ROBUST OPTIMIZATION APPROACH IN DATA ENVELOPMENT ANALYSIS MODELS: EXTENSION TO THE CASES WITH UNCERTAIN PRODUCTION TRADE-OFFS, INTEGER DATA AND NEGATIVE DATA

YOUSEF ZEHI ROKHSANEH

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by

YOUSEF ZEHI ROKHSANEH

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I dedicate this thesis to my son Sadra, the greatest blessing in my life.

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LIST OF ABBREVIATIONS

AHP	Analytic Hierarchy Process
BCC	Banker, Charnes and Cooper
CCR	Charnes, Cooper and Rhodes
CE	Cost efficiency measure
CRS	Constant returns to scale
DEA	Data envelopment analysis
DFA	Distribution free approach
DMU	Decision making unit
FDH	Free disposal hull
HRS	Hybrid returns to scale
MIDEA	Mixed-integer DEA
$MIDEA_E$	The equivalent mixed-integer DEA
MILP	Mixed-integer linear programming
P_{TO}	Production possibility set in the presence of production trade-offs
PPS	Production possibility set
RDM	Range directional measure
RM	Russell measure
R-MIDEA	Robust mixed-integer DEA
RSORM	Robust semi-oriented radial measure
$RSORM_M^\Gamma$	Robust counterpart of the multiplier SORM model under a box-polyhedral uncertainty set
$RSORM_E^\Gamma$	Robust counterpart of the envelopment SORM model under a box-polyhedral uncertainty set
$RSORM_M^\varphi$	Robust counterpart of the multiplier SORM model under a box-ellipsoidal uncertainty set
SBM	Slack base measure of efficiency
SFA	Stochastic frontier approach
SORM	Semi-oriented radial measure
TE	Technical efficiency measure
VRS	Variable returns to scale

PENDEKATAN PENGOPTIMUMAN TEGUH UNTUK MODEL ANALISIS PENYAMPULAN DATA: PERLUASAN KEPADA KES DENGAN PENGELUARAN PERDAGANGAN TIDAK PASTI, DATA INTEGER DAN DATA NEGATIF

ABSTRAK

Analisis Penyampulan Data (APD) merupakan teknik mengukur prestasi yang popular dan semenjak diperkenalkan, model APD telah diaplikasikan secara meluas dalam masalah pengurusan dunia nyata. Salah satu cabaran dalam mengaplikasikan model APD dalam masalah dunia nyata ialah ketidakpastian dan ketidaktepatan data yang boleh menyebabkan ralat dalam pengukuran, pengiraan, jangkaan dan lain-lain. Oleh kerana ketidakpastian merupakan faktor yang tidak dapat dielakkan dalam kebanyakan masalah pengoptimuman, ketidakpastian dalam data perlu dipertimbangkan untuk memastikan penyelesaian optimum dan penanda aras yang boleh dipercayai. Pengoptimuman teguh merupakan salah satu pendekatan terbaru dalam mengendalikan ketidakpastian dalam model APD yang memberi imunisasi kepada parameter tidak pasti dengan set tidak pasti yang telah ditentukan untuk menentukan penyelesaian optimum yang dijamin menjadi terbaik bagi kebanyakan kes parameter tidak pasti. Aplikasi pendekatan pengoptimuman teguh pada model APD telah memperkenalkan bidang APD teguh pada tahun 2008, yang merupakan cabang baru dan sedang berkembang dalam APD. Matlamat tesis ini adalah untuk memenuhi beberapa jurang teori dan praktikal dalam bidang APD teguh. Penyelidikan terdahulu dalam APD teguh hanya mempertimbangkan data masuk dan data keluar yang tidak pasti. Oleh itu, salah satu objektif tesis ini adalah untuk mengakses kesan

