for developability are derived geometrically from the de Casteljau algorithm and expressed as a set of equations that must be fulfilled by the Bézier control points (Chu and Séquin, 2002). The designing of various kinds of developable surfaces such as Enveloping GHT-Bézier developable surfaces, Spine GHT-Bézier developable surfaces and geodesic interpolation curve on GHT-Bézier developable surfaces has been studied by Hu et al. (2020a), Ammad et al. (2021) and Maqsood et al. (2021b). Rational $(1,n)$-Bézier surfaces are ruled surfaces which are generated by a one parameter set of straight lines and play a special role in technical use, as described in Lang and Röschel (1992). Fernández-Jambrina (2017) addressed the issue of designing of developable surfaces with Bézier patches, and also show that the developable surfaces with a polynomial edge of regression are the set of developable surfaces which can be constructed with Aumann's algorithm.

The designing techniques for developable surfaces have two divisions, Point Geometric Representation (PGR) and, Line and Plane Geometric Representation (LPGR). Further, two particular approaches are there in PGR to design various kind of developable surfaces. The first approach is to build up a developable surface based on the original direction and given directrix. The second approach is to formulate it by two interpolating boundary curves. Taking a Bézier space curve as directrix, Zhang and Wang (2006) investigated the geometric design of Bézier developable surfaces.

As a generalization of algorithm in Aumann (2003), Fernández-Jambrina (2007) provided a linear algorithm for constructing B-Spline control nets of developable surfaces of random order. Chu et al. (2008) proposed a technique to interpolate a strip in a conical form specified by two space curves with developable patches. However, PGR results in tough computation due to ambiguous descriptions of developable surface and non-linearity of characteristic equations that limit its application area. Contrarily, LPGR which is also known as dual representation, presents a developable surface as a curve in dual projective space, which removes the flaws of PGR.

For the first time, Bodduluri and Ravani (1993) developed the dual Bézier and B-Spline interpolations to create developable surfaces and made practical and effective use of cubic Bézier and B-Spline basis functions to design developable surface with explicit expression. Encouraged from study of Bodduluri and Ravani (1993), Pottmann and Farin (1995) defined rational developable Bézier and B-Spline surfaces. Afterward, the LPGR technique was applied to approximate and construct developable NURBS surfaces by Pottmann and Wallner (1999). Zhou et al. (2006) constructed quartic and quintic developable Bézier surfaces.

Nevertheless, these developable Bézier surfaces have only one shape parameter, which leads to the limited shape control in the composition of complicated developable surfaces. In recent times, Hu et al. (2017c), Hu et al. (2018b), Hu and Wu (2019), and Hu et al. (2020a) defined some straightforward schemes for the CAD of developable

Bézier-like, H-Bézier, generalized quartic H-Bézier and generalized C-Bézier developable surfaces respectively with multiple shape parameters. They also calculated their G^2 continuity conditions along with application in geometric modeling.

2.4 Optimization Techniques

To construct the developable surfaces with highly accurate developability degree, many Bio-inspired optimization techniques such as Bonabeau et al. (1999), Dorigo et al. (2006), Mirjalili et al. (2014), and Haldurai et al. (2016) are studied. The generation of quasi developable Q-Bézier strip via PSO-based shape parameters are presented by Cao et al. (2022). The developability degree was also determined. The shape optimization of developable surfaces by using Cukoo search algorithm is presented in Hu et al. (2020b). Similarly, the generation of piecewise developable free form grid surface is presented by using the plate components to overcome various difficulties in the manufacturing process (Cui et al., 2021). Nadimi-Shahraki et al. (2021) and Hou et al. (2022) proposed an Improved Grey Wolf Optimizer (I-GWO) for solving global optimization and engineering design problems. This improvement is proposed to alleviate the lack of population diversity, the imbalance between the exploitation and exploration, and premature convergence of the GWO algorithm. The salp swarm algorithm (SSA) was proposed by Lu et al. (2022) which describes the foraging habits of a class of marine organisms. SSA has been widely noticed and applied in various fields due to its advantages such as simplicity and strong robustness, and has achieved good results. The implementation of particle swarm optimization (PSO) algorithm to the electromagnetic system is presented by Robinson and Rahmat-Samii (2004).

In response to existing approaches, our research work based on the construction of new Bézier type curves called GHT-Bézier curves of order $m (m \geq 2)$ with similar features to the classical Bézier curves. As an alternative technique of representing curves, these proposed curves not only have the valuable features of Bézier curves and surfaces but also have an efficient shape modification feature. Moreover, to resolve the problem of not being able to construct complex curves/developable surfaces using a single curve/developable surface, C^m as well as G^m continuity conditions between two adjacent GHT-Bézier curves/developable surfaces have been derived to make a smooth connection among them, which is a novel method to design complicated curves/surfaces. The developable surfaces with higher developability degree is an important requirement in industrial designing. Therefore, the Bio-inspired optimization techniques such as PSO and I-GWO techniques are used to construct the developable surfaces with higher developability degree.

CHAPTER 3

PRELIMINARIES

This chapter includes some important mathematical terms that will provide a better understanding of the current research work.

3.1 Curves

Mathematics has played a central role in the progress of curves in product design, especially with CAGD technology development, particularly CAD, which has changed product design in recent years. Designing curves, especially robust curves, which are controllable, well behaved and easily worked out, contributes a special role in computer graphics and geometric modeling.

3.1.1 Binomial Coefficients

For an integer $k \geq 0$, the binomial coefficients usually derived from Pascal's triangle as in Farin (2002),

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}. \tag{3.1.1}$$

Note that particularly, $\binom{m}{0} = 1$, and $\binom{0}{0} = 1$.

