CONVOLUTION OPERATORS WITH SPLINE KERNELS

by

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Abstrak

Pengoperasi Konvolusi Berinti Splin

Dalam projek ini, kita akan memperolehi rumusan asimptot bagi pengoperasi konvolusi dengan inti splin untuk fungsi terbezakan peringkat tertinggi. Dua kelas pengoperasi yang akan dipertimbanganakan ialah pengoperasi de la Vallée Poussin-Schoenberg $T_{m,k}$ dengan inti B-splin trigonometri darjah m dan kamiran singular Riemann- Lebesgue $R_{n,k}$ dengan inti B-splin berkala darjah $n-1$. Rumus-rumus ini adalah analog perluasan Bernstein bagi anggaran Voronovskaya untuk polinomial Bernstein dan perluasan Marsden dan Riemenschneider untuk pengoperasi Bernstein-Schoenberg bagi fungsi terbezakan peringkat tinggi.

Dalam Bab 1, kita akan mempertimbangkan anggaran asimptot untuk $T_{m,k}$ dengan $mh \to 0$, m tetap dan $R_{n,k}$ dengan $nh \to 0$, n tetap serta $n \to \infty$. Untuk memperolehi rumus-rumus ini, kita hendaklah mengkaji telatah asimptot momen trigonometri bagi inti-intinya yang boleh diungkapkan sebagai pekali Fourier. Ungkapan ini masing-masing boleh dinilai sebagai suatu polinomial dalam m dan n dengan menggunakan suatu algoritma.

Dalam Bab 2, kita akan memperolehi anggaran bagi $T_{m,k}$ apabila $mh \to \alpha \in (0, \pi]$ dan bagi $R_{n,k}$ apabila $nh \to \beta \neq 0$. Min de la Vallée Poussin merupakan kes khas bagi $T_{m,k}$ apabila $\alpha = \pi$. Hasil bagi $R_{n,k}$ adalah ekoran dari Bab 1, sementara untuk $T_{m,k}$, kita perlu menganggarkan momen trigonometri bagi B-splin trigonometri dengan menggunakan hubungan rekursinya.

Dalam Bab 3, kita akan mempertimbangkan pengoperasi konvolusi dwipengubah $t_K^{(\alpha)}$ dengan K merupakan matrik tak singular 2×2, $H = 2\pi K^{-1}$ dengan jualatnya $V(K)$ direntangi oleh $\{\phi_K(\cdot - H\mathbf{n})\}_{\mathbf{n}\in I}$ di mana $I = \{\mathbf{n}_0, \mathbf{n}_1, ..., \mathbf{n}_{\Delta-1}\}$ mewakili koset bagi $\mathbb{Z}^2/K\mathbb{Z}^2$. Kita akan tunjukkan bahawa fungsi eigen bagi

pengoperasi ini, yang tak bersandar pada α membentuk suatu asas otogon untuk $V(K)$. Kita juga akan mempertimbangkan telatah penghad bagi $t_K^{(\alpha)}$ bila $\|H\| \to 0$ dan memperolehi semi-bulatannya. Contoh yang akan dipertimbangkan ialah splin kotak berkala $B_{\boldsymbol{\alpha}}(\mathrm{x})$, $\boldsymbol{\alpha} \in (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in \mathbb{N}_0^4$ dengan saring berarah empat di mana $\alpha_3 = \alpha_4$ dan H adalah matrik pepenjuru.

Abstract

Convolution Operators with Spline Kernels

In this project, we shall derive the asymptotic formulas for convolution operators with spline kernels for higher order differentiable functions. The two classes of operators which will be considered are the de la Vallée Poussin-Schoenberg operators $T_{m,k}$ with trigonometric B-spline kernel of degree m and the singular integrals of Riemann-Lebesgue $R_{n,k}$ with the periodic B-spline kernel of degree $n-1$. These formulas are analogous to the Bernstein's extension of Voronovskaya's estimate for Bernsteins polynomials and Marsden and Riemenschneider's extension of Bernstein-Schoenberg operators for higher order derivatives.

In Chapter 1, we shall derive the asymptotic formulas for $T_{m,k}$ by taking limit as $mh \to 0$ with m fixed and $R_{n,k}$ as $nh \to 0$ with n fixed as well as $n \to \infty$. In order to derive these formulas we need to study the asymptotic behaviour of the trigonometric moments of their kernels which can be expressed in terms of their Fourier coefficients and also as polynomials in m and n respectively which can be evaluated using an algorithm.

In Chapter 2, we shall derive the asymptotic estimates for $T_{m,k}$ as $mh \to \alpha \in (0, \pi]$ and for $R_{n,k}$ as $nh \to \beta \neq 0$. The former includes the de la Vallée Poussin means as a special case when $\alpha = \pi$. The result for $R_{n,k}$ follows from Chapter 1, while for $T_{m,k}$, we have to estimate the trigonometric moments for the trigonometric B-spline using its recurrence relation.

In Chapter 3, we shall consider discrete bivariate convolution operators $t_K^{(\alpha)}$ where K is a 2×2 nonsingular matrix over $\mathbb{Z}$, $H = 2\pi K^{-1}$ whose range is a space $V(K)$ spanned by $\{\phi_K(\cdot - H\mathrm{n})\}_{\mathrm{n} \in I}$ where $I = \{\mathrm{n}_0, \mathrm{n}_1, ..., \mathrm{n}_{\Delta-1}\}$ denotes the representatives of the cosets of $\mathbb{Z}^2/K\mathbb{Z}^2$. We shall show that the eigenfunctions

of these operators which are independent of α form an orthogonal basis for $V(K)$. We shall also study the limiting behaviour of $t_K^{(\alpha)}$ as $\|H\| \to 0$ and compute the corresponding limiting semi-groups. The example considered here is the periodic box-spline $B_{\boldsymbol{\alpha}}(\mathrm{x})$, $\boldsymbol{\alpha} \in (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in \mathbb{N}_0^4$ on a 4-directional mesh with $\alpha_3 = \alpha_4$ and H is a diagonal matrix.

Chapter 1

Asymptotic Formulas for Convolution Operators with Kernels of Fixed Degree

1.1. INTRODUCTION

The Bernstein-Schoenberg operators introduced by Schoenberg for continuous functions on a finite interval are a spline extension of the Bernstein polynomial operators. He also stated in [19] an analogue of Voronovskaya's formula for the asymptotic behaviour of the operators for twice differentiable functions. An extension to functions with higher order derivatives was made by Marsden and Riemenschneider ([14],[15]). This extension was in line with Bernstein's extension of Voronovskaya's result for Bernstein polynomial operators (see also [8]).

The de la Vallée Poussin means of a 2π-periodic function f,

$$V_m(f;x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \omega_m(x-t)f(t)dt, \quad x \in [0,2\pi), \tag{1.1.1}$$

where

$$\omega_m(x) := \sum_{\nu=-m}^{m} \frac{(m!)^2}{(m-\nu)!(m+\nu)!}e^{i\nu x}, \quad x \in \mathbb{R}, \tag{1.1.2}$$

and m is a positive integer, are trigonometric counterparts of the Bernstein polynomials (see [1],[3]). They are shape preserving trigonometric convolution operators [17]. A spline extension of the de la Vallée Poussin means consists of the convolution operators

$$T_m(f;x) \equiv T_{m,k}(f;x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \tau_{m,k}(x-t)f(t)dt, \tag{1.1.3}$$

where m, k are positive integers with $k \geq 2m+1$ and

(1.1.4) $$\tau(x) \equiv \tau_{m,k}(x) := \sum_{\nu \in \mathbb{Z}} \hat{\tau}(\nu) e^{i\nu x}$$

with Fourier coefficients

(1.1.5)
$$\hat{\tau}(\nu) = \begin{cases} \dfrac{(m!)^2 \left(sin(m-\nu)\frac{h}{2} \dots sin\frac{h}{2}\right)\left(sin(m+\nu)\frac{h}{2} \dots sin\frac{h}{2}\right)}{(m-\nu)!(m+\nu)!\left(sin\frac{mh}{2} \dots sin\frac{h}{2}\right)^2}, & |\nu| \leq m \\ \dfrac{k(m!)^2 sin(\nu-m)\frac{h}{2} sin(\nu-m+1)\frac{h}{2} \dots sin(\nu+m)\frac{h}{2}}{\pi(\nu-m)(\nu-m+1)\dots(\nu+m)\left(sin\frac{mh}{2} \dots sin\frac{h}{2}\right)^2}, & |\nu| > m, \end{cases}$$

where $h = \frac{2\pi}{k}$. The function $\tau_{m,k}$ is the trigonometric B-spline of degree m ([6],[20],[21]). We shall call T_m the *de la Vallée Poussin-Schoenberg operators*. If $k = 2m+1$, T_m reduces to the de la Vallée Poussin means.