ketidakpastian pada parameter lain yang terlibat dalam pengoptimuman seperti pemberat yang berkait dengan data masuk dan keluar dan pengeluaran perdagangan. Tambahan pula, analisis perbandingan antara model APD teguh yang dicadangkan dengan pendekatan lain untuk pengendalian ketidakpastian dalam data seperti APD selang, akan disertakan. Objektif lain tesis ini adalah mencadangkan model APD teguh yang mampu untuk menangani jenis data tidak pasti tertentu seperti data integer dan data negatif. Salah satu cabaran dalam membina kaunterpart teguh untuk model APD yang mengandungi data sedemikian adalah kewujudan pembatas kesamaan dalam model-model ini yang boleh menyebabkan penyelesaian tidak tersaur atau rantau tersaur yang terbatas. . Oleh itu, model-model setara yang mengandungi data tidak pasti integer dan negatif tanpa sebarang pembatas kesamaan telah diformulasi untuk mengatasi masalah ini. Sebagai tambahan, ciri-ciri set tidak pasti dalam kewujudan data integer tidak pasti akan dikaji dalam tesis ini. Juga, model-model APD teguh yang dibina berdasarkan set tidak pasti berlainan seperti set tidak pasti kotak polyhedron dan kotak ellipsoid akan dibandingkan dari segi aspek pengiraan dan ciri-ciri dan had pendekatan-pendekatan ini juga telah dikaji. Daripada sudut pandangan praktikal, beberapa aplikasi dalam dunia nyata disediakan untuk mengesahkan kebolehgunaan model-model yang dibangunkan. . Sebagai contoh, tiga kajian kes berbeza disediakan untuk mengukur kepekaan seperti penyelidikan bertaja, universiti dan bank untuk menunjukkan kebolehgunaan model-model yang dicadangkan sekaligus menunjukkan peluang bahawa model-model tersebut boleh diaplikasikan untuk mengendalikan pelbagai kes analisis kepekaan teguh.

ROBUST OPTIMIZATION APPROACH IN DATA ENVELOPMENT ANALYSIS MODELS: EXTENSION TO THE CASES WITH UNCERTAIN PRODUCTION TRADE-OFFS, INTEGER DATA AND NEGATIVE DATA

ABSTRACT

Data envelopment analysis (DEA) is a popular performance measurement technique and since it was first introduced, DEA models have been extensively applied in real-world managerial problems. One of the challenges in applying DEA models in real-world problems is uncertainty and inaccuracy in data which can be due to error in measurement, calculation, prediction etc. As uncertainty is an inevitable factor in many optimization problems, therefore the uncertainty in data should be taken into consideration to ensure reliable optimal solutions and benchmarking. Robust optimization is one of the most recent approaches for handling uncertainty in DEA models which immunize the uncertain parameters over a pre-specified uncertainty set to determine an optimal solution which is guaranteed to be the best for all or most of the possible realizations of the uncertain parameters. Applying robust optimization approach in DEA models resulted to Robust DEA field which is a relatively young yet growing field in DEA, introduced in 2008. The goal of this thesis is to fulfil some of the theoretical and practical gaps in robust DEA field. The previous works on robust DEA models only considered inputs and outputs data to be uncertain, thus one of the objectives of this thesis is to assess the effect of uncertainty in the other involved parameters in the optimization such as weights assigned to inputs and outputs and production trade-offs. Moreover, a comparative analysis between the proposed robust DEA model and other approaches of handling uncertainty in data such as interval DEA will be provided. Another objective of this thesis is to propose robust DEA models that

are capable to handle special type of uncertain data such as integer data and negative data. One of main challenges to construct a robust counterpart for the DEA models containing such data is the presence of equality constraints in these models which can lead to an infeasible solution or a restricted feasible region. Therefore, equivalent models containing integer or negative uncertain data without any equality constraints are formulated to overcome this problem. In addition, the characteristic of uncertainty set in the presence of uncertain integer-valued data is investigated in this thesis. Also, robust DEA models constructed based on different uncertainty sets such as polyhedral-box and ellipsoidal-box uncertainty sets are compared from a computational aspect and the characteristic and limitations of these approaches have been studied. From the practical point of view, several real-world applications are provided to validate the applicability of the developed models in this thesis. For example, three different efficiency measurement case studies such as funded research projects, universities and banks are provided to show the applicability of the proposed models and the opportunities that the proposed models can be applied to handle various case of robust efficiency analysis.

CHAPTER 1

INTRODUCTION

Performance measurement and benchmarking are critical procedures for all type of organizations, by which an organization monitors important aspects of its programs, systems, and processes. Performance assessment helps organizations to improve their efficiency and set goals, using the data which have been derived in the process of performance measurement. Performance assessment provides a reliable process for the organizations to determine if their current systems and processes are working efficiently and as a result, best and worst performers in the organization can be detected. Analysing the result of performance assessment can provide important information and data on how an organization can use its resources in order to optimize its efficiency and productivity. There are several methods in operations research literature that can be applied for the assessment of efficiency in different organizations.