3.1.2 Bernstein Basis Functions

Over 150 years ago, the Bernstein functions were originally defined by Bernstein to prove the famous Weierstrass Theorem and are formally expressed as (Farin, 2002),

$$b_{k,m}(z) = \binom{m}{k} z^k (1-z)^{m-k}, \quad 0 \leq k \leq m, \ \ z \in [0,1], \tag{3.1.2}$$

where, $\binom{m}{k}$ is the binomial coefficient. The Bernstein functions are a key to understanding Bézier curves. The important properties that make Bézier curves useful in design, mostly come from these basis functions. The Bernstein basis have many useful properties for curve generation descried as Farin (2002).

1. **Non-negativity:** Bernstein polynomials are non-negative, that is, $b_{k,m}(z) \geq 0$, $z \in [0,1]$, $k = 0, \ldots, m$.

2. **Partition of unity:**

$$\sum_{k=0}^{m} b_{k,m}(z) = 1. \tag{3.1.3}$$

3. **Symmetry:**

$$b_{m-k,m}(z) = b_{k,m}(1-z). \tag{3.1.4}$$

4. **Recursion:**

$$b_{k,m}(z) = (1-z) b_{k,m-1}(z) + z b_{k-1,m-1}(z), \tag{3.1.5}$$

   where, $b_{0,0}(z) = 1$ and $b_{k,m}(z) = 0$ for $k = -1$ and $k > m$.

5. **Degree Elevation:**

$$b_{k,m}(z) = (1 - \frac{k}{m+1}) b_{k,m+1}(z) + \frac{k+1}{m+1} b_{k+1,m+1}(z). \tag{3.1.6}$$

6. **Derivative at end points:**

$$\frac{d\ b_{k,m}(z)}{dz} = m(b_{k-1,m-1}(z) - b_{k,m-1}(z)), \tag{3.1.7}$$

where $b_{-1,m-1}(z) = b_{m,m-1}(z) = 0$.

3.1.3 Bézier Curve

Bézier curve is named after Pierre Bézier, who used it in 1960s for designing the bodywork of Renault cars. A classical Bézier curve of degree m is described as,

$$B_{k,m}(z) = \sum_{k=0}^{m} P_k b_{k,m}(z), \quad z \in [0,1], \tag{3.1.8}$$

where P_k are the control points or Bézier points of classical Bézier curve and $b_{k,m}(z)$ are the Bernstein basis functions.

The number of control points of Bézier curve increases as the degree of Bézier curve increased.

3.1.4 Properties of Bézier Curve

Bézier curve has various properties that describe their nice behavior in geometric modeling (Farin, 2002),

1. **Endpoint Interpolation:** Bézier curve interpolates the first and last control points P_0 and P_m as given below,

$$B_{k,m}(0) = P_0 \tag{3.1.9}$$

$$B_{k,m}(1) = P_m \tag{3.1.10}$$

2. **Tangent Point Interpolation:** Bézier curve has tangent to the first and last segments of the control polygon at the first and last control points respectively, in fact

$$B'_{k,m}(0) = m(P_1 - P_0) \tag{3.1.11}$$

$$B'_{k,m}(1) = m(P_m - P_{m-1}) \tag{3.1.12}$$

3. **Symmetry:**

$$B_{m-k,m}(z) = B_{k,m}(1-z). \tag{3.1.13}$$

With the same control points for a Bézier curve and specified in the opposite direction, the same Bézier curve shape is achieved. The only difference will be the parametric direction of the curve. The direction of increasing parameter reverses when the control points are specified in the reverse order.

4. **Affine Invariance:** The property of Bernstein functions assure that the Bézier curves are affine invariant concerning their control points. This means that any linear transformation (such as rotation or scaling) or translation of the control points define a new curve which is just the transformation or translation of the original curve.

5. **Affine Transformation:** Bézier curves are invariant under affine transformations and are also invariant under affine parameter transformations. That is, while the curve is usually defined on the parametric interval $[0,1]$, an affine transformation mapping $[0,1]$ to the interval $[a,b]$, $a \neq b$, yields the same curve.

6. **Variation Diminishing:** For a planar Bézier curve, the number of intersections of a given line on the curve is less than or equal to the number of intersections of that line with the control polygon.

7. **Convex Hull Property:** A Bézier curve always lies inside the convex hull spanned by its control points.

3.1.5 Classical Bézier Surface

When the set of control points of the Bézier curve moving in three dimensionsal space, new curves are generated. When these curves are moved smoothly, then they formed a surface, which may be thought of as a bundle of curves. If each of the control point is moved along a Bézier curve, then a Bézier surface patch is created. A tensor product Bézier surface of degree (m,n) is defined by a set of $(m+1)(n+1)$ control points $P_{k,j}, \in R^2 \; or \; R^3 (k = 0,1,...,m; j = 0,1,...,n)$ as,

$$W_{m,n}(z,z1) = \sum_{k=0}^{m}\sum_{j=0}^{n} P_{k,j} b_{k,m}(z) b_{j,n}(z1), \quad z,z1 \in [0,1] \tag{3.1.14}$$

where $b_{k,m}(z)$ and $b_{j,n}(z1)$ (i = 0,1,.. m; j = 0,1,..., n) are Bernstein basis functions. As the classical Bézier surfaces are also defined using Bernstein functions, so they carry all geometric features that the classical Bézier curves have. The classical Bézier surfaces possess convex polygon property, angular point interpolation property, shape