A related sequence of operators is the sequence of *singular integrals of Riemann-Lebesgue* (see [3], pg.54),

(1.1.6) $$R_n(f;x) \equiv R_{n,k}(f;x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} b_{n,k}(x-t) f(t)dt,$$

which are convolution operators in which the kernels are defined by their Fourier series expansions

(1.1.7) $$b_{n,k}(x) := \sum_{\nu \in \mathbb{Z}} \left(\frac{sin\ \nu h/2}{\nu h/2} \right)^n e^{i\nu x}, \quad x \in \mathbb{R},$$

where n, k are positive integers and $h = \dfrac{2\pi}{k}$. The functions $b_{n,k}$ are the periodic polynomial B-splines of degree $n-1$ ([16] and see Section 6).

The following asymptotic formula for the de la Vallée Poussin means of a twice differentiable function is due to Natanson (see [3], pg.115).

Theorem 1.1.1. (Natanson) *If $f^{(2)}(x)$ exists,*

(1.1.8) $$\lim_{m\to\infty} (m+1)\{V_m(f;x) - f(x)\} = f^{(2)}(x).$$

This is the trigonometric analogue of Voronovskaya's estimate for Bernstein polynomials. In line with Schoenberg's extension of Voronovskaya's theorem to Bernstein-Schoenberg operators, it was shown in [7] that the following holds for the de la Vallée Poussin -Schoenberg operators $T_{m,k}(f;\cdot)$ if $f^{(2)}(x)$ exists:

$$(1.1.9) \qquad \lim_{\substack{m\to\infty \\ mh\to\alpha\in(0,\pi]}} (m+1)\{T_{m,k}(f;x)-f(x)\} = \left(1-\frac{\alpha}{2}\cot\frac{\alpha}{2}\right)f^{(2)}(x).$$

The objective of this chapter is to derive the asymptotic formulas for the de la Vallée Poussin-Schoenberg operators and the singular integrals of Riemann-Lebesgue for higher order differentiable functions, in line with Bernstein's extension of Voronovskaya's estimate for Bernstein polynomials, and Marsden and Riemenschneider's extension of Schoenberg's result on Bernstein-Schoenberg operators. We will first consider the cases where $mh \to 0$, m fixed, for the de la Vallée Poussin-Schoenberg operators and $nh \to 0$ with n fixed, as well as $nh \to 0$ with $n \to \infty$ for the singular integrals of Riemann-Lebesgue. The main theorems are stated in Section 1.2. A preliminary result on the asymptotic behaviour of positive convolution operators with even kernels is given in Section 1.3. The proofs of the main theorems which require the trigonometric moments of the kernels are obtained in Sections 1.5 and 1.6 for the de la Vallée Poussin-Schoenberg operators and singular integrals of Riemann-Lebesgue operators respectively.

1.2. THE MAIN THEOREMS

To state the main theorems we shall first introduce the combinatorial numbers which are coefficients in the expansion of the central factorial polynomials,

$$(1.2.1) \qquad x^{[n]} := \begin{cases} x\prod_{j=1}^{n-1}\left(x-\frac{n}{2}+j\right), & n>0 \\ 1, & n=0, \end{cases}$$

where $n > 0$ is the degree of the polynomial $x^{[n]}$. The coefficients $t(n,j)$ in the expansion

$$(1.2.2) \qquad x^{[n]} = \sum_{j=0}^{n} t(n,j)x^j, \quad n \in \mathbb{N}_0,$$

are called the *central factorial numbers of the first kind* (see [18], pg.213). In (1.2.2), $\mathbb{N}_0$ denotes the set of nonnegative integers. We use $\mathbb{N}$ to denote the natural numbers.

The asymptotic formulas for the convolution operators involve the trigonometric moments of their kernels. For an even 2π-periodic kernel ϕ, its trigonometric moment of order $2j$, $j \in \mathbb{N}_0$, is defined by

$$M_{2j}(\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 \sin \frac{1}{2} t)^{2j} \phi(t) dt. \tag{1.2.3}$$

For s, $\nu \in \mathbb{N}$, let

$$a_{s,\nu}(\phi) = \sum_{j=\nu}^{s} \frac{(-1)^{j+\nu} t(2j, 2\nu)}{(2j)!} M_{2j}(\phi) \tag{1.2.4}$$

and let $C_{2\pi}$ denote the class of continuous 2π-periodic functions. We are now in the position to state the main theorems.

Theorem 1.2.1. *For a fixed $m \in \mathbb{N}$, $s < m$, and $f \in C_{2\pi}$ for which the derivatives up to order $2s$ exist at $x \in (-\pi, \pi)$,*

$$\lim_{h \to 0} \frac{1}{h^{2s}} \left\{ T_{m,k}(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(\tau_{m,k}) f^{(2\nu)}(x) \right\} = (-1)^s \alpha_s^m f^{(2s)}(x), \tag{1.2.5}$$

where α_s^m is a polynomial in m of degree s with leading coefficient $\frac{(-1)^s}{(3!2)^s s!}$. Further, α_s^m can be evaluated by the following algorithm:

For $k \in \mathbb{N}$,

$$\alpha_k^0 := \frac{(-1)^k}{(2k+1)! 2^{2k}},$$

and for $r = 1, ..., m$,

$$\alpha_k^r := \sum_{\nu=0}^{k} \frac{2^{2\nu} \alpha_{k-\nu}^{r-1} \alpha_\nu^0}{\nu + 1}.$$

Theorem 1.2.2. *For a fixed $n \in \mathbb{N}$, $s < n$, and $f \in C_{2\pi}$ for which the derivatives up to order $2s$ exist at $x \in (-\pi, \pi)$,*

$$\lim_{h \to 0} \frac{1}{h^{2s}} \left\{ R_n(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = (-1)^s \beta_s^n f^{(2s)}(x), \tag{1.2.6}$$

where β_s^n *is a polynomial in* n *of degree* s *with leading coefficient* $\frac{(-1)^s}{(4!)^s s!}$. *Further,* β_s^n *can be evaluated by the following algorithm:*

For $k \in \mathbb{N}$,

$$\beta_k^1 := \frac{(-1)^k}{(2k+1)!2^{2k}}$$

and for $r = 2, 3, \ldots, n$

$$\beta_k^r := \sum_{\nu=0}^{k} \beta_{k-\nu}^{r-1} \beta_\nu^1.$$

Theorem 1.2.3. *Suppose* $s \in \mathbb{N}$. *For* $f \in C_{2\pi}$ *for which the derivatives up to order* $2s$ *exist at* $x \in (-\pi, \pi)$,

$$\lim_{\substack{nh\to 0\\ n\to\infty}} \frac{1}{(nh^2)^s} \left\{ R_n(f;x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = \left(\frac{1}{4!}\right)^s \frac{f^{(2s)}(x)}{s!}. \tag{1.2.7}$$

1.3. POSITIVE CONVOLUTION OPERATORS WITH EVEN KERNELS

For $n \in \mathbb{N}$, let

$$K_n(f;x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) k_n(x-t)dt, \quad f \in C_{2\pi}, \tag{1.3.1}$$

be a sequence of positive convolution operators with even kernels k_n which are nonnegative and normalised so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t)dt = 1. \tag{1.3.2}$$

The asymptotic formula for K_n will involve the trigonometric moments of its kernel k_n. We shall require the following Taylor expansion (see [16],[24]).