Efficiency assessment methods are categorized into two basic groups based on the estimation of production frontier: parametric and non-parametric methods. In the parametric methods the shape of the frontier will be estimated by identifying the relationship between inputs and outputs based on a production function that shows how a maximum output can be attained by using a certain level of input. Parametric frontiers are based on specific functional forms and can be either deterministic or stochastic. In the Non-parametric techniques such as Data Envelopment Analysis (DEA), the efficiency frontier will be identified using linear programming methods. In fact, all decision-making units (DMUs) in the technology will be compared with those which are placed on the efficient frontier and called efficient DMUs.

1.1 Data Envelopment Analysis and Uncertainty

Data Envelopment Analysis (DEA) is a popular non-parametric technique for the assessment of efficiency of a set of homogeneous decision-making units (DMUs) with the same set of inputs and outputs. DEA pioneered by Farrell (1957), who proposed a non-parametric frontier analysis for solving a linear programming to measure productive efficiency. Later the first DEA model which is called Charnes, Cooper Rhodes (CCR) model was developed by Charnes et al., (1978). Since introducing the first DEA model, there has been a massive growth in the theory and application of DEA. From the theoretical aspect, various new DEA models have been proposed to improve and extend DEA methodology and its applicability. In managerial applications, DEA has been widely used in different areas, such as health care, education, banking, agriculture, marketing, hospitality and many more. One of the challenges in applying DEA models in real world applications is uncertainty and imprecision in data which is inevitable in many production technologies. The ambiguity in data can be due to errors in measurement and calculations, errors in predictions, unachievable or inadequate information, quantifying qualitative measures or environmental conditions. Hence, the uncertainty in data can lead to unreliable efficiency scores and ranking for the DMUs and the decisions made based on the unreliable efficiency scores, ranking and benchmarking will lead to unreliable and practically unattainable management decisions.

1.2 Robust Optimization

Data uncertainty seems to be an unavoidable issue in many real-world optimization problems and a small uncertainty or perturbation in data may easily result in a completely meaningless and misleading nominal solution for the optimization problem. Hence, various approaches have been developed to handle the uncertainty in data in the mathematical optimization problems such as stochastic optimization, fuzzy theory, robust optimization. Currently robust optimization is one of the popular optimization methodologies which has been proposed to tackle the issue of uncertainty and inexactness in data in mathematical programming problems. Robust optimization was first introduced by Soyster (1978) and was developed and extended by many researchers such as Ben-Tal and Nemirovski (1998; 1999; 2000) and Bertsimas and Sim (2003; 2004). One of the advantages of robust optimization approach in comparison with the other approaches is the tractability of formulations based on this approach. Moreover, in robust optimization the probability distributions of the uncertain data are assumed to be unknown which is also an important point. This is because in many optimization problems the historical information for an uncertain data or event might not be available or accessible. In robust optimization the uncertain parameters are immunized over a pre-specified uncertainty set to determine an optimal solution which is guaranteed to be the best for all or most of the possible realizations of the uncertain parameters.

Sadjadi and Omrani (2008) were the first to introduce robust optimization in DEA models to handle the uncertainty on inputs and outputs data for DMUs. Afterwards, many researchers developed more robust DEA models and the Robust DEA field started to form. The robust DEA field is relatively young, yet popular, as various robust DEA models are being developed and introduced in this research area in recent years.

1.3 Problem Statement

Since the introduction of the first robust DEA model by Sadjadi and Omrani (2008), the robust DEA field has been growing rapidly and various robust DEA models have been developed under different uncertainty sets. However, to the best of our knowledge almost all these models considered the uncertainty to appear in inputs and outputs data in DMUs, where other factors that are involving in the efficiency assessment process such as weights assigned to the inputs and outputs might also be subjected to uncertainty. In weight restriction approaches such as super efficiency and assurance regions type I and II**,** the technological meaning of efficiency as a realistic input or output improvement factor does not remain clear and interpretable. The main reason is that the conventional weight restrictions in approaches such as assurance region type I and II, are constructed based on value judgments, monetary values or perceived importance of inputs or outputs. One of the recent methods to overcome this issue is incorporating production trade-offs as simultaneous changes to the inputs and outputs which naturally exist in any real production technology (Podinovski, 2004). However, in many real-world cases, the trade-offs between inputs and outputs cannot be express precisely. Therefore, by ignoring the uncertainty and perturbation in production trade-offs, the results will not be reliable.