$$(\arcsin x)^p = \sum_{j=0}^{\infty} (-1)^j 2^{2j} \frac{p!}{(p+2j)!} t(p+2j, p) x^{p+2j}, \quad |x| < 1, \tag{1.3.3}$$

where $p \in \mathbb{N}$ and $t(n,j)$ are the central factorial numbers. For even p, $p = 2\nu$, (1.3.3) can be written in the form

$$t^{2\nu} = \sum_{j=\nu}^{\infty} (-1)^{j+\nu} \frac{(2\nu)!}{(2j)!} t(2j, 2\nu) (2 \sin \frac{1}{2} t)^{2j}, \quad |t| < \pi. \tag{1.3.4}$$

We observe that for $j \in \mathbb{N}$, $x^{[2j]} = \prod_{\ell=0}^{j-1}(x^2 - \ell^2)$. Therefore

$$\sum_{\nu=0}^{2j} t(2j,\nu)x^\nu = \prod_{\ell=0}^{j-1}(x^2 - \ell^2). \tag{1.3.5}$$

It follows that

$$t(2j,0) = t(2j,2\nu-1) = 0.$$

Furthermore, $t(2m,2\nu)$ satisfies the following partial difference equations:

$$t(2m+2,2\nu) = t(2m,2\nu-2) - m^2 t(2m,2\nu) \tag{1.3.6}$$

with initial conditions

$$t(2,0) = 0, \quad t(2,2) = 1. \tag{1.3.7}$$

In equation (1.3.6), which we obtained readily from (1.3.5), we have assumed that $t(2j,2\nu) = 0$ for $\nu < 0$ or $\nu > m$. It follows easily from (1.3.5), by induction, that

$$\operatorname{sgn}(t(2j,2\nu)) = (-1)^{j+\nu}, \quad \nu = 1,2,\ldots,j.$$

Hence the series (1.3.4) is a positive series.

Theorem 1.3.1. *Suppose for $j \in \mathbb{N}$, the limit $\lim_{n\to\infty} n^j M_{2j}(k_n)$ exists and*

$$\lim_{n\to\infty} n^j M_{2j}(k_n) = \lambda_j\ . \tag{1.3.8}$$

If $f \in C_{2\pi}$ and its derivatives up to order $2s$ exist at $x \in (-\pi,\pi)$, then

$$\lim_{n\to\infty} n^s\{K_n(f;x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(k_n) f^{(2\nu)}(x)\} = \frac{\lambda_s f^{(2s)}(x)}{(2s)!}. \tag{1.3.9}$$

Proof. For $t \in (-\pi,\pi)$, Taylor's formula about x gives

$$f(x+t) = \sum_{\nu=0}^{s} \frac{f^{(2\nu)}(x)}{(2\nu)!} t^{2\nu} + \sum_{\nu=0}^{s-1} \frac{f^{(2\nu+1)}(x)}{(2\nu+1)!} t^{2\nu+1} + g(t)t^{2s}, \tag{1.3.10}$$

where g is continuous and $\lim_{t\to 0} g(t) = 0$. Using (1.3.4), one can express

$$(1.3.11)\quad f(x+t) = \sum_{\nu=0}^{s} f^{(2\nu)}(x) \sum_{j=\nu}^{s} (-1)^{j+\nu} \frac{t(2j,2\nu)}{(2j)!} (2\, sin\, \frac{1}{2}\, t)^{2j}$$
$$+ \sum_{\nu=0}^{s} f^{(2\nu)}(x) \sum_{j=s+1}^{\infty} (-1)^{j+\nu} \frac{t(2j,2\nu)}{(2j)!} (2\, sin\, \frac{1}{2}\, t)^{2j}$$
$$+ \sum_{\nu=0}^{s-1} \frac{f^{(2\nu+1)}(x)}{(2\nu+1)!} t^{2\nu+1} + g(t) t^{2s}.$$

Since k_n is even, (1.3.11) leads to

$$(1.3.12)\quad K_n(f;x) = \sum_{\nu=0}^{s} f^{(2\nu)}(x) \sum_{j=\nu}^{s} (-1)^{j+\nu} \frac{t(2j,2\nu)}{(2j)!} M_{2j}(k_n) + S_{1,n} + S_{2,n},$$

where

$$S_{1,n} := \sum_{\nu=0}^{s} f^{(2\nu)}(x) \sum_{j=s+1}^{\infty} (-1)^{j+\nu} \frac{t(2j,2\nu)}{(2j)!} M_{2j}(k_n)$$

and

$$S_{2,n} := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) k_n(t) t^{2s} dt.$$

Therefore

(1.3.13)

$$\lim_{n\to\infty} n^s \left\{ K_n(f;x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(k_n) f^{(2s)}(x) \right\} = \lambda_s \frac{f^{(2s)}(x)}{(2s)!} + \lim_{n\to\infty} n^s S_{1,n} + \lim_{n\to\infty} n^s S_{2,n}.$$

Since the series

$$\sum_{\nu=0}^{s} f^{(2\nu)}(x) \sum_{j=s+1}^{\infty} (-1)^{j+\nu} \frac{t(2j,2\nu)}{(2j)!} M_{2j}(k_n)$$

converges absolutely and

$$\lim_{n\to\infty} n^s M_{2j}(k_n) = 0 \quad \text{for } s < j,$$

by (1.3.8), the first limit on the right of (1.3.13) vanishes. We will show that the second limit is also zero.

For all $\epsilon > 0$, we can choose $\delta > 0$ such that $|g(t)| < \epsilon$ whenever $|t| < \delta$, and write

$$n^s S_{2,n} = I_1 + I_2, \tag{1.3.14}$$

where

$$I_1 := \frac{n^s}{2\pi} \int_{|t|<\delta} g(t) t^{2s} k_n(t) dt$$

and

$$I_2 := \frac{n^s}{2\pi} \int_{\delta \le |t| \le \pi} g(t) t^{2s} k_n(t) dt.$$

Because of the inequality $t < \pi \, sin \frac{1}{2} t, \;\; t \in [0, \pi]$,

$$\begin{aligned} |I_1| &\le \frac{n^s}{2\pi} \epsilon \int_{|t|<\delta} |(\pi \, sin \, \frac{1}{2} \, t)^{2s} k_n(t)| dt \\ &= \frac{n^s}{2\pi} \epsilon \int_{|t|<\delta} (\pi \, sin \, \frac{1}{2} \, t)^{2s} k_n(t) dt \\ &= n^s \epsilon \left(\frac{\pi}{2}\right)^{2s} \frac{1}{2\pi} \int_{|t|<\delta} \left(2 \, sin \, \frac{1}{2} \, t\right)^{2s} k_n(t) dt. \\ &= \epsilon \left(\frac{\pi}{2}\right)^{2s} n^s M_{2s}(k_n). \end{aligned} \tag{1.3.15}$$

which is arbitrarily small, since $n^s M_{2s}(k_n)$ is bounded. On the other hand,

$$\begin{aligned} |I_2| &\le n^s \|g\| \frac{1}{2\pi} \int_{\delta \le |t| \le \pi} \left(\frac{t}{\delta}\right)^2 t^{2s} k_n(t) dt \\ &= \frac{n^s \|g\|}{\delta^2} \frac{1}{2\pi} \int_{\delta \le |t| \le \pi} t^{2s+2} k_n(t) dt \\ &\le \frac{n^s \|g\|}{\delta^2} \left(\frac{\pi}{2}\right)^{2(s+1)} \frac{1}{2\pi} \int_{\delta \le |t| \le \pi} (2 \, sin \, \frac{1}{2} \, t)^{2(s+1)} k_n(t) dt \\ &= \frac{\|g\|}{\delta^2} \left(\frac{\pi}{2}\right)^{2(s+1)} n^s M_{2(s+1)}(k_n) \end{aligned} \tag{1.3.16}$$

which tends to zero as $n \to \infty$ by (1.3.8). □

1.4. TRIGONOMETRIC B-SPLINES

Let n, k be positive integers, $h := \dfrac{2\pi}{k}$ and define a sequence $(c_{n,\nu})_{\nu \in \mathbb{Z}}$, by

$$c_{n,\nu} := \frac{1}{2\pi i} \prod_{j=0}^{n} \left(\frac{1 - e^{i(j-\nu)h}}{\nu - j}\right), \quad \nu \in \mathbb{Z}, \tag{1.4.1}$$

where the factor in the product is taken to be ih when its denominator equals zero. The terms of the sequence $c_{n,\nu} = 0$ if and only if $\nu = kp + j$, $j = 0, 1, ..., n$, $p \in \mathbb{Z}\backslash\{0\}$. It is known (see Schoenberg [20]) that

$$M_n(e^{ix}) := \sum_{\nu\in\mathbb{Z}} c_{n,\nu} e^{i\nu x}, \quad x \in [0, 2\pi],$$

is a piecewise polynomial function in e^{ix} of degree n, with knots at jh, $j = 0, 1, ..., k-1$, which possesses continuous derivatives up to order $n-1$. It is supported on $[0, (n+1)h]$.

A straightforward computation gives

$$c_{n,\nu} = i^n e^{i(n+1)\left(\left(\frac{1}{2}n-\nu\right)h/2\right)} t_\nu,$$

where

$$t_\nu := \frac{2^n}{\pi} \prod_{j=0}^{n} \frac{sin(\nu - j)h/2}{(\nu - j)}, \quad 0 \le \nu \le n, \tag{1.4.2}$$

and the factor in the product is taken to be $\frac{h}{2}$ when its denominator equals zero. Hence

$$M_n(e^{ix}) = i^n e^{inx/2} \sum_{\nu\in\mathbb{Z}} t_\nu e^{i(\nu-n/2)(x-(n+1)h/2)}. \tag{1.4.3}$$

Since $t_\nu = t_{n-\nu}$, $\nu \in \mathbb{Z}$, the function

$$p_n(x) := \sum_{\nu\in\mathbb{Z}} t_\nu e^{i(\nu-n/2)(x-(n+1)h/2)}, \quad x \in [0, 2\pi), \tag{1.4.4}$$

is a real function supported on $[0, (n+1)h]$. It is called a trigonometric B-spline with uniform knots at νh, $\nu = 0, 1, ..., n+1$. From (1.4.3) and (1.4.4), we have

$$p_n(x) = (-i)^n e^{-inx/2} M_n(e^{ix}), \quad x \in [0, 2\pi).$$

We are interested in the case $n = 2m$ is an even integer, $m = 1, 2, ...$, where we define

$$\tau_{m,k}(x) := p_{2m}(x + (2m+1)h/2)/t_m, \quad x \in \mathbb{R}. \tag{1.4.5}$$

Then

$$\tau_{m,k}(x) := \sum \hat{\tau}(\nu) e^{i\nu x}, \quad x \in \mathbb{R},$$

where

$$\hat{\tau}(\nu) = t_{\nu+m}/t_m$$
$$= \begin{cases} \dfrac{(m!)^2(sin(m-\nu)h/2 \ldots sinh/2)(sin(m+\nu)h/2 \ldots sinh/2)}{(m-\nu)!(m+\nu)!(sin\frac{mh}{2} \ldots sin\frac{h}{2})^2}, & |\nu| \leq m \\ \dfrac{k(m!)^2 sin(\nu-m)h/2 sin(\nu-m+1)h/2 \ldots sin(\nu+m)h/2}{\pi(\nu-m)(\nu-m+1)\ldots(\nu+m)(sin\frac{mh}{2} \ldots sin\frac{h}{2})^2}, & |\nu| > m. \end{cases}$$

This is the trigonometric B-spline kernel defined by (1.1.4) and (1.1.5) (see [6]). The sequence p_n satisfies the recurrence relation

(1.4.6)

$$np_n(x) = 2\ sin\ \frac{1}{2}\ xp_{n-1}(x) + 2\ sin\ \frac{1}{2}((n+1)h - x)p_{n-1}(x-h), \quad n \in \mathbb{N}$$

(see [6],[13]).

1.5. ASYMPTOTIC ESTIMATE FOR $T_{m,k}$ WHEN $mh \to 0$, m FIXED

In order to prove Theorem 1.2.1, we need a precise estimate for the trigonometric moments of the trigonometric B-spline kernels. For an even 2π-periodic integrable function k_n, its trigonometric moment of order 2σ, $\sigma \in \mathbb{N}$, can be expressed as follows.

(1.5.1)

$$\begin{aligned} M_{2\sigma}(k_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(2\ sin\ \frac{1}{2}\ t\right)^{2\sigma} k_n(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2(e^{i\frac{1}{2}t} - e^{-i\frac{1}{2}t})}{2i}\right)^{2\sigma} k_n(t)dt \\ &= (-1)^{\sigma}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{i\frac{1}{2}t} - e^{-i\frac{1}{2}t}\right)^{2\sigma} k_n(t)dt \\ &= (-1)^{\sigma}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\sigma}(1-e^{-it})^{2\sigma} k_n(t)dt \end{aligned}$$

$$= (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^{2\sigma-j} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(\sigma-j)} k_n(t)dt$$

$$= (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j \widehat{k_n}(\sigma - j).$$

For the trigonometric B-splines, $\tau_{m,k}$, its Fourier coefficients $\widehat{\tau_{m,k}}\ (\sigma - j)$, $\sigma \le m$, $j = 0, ..., 2\sigma$, can be expressed as

$$(1.5.2)\quad \widehat{\tau_{m,k}}(\sigma - j) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\tfrac{1}{2}mh \ldots 2\, sin\tfrac{1}{2}h\right)^2} \prod_{\ell=0}^{2m} \frac{sin\tfrac{1}{2}(m+\sigma-j-\ell)h}{\tfrac{1}{2}(m+\sigma-j-\ell)h},$$

where Π denotes the product in which the undefined factor (in the above equation the factor corresponds to $\ell = \sigma - j + m$) is taken to be 1. Hence

$$M_{2\sigma}(\tau_{m,k}) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\ \tfrac{1}{2}mh \ldots 2\, sin\ \tfrac{1}{2}h\right)^2}(-1)^\sigma \times \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j \prod_{\ell=0}^{2m} \frac{sin\tfrac{1}{2}(m+\sigma-j-\ell)h}{\tfrac{1}{2}(m+\sigma-j-\ell)h}$$

$$(1.5.3)\quad = \frac{(m!)^2 h^{2m}}{\left(2\, sin\tfrac{1}{2}mh \ldots 2\, sin\tfrac{1}{2}h\right)^2}(-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j A(m, j; h),$$

where

$$(1.5.4)\quad A(r, j; h) = \prod_{\ell=-r}^{r} \frac{sin\tfrac{1}{2}(\sigma-j-\ell)h}{\tfrac{1}{2}(\sigma-j-\ell)h}, \quad r = 0, 1, ..., m,\ j = 0, 1, ..., 2\sigma.$$

Hence forward we shall assume that $\sigma \le m$. For $r = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, 2\sigma$, we define a sequence $(A_{2k}(r, j))_{k \in \mathbf{N}_0}$ by

$$(1.5.5)\qquad A(r, j; h) = \sum_{k=0}^{\infty} A_{2k}(r, j) h^{2k}.$$

Lemma 1.5.1. *For* $r = 0, 1, \ldots, m$ *and* $j = 0, 1, \ldots, 2\sigma$,

$$(1.5.6)\qquad A_{2k}(r, j) = \alpha_k^r (\sigma - j)^{2k} + polynomial\ in\ (\sigma - j)\ of\ degree\ < 2k,$$

where

$$(1.5.7) \qquad \alpha_k^0 := \frac{(-1)^k}{(2k+1)!2^{2k}}, \quad k \in \mathbb{N}_0,$$

and for $r = 1, ..., m$, α_k^r *are defined recursively by*

$$(1.5.8) \qquad \alpha_k^r = \sum_{\nu=0}^{k} \frac{2^{2\nu}\alpha_{k-\nu}^{r-1}\alpha_\nu^0}{(\nu+1)}.$$