One of the difficulties in construction of robust counterpart for different DEA models is the existence of equality constraints in some of DEA models. Such constraints containing uncertain parameters restrict the feasible region and may lead to infeasible solutions for the robust analysis. One of the popular DEA models containing equality constraints is the mixed-integer DEA (MIDEA) model by Kuosmanen and Matin (2009) which was proposed to handle integer-valued data in DEA models and ensure feasible integer-valued targets for inefficient DMUs. Another problem for developing a robust integer DEA model is the construction of uncertainty set in the presence of integer-valued data, as the general assumption is that the uncertain parameters are real-valued.

One of the assumptions in the proposed robust DEA models so far, is that the inputs and outputs variables are non-negative, however in many applications such as banking cases some variables may take negative values and such negative values can also be affected by perturbation and errors. Therefore, the conventional robust DEA models are not capable to handle negative data and appropriate robust DEA model can be proposed which are capable to cope with uncertain negative data. Also, the presence of equality constraint in DEA models in the presence of negative data such as semi-oriented radial measure (SORM) (Emrouznejad et al., 2010), might be problematic and cause difficulty in constructing an appropriate robust counterpart, hence these models should be modified to an equivalent model without any equality constraints.

A comparative analysis between the proposed robust DEA models and other DEA approaches for handling uncertainty such as interval DEA and a comparative study between robust DEA models constructed based on different uncertainty sets provides a computational comparison as well as an insight on the characteristic and limitations of these approaches.

1.4 Research Objectives

The objectives of this research are as follows:

- To propose a robust weight restriction model in which the weight restrictions are constructed based on production trade-offs and provide a comparison between the methods that are able to cope with uncertainty in production trade-offs.

- To construct an uncertainty set in the presence of uncertain integer valued parameters and investigate difficulties for constructing a robust counterpart for the conventional MIDEA model.
- To propose a robust DEA model with the ability to be applied in cases with integer-valued uncertain parameters.
- To modify robust DEA models in the presence of uncertain negative data under different uncertainty sets.

1.5 Research Contributions

This thesis addresses some of the theoretical gaps in the robust DEA field to expand this field to be more applicable for real-life optimization problems. First, this research proposes a robust weight restriction model where the weight restrictions are constructed based on production trade-offs. Additionally, an interval weight restriction model is modified to handle the uncertainty in production trade-offs and to provide an insightful comparison between the interval weight restriction model and the robust weight restriction model. The proposed models are examined with a case study of funded research projects in engineering discipline from Universiti Sains Malaysia.

Next, the gap in the robust DEA models in the presence of integer-valued data is addressed and the difficulties in constructing a robust counterpart for the conventional MIDEA model are studied. We proposed an equivalent integer DEA model without any equality constraint based on the fuzzy interpretation of efficiency to cope with the arise problem from the equality constraints. Moreover, we studied the properties of the uncertainty set constructed for the uncertain integer-valued parameters Then, based on the developed uncertainty set, the robust counterpart of the proposed equivalent integer DEA model is proposed. To demonstrate the applicability our proposed model, an application of efficiency assessment in Malaysian universities is provided.

Then the robust DEA models in the presence of uncertain negative data are studied. In robust DEA literature many of the proposed robust DEA models ignored the uncertainty in either inputs or outputs to avoid the uncertainty in the normalization constraint. To be clear, in an input-oriented DEA model only output variables are assumed as uncertain, hence we modified an equivalent SORM model that can be applied in cases where inputs and outputs are simultaneously subjected to uncertainty. Aside from the proposed robust DEA models under a box-polyhedral uncertainty set, we studied and formulated the robust counterpart of SORM model considering a box-ellipsoidal uncertainty set.