Proof. The proof is by induction on r. For $r = 0$, consider

$$A(0,j;h) = \begin{cases} \frac{\sin\frac{1}{2}(\sigma-j)h}{\frac{1}{2}(\sigma-j)h}, & j = 0,1,...,2\sigma, \\ 1, & j = \sigma. \end{cases}$$

Expanding the sine function in powers of h gives

$$A(0,j;h) = \sum_{k=0}^{\infty} \frac{(-1)^k(\sigma-j)^{2k}}{(2k+1)!}\left(\frac{h}{2}\right)^{2k}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k(\sigma-j)^{2k}}{(2k+1)!2^{2k}} h^{2k}.$$

It follows that

$$A_{2k}(0,j) = \frac{(-1)^k(\sigma-j)^{2k}}{2^{2k}(2k+1)!} = \alpha_k^0(\sigma-j)^{2k},$$

where

$$\alpha_k^0 = \frac{(-1)^k}{2^{2k}(2k+1)!}.$$

Suppose that for $r < m$,

$$A_{2k}(r,j) = \alpha_k^r(\sigma-j)^{2k} + \text{a polynomial in } (\sigma-j) \text{ of degree } < 2k.$$

By (1.5.4),

(1.5.9)

$$A(r+1,j;h) = \prod_{\ell=-(r+1)}^{r+1} \frac{\sin\frac{1}{2}(\sigma-j-\ell)h}{\frac{1}{2}(\sigma-j-\ell)h}$$
$$= A(r,j;h)\frac{\sin\frac{1}{2}(\sigma-j+r+1)h}{\frac{1}{2}(\sigma-j+r+1)h}\,\frac{\sin\frac{1}{2}(\sigma-j-r-1)h}{\frac{1}{2}(\sigma-j-r-1)h}.$$

The second and third factors of (1.5.9) can be expressed using the identity

$$cos(a-b) - cos(a+b) = 2\ sin\ a\ sin\ b,$$

as

$$\frac{sin\frac{1}{2}(\sigma - j + r + 1)h\ sin\frac{1}{2}(\sigma - j - r - 1)h}{\frac{1}{2}(\sigma - j + r + 1)h\ \frac{1}{2}(\sigma - j - r - 1)h} = \frac{cos(r+1)h - cos(\sigma - j)h}{2[(\sigma-j)^2 - (r+1)^2](\frac{1}{2}h)^2}.$$

Expanding $cos(r+1)h - cos(\sigma - j)h$ in terms of powers of h gives

(1.5.10)
$$\begin{aligned} &cos(r+1)h - cos(\sigma - j)h \\ &= \left[1 - \frac{(r+1)^2h^2}{2!} + \frac{(r+1)^4h^4}{4!} - \frac{(r+1)^6h^6}{6!} + \dots\right] \\ &\quad - \left[1 - \frac{(\sigma-j)^2h^2}{2!} + \frac{(\sigma-j)^4h^4}{4!} - \frac{(\sigma-j)^6h^6}{6!} + \dots\right] \\ &= \sum_{k=1}^{\infty}(-1)^k\frac{h^{2k}}{(2k)!}[(r+1)^{2k} - (\sigma-j)^{2k}]. \end{aligned}$$

Using (1.5.5) and (1.5.10) on the right of (1.5.9) leads to

$$\begin{aligned} A(r+1,j;h) &= \left(\sum_{k=0}^{\infty} A_{2k}(r,j)h^{2k}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^k h^{2k}[(r+1)^{2k} - (\sigma-j)^{2k}]}{2[(\sigma-j)^2 - (r+1)^2](\frac{h}{2})^2(2k)!}\right) \\ &= \left(\sum_{k=0}^{\infty} A_{2k}(r,j)h^{2k}\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^k 2h^{2(k-1)}[(r+1)^{2k} - (\sigma-j)^{2k}]}{(2k)![(\sigma-j)^2 - (r+1)^2]}\right) \\ &= \left(\sum_{k=0}^{\infty} A_{2k}(r,j)h^{2k}\right) \\ &\qquad \times \left(\sum_{k=0}^{\infty}\frac{(-1)^k 2}{(2k+2)!}\left\{\frac{(r+1)^{2(k+1)} - (\sigma-j)^{2(k+1)}}{(r+1)^2 - (\sigma-j)^2}\right\}h^{2k}\right). \end{aligned}$$

Taking the Cauchy product leads to

(1.5.11)
$$A(r+1,j;h) = \sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{k} A_{2(k-\nu)}(r,j)\frac{2(-1)^{\nu}}{(2\nu+2)!}\left\{\frac{(r+1)^{2\nu+2} - (\sigma-j)^{2\nu+2}}{(r+1)^2 - (\sigma-j)^2}\right\}\right)h^{2k}.$$

It follows from (1.5.5) and (1.5.11) that

$$A_{2k}(r+1,j) =$$

$$\sum_{\nu=0}^{k} A_{2(k-\nu)}(r,j)\frac{2(-1)^\nu}{(2\nu+2)!}\left\{\frac{(r+1)^{2\nu+2}-(\sigma-j)^{2\nu+2}}{(r+1)^2-(\sigma-j)^2}\right\}$$

$$= \sum_{\nu=0}^{k}\{\alpha_{k-\nu}^{r}(\sigma-j)^{2(k-\nu)} + \text{a polynomial in } (\sigma-j) \text{ of degree } < 2(k-\nu)\}$$

$$\times \frac{(-1)^\nu 2}{(2\nu+2)!}\{(\sigma-j)^{2\nu}+(\sigma-j)^{2\nu-2}(r+1)^2+\ldots+(\sigma-j)^2(r+1)^{2\nu-2}+(r+1)^{2\nu}\}$$

$$= \sum_{\nu=0}^{k}\frac{\alpha_{k-\nu}^{r}(-1)^\nu 2(\sigma-j)^{2k}}{(2\nu+2)!} + \text{a polynomial in } (\sigma-j) \text{of degree } < 2k$$

$$= \sum_{\nu=0}^{k}\frac{2^{2\nu}\alpha_{k-\nu}^{r}\alpha_{\nu}^{0}}{\nu+1}(\sigma-j)^{2k} + \text{a polynomial in } (\sigma-j) \text{of degree } < 2k.$$

Hence

$$A_{2k}(r+1,j) = \alpha_k^{r+1}(\sigma-j)^{2k} + \text{a polynomial in } (\sigma-j) \text{ of degree } < 2k,$$

where

$$\alpha_k^{r+1} = \sum_{\nu=0}^{k}\frac{2^{2\nu}\alpha_{k-\nu}^{r}\alpha_{\nu}^{0}}{\nu+1}. \qquad \square$$

Lemma 1.5.2. *Let $j \in \mathbb{N}$, and for any $r \in \mathbb{N}$, let*

$$S_j(r) = \sum_{\nu=1}^{r} \nu^{j-1}.$$

Then $S_j(r)$ is a polynomial in r of degree j with leading coefficient $\frac{1}{j}$.

Proof. For $r \geq 1$,

$$S_j(r) - S_j(r-1) = \sum_{\nu=1}^{r}\nu^{j-1} - \sum_{\nu=1}^{r-1}\nu^{j-1} = r^{j-1}$$

which is a difference equation for which the general solution is of the form

$$S_j(r) = \sum_{\nu=0}^{j} a_\nu r^\nu,$$

where a_ν are constants. Therefore

$$\begin{aligned} r^{j-1} = S_j(r) - S_j(r-1) &= \sum_{\nu=0}^{j} a_\nu r^\nu - \sum_{\nu=0}^{j} a_\nu (r-1)^\nu \\ &= \sum_{\nu=1}^{j} a_\nu (r^\nu - (r-1)^\nu) \\ &= \sum_{\nu=0}^{j} a_\nu \sum_{\ell=0}^{\nu-1} \binom{\nu}{\ell} (-1)^{\nu-\ell-1} r^\ell \\ &= \sum_{\ell=0}^{j-1} \Big\{ \sum_{\nu=\ell+1}^{j} (-1)^{\nu-1} a_\nu \binom{\nu}{\ell} \Big\} (-1)^\ell r^\ell . \end{aligned}$$

Equating the coefficient of r^{j-1}, we have

$$\begin{aligned} r^{j-1} &= (-1)^{j-1} a_j \binom{j}{j-1} (-1)^{j-1} r^{j-1} \\ &= j a_j r^{j-1}. \end{aligned}$$

Hence

$$a_j = \frac{1}{j}. \qquad \square$$

Lemma 1.5.3. *For $r = 1, \ldots, m$, $k \in \mathbb{N}$,*

$$\alpha_k^r = \frac{(-1)^k r^k}{(3!2)^k k!} + \textit{a polynomial in } r \textit{ of degree } < k. \tag{1.5.12}$$

Proof. We shall establish the result by induction on k using (1.5.7) and (1.5.8). For $k = 0$, $A_0(r,j) = 1$. Hence $\alpha_0^r = 1$, for all r. By (1.5.8),

$$\begin{aligned} \alpha_1^r &= \sum_{\nu=0}^{1} \frac{2^{2\nu} \alpha_{1-\nu}^{r-1} \alpha_\nu^0}{\nu+1} \\ &= \alpha_1^{r-1} - \frac{1}{3!2} \end{aligned} \tag{1.5.13}$$

for $r \geq 1$. Repeated application of (1.5.13) gives

$$\alpha_1^r = \frac{-r}{3!2} - \frac{1}{3!4}$$

for all $r \geq 1$, by (1.5.7). Hence (1.5.12) holds for $k = 1$.

Suppose that (1.5.12) holds for all $k < \ell$ and for $r \geq 1$. Using the fact that $\alpha_0^0 = 1$, equation (1.5.8) can be written as

$$\alpha_\ell^j = \sum_{\nu=0}^{\ell} \frac{2^{2\nu}\alpha_{\ell-\nu}^{j-1}\alpha_\nu^0}{\nu+1} = \alpha_\ell^{j-1} + \sum_{\nu=1}^{\ell} \frac{2^{2\nu}\alpha_{\ell-\nu}^{j-1}\alpha_\nu^0}{\nu+1}.$$

Hence

$$\alpha_\ell^j - \alpha_\ell^{j-1} = \sum_{\nu=1}^{\ell} \frac{2^{2\nu}\alpha_{\ell-\nu}^{j-1}\alpha_\nu^0}{\nu+1} \tag{1.5.14}$$

for any integer $j \geq 1$. Summing (1.5.14) for j from 1 to r leads to

$$\alpha_\ell^r - \alpha_\ell^0 = \sum_{\nu=1}^{\ell}\left(\sum_{j=1}^{r} \alpha_{\ell-\nu}^{j-1}\frac{2^{2\nu}\alpha_\nu^0}{\nu+1}\right). \tag{1.5.15}$$

Applying the inductive hypothesis on the summand of (1.5.15) leads to

(1.5.16)

$$\begin{aligned}
\alpha_\ell^r - \alpha_\ell^0 &= 2\alpha_1^0 \sum_{j=1}^{r} \alpha_{\ell-1}^{j-1} + \text{a polynomial in } r \text{ of degree } < \ell \\
&= \frac{2(-1)}{3!2^2}\sum_{j=1}^{r} \frac{(-1)^{\ell-1}(j-1)^{\ell-1}}{(3!2)^{\ell-1}(\ell-1)!} + \text{a polynomial in } r \text{ of degree } < \ell \\
&= \frac{(-1)^\ell}{(3!2)^\ell(\ell-1)!}\sum_{j=1}^{r}(j-1)^{\ell-1} + \text{a polynomial in } r \text{ of degree } < \ell.
\end{aligned}$$

By Lemma 1.5.2, the leading terms in $\sum_{j=1}^{r}(j-1)^{\ell-1}$ is $\frac{r^\ell}{\ell}$. It follows from (1.5.16) that

$$\begin{aligned}
\alpha_\ell^r - \alpha_\ell^0 &= \frac{(-1)^\ell}{(3!2)^\ell(\ell-1)!}\frac{r^\ell}{\ell} + \text{a polynomial in } r \text{ of degree } < \ell \\
\alpha_\ell^r &= \frac{(-1)^\ell r^\ell}{(3!2)^\ell \ell!} + \text{a polynomial in } r \text{ of degree } < \ell.
\end{aligned}$$

□

Lemma 1.5.4. *For any $\sigma, m \in \mathbb{N}$,*

(1.5.17)

$$M_{2\sigma}(\tau_{m,k}) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{1}{2}mh \ldots 2\, sin\, \frac{1}{2}h\right)^2}\,(-1)^\sigma(2\sigma)!\alpha_\sigma^m h^{2\sigma} + O(h^{2\sigma+2}), \text{ as } h \to 0.$$

Proof. Using (1.5.3), (1.5.4) and (1.5.5)

$$M_{2\sigma}(\tau_{m,k}) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{1}{2}mh \ldots 2\, sin\, \frac{1}{2}h\right)^2}(-1)^\sigma \sum_{j=0}^{2\sigma}(-1)^j \binom{2\sigma}{j} \sum_{k=0}^{\infty} A_{2k}(m,j)h^{2k}$$

$$= \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{1}{2}mh \ldots 2\, sin\, \frac{1}{2}h\right)^2}(-1)^\sigma \sum_{j=0}^{2\sigma}\sum_{k=0}^{\sigma}(-1)^j \binom{2\sigma}{j} A_{2k}(m,j)h^{2k} + O(h^{2\sigma+2}).$$

The last estimate follows from the fact that the power series $\sum_{k=0}^{\infty} A_{2k}(m,j)h^{2k}$ is absolutely convergent for sufficiently small h, and in particular

$$\sum_{k=\sigma+1}^{\infty} A_{2k}(m,j)h^{2k} = O(h^{2(\sigma+1)}).$$

It follows that

$$M_{2\sigma}(\tau_{m,k}) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{1}{2}mh \ldots 2\, sin\, \frac{1}{2}h\right)^2}(-1)^\sigma \sum_{j=0}^{2\sigma}\sum_{k=0}^{\sigma}(-1)^j \binom{2\sigma}{j}$$

$$\times \{\alpha_k^m j^{2k} + \text{a polynomial in } j \text{ of degree } < 2k\}h^{2k} + O(h^{2\sigma+2}). \tag{1.5.18}$$

Since

$$\sum_{j=0}^{2\sigma}(-1)^j \binom{2\sigma}{j} j^\nu = 0, \quad \nu = 0, 1, ..., 2\sigma - 1,$$

and

$$\sum_{j=0}^{2\sigma}(-1)^j \binom{2\sigma}{j} j^{2\sigma} = (2\sigma)!,$$

it follows from (1.5.18) that

$$M_{2\sigma}(\tau_{m,k}) = \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{mh}{2} \ldots 2\, sin\, \frac{1}{2}h\right)^2}\alpha_\sigma^m h^{2\sigma}(-1)^\sigma \sum_{j=0}^{2\sigma}(-1)^j \binom{2\sigma}{j} j^{2\sigma} + O(h^{2\sigma+2})$$

$$= \frac{(m!)^2 h^{2m}}{\left(2\, sin\, \frac{1}{2}mh \ldots 2\, sin\, \frac{1}{2}h\right)^2}(-1)^\sigma(2\sigma)!\alpha_\sigma^m h^{2\sigma} + O(h^{2\sigma+2}). \qquad \square$$

Proof. [Proof of Theorem 1.2.1].