1.6 Thesis Outline

This thesis is organized in the following way:

Chapter 2 provides a review on the basic DEA models and particularly weight restriction approaches. The data uncertainty in DEA is discussed and a review of approaches for handling uncertainty in DEA is provided. In addition, an overview of robust optimization approach under various uncertainty sets is given. A literature review on the previous studies on robust DEA and proposed models is presented to find out the major developments in the robust DEA research field.

Chapter 3 develops a robust weight restriction model where production trade-offs are applied to construct weight restriction constraints to tackle the uncertainty in production trade-offs. Additionally in order to provide a comparison between the methods that are capable in handling the uncertainty in production trade-offs an interval weight restriction model is modified.

In Chapter 4 we recall a short background of fuzzy concept and relationship of fuzzy concept with the conventional CCR model. This chapter presents a new mixed-integer DEA model without any equality constraint based on the fuzzy interpretation of efficiency, as well as a modified uncertainty set constructed in the presence of uncertain integer-valued parameters. Next a robust DEA model to handle uncertain integer-valued data is presented.

Chapter 5 presents a review on the presence of negative data and proposed models in the literature to take negative data into consideration in the efficiency assessment. Moreover, the conventional DEA models in the presence of negative data are modified to provide equivalent models without the normalization constraint. Then the chapter focuses on the robust DEA models with uncertain negative data under box-polyhedral and box-ellipsoidal uncertainty sets.

Finally in Chapter 6 we discuss the concluding remarks of this thesis and future research directions.

CHAPTER 2

BACKGROUND OF THE STUDY

This chapter presents a research background on the models and approaches applied in this thesis, as well as a specific review on previous relevant studies on the robust DEA models and applications. Firstly, a background on traditional and basic DEA models is given and then the weight discrimination problem and approaches to handle this shortcoming in the conventional DEA models are discussed. Also, the arising problems, resulting from data uncertainty and perturbation are discussed and a review on approaches which are able to cope with data uncertainty is presented. Special attention is given to robust optimization which is the applied approach in this thesis to handle data uncertainty. Finally, the previous studies on the application of robust optimization in different DEA models have been studied.

2.1 Data Envelopment Analysis

Data envelopment analysis is a decision making tool for evaluating relative efficiency of a set of homogeneous DMUs (Charnes et al., 1978). The reason for using relative term is that the efficiency of each DMU under evaluation will be obtained by comparing it with all other DMUs with a simple restriction that all DMUs lie on or below an efficient frontier. A Production Possibility Set (PPS) will be constructed, which contains all input-output correspondences which are feasible in principle including those observed units being assessed (Thanassoulis, 2001). In DEA the aim is to determine which DMU is performing efficiently in comparison with other units and to benchmark the other DMUs relative to the efficient units in the defined PPS. Such an aim will be succeeded by calculating the efficiency scores with linear programming approaches which the calculated efficiency scores determine the units on and below the efficient frontier. To clarify the meaning of efficiency or a relative efficient DMU, the following definitions can be considered.

Definition 2.1. (Efficiency) A DMU can attain full efficiency if and only if none of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

Definition 2.2. (Relative Efficiency) A DMU is relatively efficient if and only if the performances of other DMUs does not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

2.1.1 Production possibility set (PPS)

In DEA a PPS is constructed to link inputs and outputs instead of functional forms. PPS is defined as the minimum set enveloping the observed units and all the input-output correspondences that are feasible. To generalize the basic assumptions underlying the PPS in DEA, suppose there are n DMUs, DMU$_j$ $j \in J = \{1, \dots, n\}$, which DMU$_j$, denoted by $\left(\boldsymbol{X}_j, \boldsymbol{Y}_j\right) \in \mathbb{R}_{+}^{m+s}$ uses m inputs $\boldsymbol{X}_j = \left(x_{ij}\right) \geq 0$ $i \in I = \{1, \dots, m\}$ to produce s outputs $\boldsymbol{Y}_j = \left(y_{rj}\right) \geq 0$ $r \in R = \{1, . . , s\}$. Here, the symbol ′≥′ indicates that at least one component of $\boldsymbol{X}_j$ or $\boldsymbol{Y}_j$ is positive while the remaining inputs and outputs are considered as non-negative. The production possibility set, denoted by P is defined as follow:

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) |\ \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}.$$

The PPS in DEA is defined as the minimum technology that satisfies the following production axioms (Banker et al., 1984).