It follows from (1.5.17), that for a fixed m and for any $s < m$,

$$\lim_{h \to 0} \frac{1}{h^{2s}} M_{2s}(\tau_{m,k}) = (-1)^s(2s)!\alpha_s^m.$$

Hence Theorem 1.2.1 follows immediately from Theorem 1.3.1. □

1.6. TRIGONOMETRIC MOMENTS OF PERIODIC B-SPLINE KERNELS

Let $M_1 := \chi_{(-\frac{1}{2},\frac{1}{2}]}$ be the characteristic function of the interval $(-\frac{1}{2},\frac{1}{2}]$ and for $n = 2,3,...,$ let $M_n := M_1 * M_{n-1}$ be the uniform B-spline of degree $n-1$ ([22],[23]). Let k be a positive integer, $h := \frac{2\pi}{k}$ and for $n = 1,2,...,$ we define

$$b_{n,k}(x) := \sum_{\nu\in\mathbb{Z}} kM_n(h^{-1}(x-2\pi\nu)), \quad x \in \mathbb{R}, \tag{1.6.1}$$

the uniform, 2π-periodic B-spline of degree $n-1$. The Fourier transform of $b_{n,k}$ can be computed using the Fourier transform of M_n. A straightforward computation gives

$$\begin{aligned}(\widehat{b_{n,k}})(\mu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b_{n,k}(x)e^{-i\mu x}dx \\ &= \sum_{\nu\in\mathbb{Z}} k\frac{1}{2\pi}\int_{-\pi}^{\pi} M_n(h^{-1}(x-2\pi\nu))e^{-i\mu x}dx \\ &= \sum_{\nu\in\mathbb{Z}} \int_{\frac{-\pi-2\pi\nu}{h}}^{\frac{\pi-2\pi\nu}{h}} M_n(t)e^{-ih\mu t}dt \\ &= \int_{-\infty}^{\infty} M_n(t)e^{-i\mu ht}dt \\ &= \widehat{M_n}\ (h\mu) = \left(\frac{\sin\ h\mu/2}{h\mu/2}\right)^n, \ \mu\in\mathbb{Z}.\end{aligned} \tag{1.6.2}$$

Hence

$$b_{n,k}(x) = \sum_{\mu\in\mathbb{Z}} \widehat{b_{n,k}}\ (\mu)e^{i\mu x}. \tag{1.6.3}$$

The function $b_{n,k}(x)$ is even, positive, 2π-periodic. Further, $(\widehat{b_{n,k}})(0) = 1$ and $(\widehat{b_{n,k}})(\mu) \to 1$, as $k \to \infty$ (i.e. $h \to 0$). Suppose for all $\mu \in \mathbb{Z}$,

$$B(\ell,j;h) = \left(\frac{\sin(\sigma-j)\frac{h}{2}}{(\sigma-j)\frac{h}{2}}\right)^{\ell}, \ \ell = 1,2,...,n, \text{ and } j = 0,1,...,2\sigma. \tag{1.6.4}$$

By (1.5.1),

$$M_{2\sigma}(b_{n,k}) = (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j B(n,j;h). \tag{1.6.5}$$

For $\ell = 1,2,...,n$ and $j = 0,1,...,2\sigma$, we define a sequence $(B_{2k}(\ell,j))_{k\in N_0}$ such that

$$B(\ell,j;h) = \sum_{k=0}^{\infty} B_{2k}(\ell,j)h^{2k}. \tag{1.6.6}$$

Lemma 1.6.1. *For $\ell = 1,2,...,n$ and $j = 0,1,...,2\sigma$,*

$$B_{2k}(\ell,j) = \beta_k^\ell(\sigma - j)^{2k}, \; k \in \mathbb{N}_0, \tag{1.6.7}$$

where

$$\beta_k^1 = \frac{(-1)^k}{(2k+1)!2^{2k}}, \tag{1.6.8}$$

and for $\ell = 2,3,...,n$

$$\beta_k^\ell = \sum_{\nu=0}^{k} \beta_{k-\nu}^{\ell-1}\beta_\nu^1. \tag{1.6.9}$$

Proof. For $\ell = 1$, we have

$$B(1,j;h) = \frac{sin(\sigma-j)\frac{h}{2}}{(\sigma-j)\frac{h}{2}}.$$

Expanding the sine function in powers of h gives

$$\begin{aligned} B(1,j;h) &= \sum_{k=0}^{\infty} \frac{(-1)^k(\sigma-j)^{2k}h^{2k}}{(2k+1)!2^{2k}} \\ &= \sum_{k=0}^{\infty} B_{2k}(1,j)h^{2k}. \end{aligned}$$

It follows that

$$B_{2k}(1,j) = \frac{(-1)^k(\sigma-j)^{2k}}{(2k+1)!2^{2k}} = \beta_k^1(\sigma-j)^{2k},$$

where

$$\beta_k^1 = \frac{(-1)^k}{2^{2k}(2k+1)!}.$$

Note that for $j = \sigma$, $B(1, \sigma; h) = 1$.

Suppose that for $\ell < n$,

$$B_{2k}(\ell, j) = \beta_k^\ell (\sigma - j)^{2k}. \tag{1.6.10}$$

From (1.6.4),

$$B(\ell + 1, j; h) = B(\ell, j; h) \frac{sin(\sigma - j)\frac{h}{2}}{(\sigma - j)\frac{h}{2}}.$$

Expanding the second factor in powers of h and using (1.6.6) leads to

$$B(\ell + 1, j; h) = \left(\sum_{k=0}^{\infty} B_{2k}(\ell, j) h^{2k} \right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\sigma - j)^{2k} h^{2k}}{(2k+1)! 2^{2k}} \right).$$

Taking the Cauchy product leads to

$$B(\ell + 1, j; h) = \sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{k} B_{2(k-\nu)}(\ell, j) \frac{(-1)^\nu (\sigma - j)^{2\nu}}{(2\nu + 1)! 2^{2\nu}} \right) h^{2k}. \tag{1.6.11}$$

It follows from (1.6.6) and (1.6.10) that

$$\begin{aligned} B_{2k}(\ell + 1, j) &= \sum_{\nu=0}^{k} B_{2(k-\nu)}(\ell, j) \frac{(-1)^\nu (\sigma - j)^{2\nu}}{(2\nu + 1)! 2^{2\nu}} \\ &= \sum_{\nu=0}^{k} \beta_{k-\nu}^\ell (\sigma - j)^{2(k-\nu)} \frac{(-1)^\nu (\sigma - j)^{2\nu}}{(2\nu + 1)! 2^{2\nu}} \\ &= \sum_{\nu=0}^{k} \frac{\beta_{k-\nu}^\ell (-1)^\nu}{(2\nu + 1)! 2^{2\nu}} (\sigma - j)^{2k} \\ &= \left(\sum_{\nu=0}^{k} \beta_{k-\nu}^\ell \beta_\nu^1 \right) (\sigma - j)^{2k}. \end{aligned}$$

Hence

$$B_{2k}(\ell + 1, j) = \beta_k^{\ell+1} (\sigma - j)^{2k},$$

where $\beta_k^{\ell+1} = \sum_{\nu=0}^{k} \beta_{k-\nu}^{\ell} \beta_{\nu}^{1}$. □

Lemma 1.6.2. *For $\ell = 1, 2, \ldots, n$, $k \in \mathbb{N}$,*

$$\beta_k^{\ell} = \frac{(-1)^k \ell^k}{(4!)^k k!} + \text{polynomial in } \ell \text{ of degree} < k. \tag{1.6.12}$$

Proof. We will establish the result by induction on k using (1.6.8) and (1.6.9).

For $k = 0$, $B_0(\ell, j) = 1$. Hence $\beta_0^{\ell} = 1$ for all ℓ. By (1.6.9),

$$\beta_1^{\ell} = \beta_1^{\ell-1} - \frac{1}{3!2^2} \tag{1.6.13}$$

for $\ell \geq 1$. Repeated application of (1.6.13) gives

$$\begin{aligned} \beta_1^{\ell} &= \beta_1^{1} - \frac{(\ell - 1)}{3!2^2} \\ &= \frac{(-1)\ell}{4!} \end{aligned}$$

for any $\ell \geq 1$, by (1.6.8).

Hence (1.6.12) holds for $k = 1$. Suppose (1.6.12) holds for $k < s$ and for all $\ell \geq 1$. Using (1.6.9), and noting that $\beta_0^1 = 1$, we have

$$\begin{aligned} \beta_s^{j} - \beta_s^{j-1} &= \sum_{\nu=0}^{s} \beta_{s-\nu}^{j-1} \beta_{\nu}^{1} - \beta_s^{j-1} \\ &= \sum_{\nu=1}^{s} \beta_{s-\nu}^{j-1} \beta_{\nu}^{1} \end{aligned} \tag{1.6.14}$$

for any integer $j \geq 1$. Summing (1.6.14) for j from 2 to ℓ leads to

$$\begin{aligned} \beta_s^{\ell} - \beta_s^{1} &= \sum_{\nu=1}^{s} \sum_{j=2}^{\ell} \beta_{s-\nu}^{j-1} \beta_{\nu}^{1} \\ &= \sum_{\nu=1}^{s} \sum_{j=2}^{\ell} \beta_{s-\nu}^{j-1} \frac{(-1)^{\nu}}{(2\nu + 1)!2^{2\nu}}. \end{aligned} \tag{1.6.15}$$

Applying the inductive hypothesis to the summand on the right of (1.6.15) leads to

(1.6.16)

$$\beta_s^\ell - \beta_s^1 = \sum_{j=2}^{\ell}\left(\beta_{s-1}^{j-1}\frac{(-1)}{4!} + \beta_{s-2}^{j-1}\frac{1}{5!2^2} + \ldots + \frac{\beta_0^{j-1}(-1)^s}{(2s+1)!2^{2s}}\right)$$

$$= \frac{(-1)}{4!}\left[\sum_{j=2}^{\ell}\frac{(-1)^{s-1}(j-1)^{s-1}}{(4!)^{s-1}(s-1)!} + \text{polynomial in } (j-1) \text{ of degree } < s-1\right]$$

$$= \frac{(-1)^s}{(4!)^s(s-1)!}\sum_{j=2}^{\ell}\{(j-1)^{s-1} + \text{polynomial in } (j-1) \text{ of degree } < s-1\}.$$

By Lemma 1.5.2, the leading term in $\sum_{j=2}^{\ell}(j-1)^{s-1}$ is $\frac{\ell^s}{s}$. It follows from (1.6.16) that

$$\beta_S^\ell = \frac{(-1)^S}{(4!)^S}\frac{1}{(s-1)!}\frac{\ell^S}{s} + \text{polynomial in } \ell \text{ of degree } < s$$

$$= \frac{(-1)^S\ell^S}{(4!)^S s!} + \text{polynomial in } \ell \text{ of degree } < s. \qquad \square$$

Lemma 1.6.3. *For any* $\sigma \in \mathbb{N}$,

(1.6.17) $$M_{2\sigma}(b_{n,k}) = (-1)^\sigma(2\sigma)!\beta_\sigma^n h^{2\sigma} + O(n^{\sigma+1}h^{2\sigma+2}), \text{ as } nh^2 \to 0.$$

Proof. Using (1.6.5), (1.6.6) and (1.6.7), we have

(1.6.18) $$M_{2\sigma}(b_{n,k}) = (-1)^\sigma\sum_{j=0}^{2\sigma}\binom{2\sigma}{j}(-1)^j\sum_{k=0}^{\infty}B_{2k}(n,j)h^{2k}$$

$$= (-1)^\sigma\sum_{j=0}^{2\sigma}\binom{2\sigma}{j}(-1)^j\sum_{k=0}^{\infty}\beta_k^n(\sigma-j)^{2k}h^{2k}$$

$$= (-1)^\sigma\sum_{j=0}^{2\sigma}\binom{2\sigma}{j}(-1)^j\left(\sum_{k=0}^{\sigma} + \sum_{k=\sigma+1}^{\infty}\beta_k^n(\sigma-j)^{2k}h^{2k}\right).$$

Since

$$\sum_{j=0}^{2\sigma}(-1)^j\binom{2\sigma}{j}j^\nu = 0, \quad \text{for } \nu = 0, \ldots, 2\sigma-1,$$

and

$$\sum_{j=0}^{2\sigma}(-1)^j \binom{2\sigma}{j} j^{2\sigma} = (2\sigma)!,$$

it follows from (1.6.18) that

(1.6.19)

$$M_{2\sigma}(b_{n,k}) = (-1)^\sigma \beta_\sigma^n (2\sigma)! h^{2\sigma} + (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j \sum_{k=\sigma+1}^{\infty} \beta_k^n (\sigma - j)^{2k} h^{2k}.$$

By (1.6.12), for $k \in \mathbb{N}$, $\beta_k^n = O(n^k)$ as $n \to \infty$. Since $\sum_{k=\sigma+1}^{\infty} \beta_k^n (\sigma - j)^{2k} h^{2k}$ is convergent for sufficiently small nh^2, for $j = 0, ..., 2\sigma$, and $n \in \mathbb{N}$, it follows that $\sum_{k=\sigma+1}^{\infty} \beta_k^n (\sigma - j)^{2k} h^{2k} = O((nh^2)^{\sigma+1})$ as $nh^2 \to 0$. Hence (1.6.17) follows from (1.6.19). □

Proof. [Proof of Theorem 1.2.2]

Suppose n is fixed and $s < n$. By (1.6.17) we have

$$\lim_{h \to 0} \frac{1}{h^{2s}} M_{2s}(b_{n,k}) = (-1)^s (2s)! \beta_s^n.$$

The result follows immediately from Theorem 1.3.1. □

Proof. [Proof of Theorem 1.2.3]

It follows from (1.6.17) and (1.6.12) that

$$\begin{aligned} M_{2s}(b_{n,k}) &= (-1)^s (2s)! \beta_s^n h^{2s} + O(n^{s+1} h^{2s+2}) \\ &= (-1)^s (2s)! \left\{ \frac{(-1)^s n^s}{(4!)^s s!} + \text{a polynomial in } n \right. \\ &\qquad \left. \text{of degree } < s \right\} h^{2s} + O(n^{s+1} h^{2s+2}). \end{aligned}$$

Hence

$$\lim_{nh \to 0} \frac{1}{(nh^2)^s} M_{2s}(b_{n,k}) = \frac{(2s)!}{(4!)^s s!},$$

and Theorem 1.2.3 follows immediately from Theorem 1.3.1. □

Chapter 2

Asymptotic Formulas for Convolution Operators as the Degree of the Kernels tends to infinity

In the previous chapter, the asymptotic estimates for the singular integral of Riemann-Lebesgue R_n were taken as $nh \to 0$ with n fixed as well as $nh \to 0$ with $n \to \infty$. In the case of the de la Vallée Poussin-Schoenberg operators, $T_{m,k}$, the asymptotic estimate was taken as $mh \to 0$ with m fixed. In this chapter, we shall derive the asymptotic formulas for R_n as $nh \to \beta \neq 0$ with $n \to \infty$ and for $T_{m,k}$ as $mh \to \alpha \in (0, \pi]$ with $m \to \infty$. The result for R_n follows from the lemmas in Chapter 1. However, for $T_{m,k}$, a new approach is required to obtain a more precise estimate for the trigonometric moments of the trigonometric B-spline kernels. Here we shall derive an asymptotic recursive formula for the moments using the recurrence relation of trigonometric B-splines. We shall use the recursive formula to obtain the asymptotic estimate for $T_{m,k}$ as $mh \to \alpha \in (0, \pi]$.

2.1. THE MAIN THEOREMS

Theorem 2.1.1. *For $f \in C_{2\pi}$ with derivatives up to order $2s$ exist at $x \in (-\pi, \pi)$,*

$$\lim_{\substack{n\to\infty \\ nh\to\beta}} n^S \left\{ R_n(f;x) - \sum_{\nu=0}^{S-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = \left(\frac{\beta^2}{4!}\right)^S \frac{f^{(2S)}(x)}{s!} \tag{2.1.1}$$

where the limit is taken as $n \to \infty$ and $nh \to \beta > 0$.

The corresponding result for the de la Vallée Poussin-Schoenberg is

Theorem 2.1.2. *For $f \in C_{2\pi}$ with derivatives up to order $2s$ exist at $x \in (-\pi, \pi)$,*

$$\lim_{\substack{m\to\infty \\ mh\to\alpha}} m^s \left\{ T_{m,k}(f;x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(\tau_{m,k}) f^{(2\nu)}(x) \right\} = \left(1 - \frac{\alpha}{2}\cot\frac{\alpha}{2}\right)^S \frac{f^{(2s)}(x)}{s!} \tag{2.1.2}$$