Axiom 2.1. Feasibility of observed data. $(\boldsymbol{X}_j, \boldsymbol{Y}_j) \in P, \forall\, j = 1,2, \dots, J$.

Axiom 2.2. *Proportionality*. Any positive proportion of a feasible pair of input and output is also feasible.

$If\ \lambda \geq 0\ and\ (\boldsymbol{x}, \boldsymbol{y}) \in P \Rightarrow (\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) \in P.$

Axiom 2.3. *Convexity*. The set *P* is convex.

$\forall\ (\boldsymbol{x}, \boldsymbol{y}) \in P,\ (\boldsymbol{x}', \boldsymbol{y}') \in P,\ 0 \leq \lambda \leq 1 \Rightarrow [\lambda(\boldsymbol{x}, \boldsymbol{y}) + (1-\lambda)(\boldsymbol{x}', \boldsymbol{y}')] \in P.$

Axiom 2.4. *Free disposability*. If a specific pair of input and output is producible, any pairs of more input and less output for the specific one are also producible.

$(\boldsymbol{x}, \boldsymbol{y}) \in P,\ \overline{\boldsymbol{x}} \geq \boldsymbol{x} \Rightarrow (\overline{\boldsymbol{x}}, \boldsymbol{y}) \in P.$

$(\boldsymbol{x}, \boldsymbol{y}) \in P, \overline{\boldsymbol{y}} \geq \boldsymbol{y} \geq 0 \Rightarrow (\boldsymbol{x}, \overline{\boldsymbol{y}}) \in P.$

$(\boldsymbol{x}, \boldsymbol{y}) \in P, \boldsymbol{y} \geq \overline{\boldsymbol{y}} \geq 0\ and\ \boldsymbol{x} \leq \overline{\boldsymbol{x}} \Rightarrow (\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \in P.$

2.1.2 DEA model classifications

2.1.2(a) Constant and variable returns to scale

DEA models can be categorized based on different aspects. One of the basic classifications is based on the returns to scale assumption which can be divided into two categories, constant returns to scale (CRS) and variable returns to scale (VRS). A brief definition of returns to scale can be given as follows:

Definition 2.3. (Returns to scale) Return of scale is an important term in production economics. It usually defined as the effect of production factors on the production. In other words, it explains the behaviour of the rate of increase in outputs (production) with respect to the associated increase in inputs (the factors of production).

Returns to scale can be either constant or variable (increasing or decreasing). If a proportional increase in all inputs result in the same proportional change in the output, returns to scale will be defined as constant returns to scale. When output increases by more than the proportional increase in all inputs, returns to scale will be defined as increasing returns to scale. When output increases by less than the proportional increase in all inputs, returns to scale will be defined as decreasing returns to scale.

2.1.2(b) Radial and non-radial DEA models

In general, DEA models can be classified into radial and non-radial models. Radial models deal with the proportional changes in inputs or outputs, in fact these models assume a proportional reduction of inputs or a proportional expansion of outputs which is common to all inputs or outputs. One of the main properties of these models is providing an efficiency score for all DMUs. The conventional radial models include CCR model which was proposed by Charnes et al. (1978) and BBC model which was proposed by Banker et al. (1984). In real world applications proportional changes in all inputs or outputs may not be possible, hence this assumption restricts radial DEA models, and this shortcoming has led to the development of non-radial DEA models. The non-radial DEA models in contrast of radial models, put aside the assumption of proportional reduction of inputs (or proportional expansion of outputs) and deal with slack variables directly. In fact, in the non-radial models the aim is to maximize the rate of reduction in inputs (or minimize the rate of expansion in outputs) which may discard varying proportions of original inputs and outputs. The additive model introduced by Charnes et al. (1985), Russell measure model by Fare and Lovell (1978) and slack base measure model (SBM) by Tone (2001) are some of the most well-known non-radial DEA models.

2.1.2(c) Input and output-oriented models

DEA models based on the orientation are classified into input-oriented and output-oriented models. In an input-oriented model, the objective is to minimize the inputs while satisfying at least the given output level. Indeed, in later models a DMU is inefficient if it is possible to decrease any input without increasing any other input and without decreasing any output. So, an inefficient DMU in an input-oriented model will become efficient through the proportional reduction of its inputs while its outputs level is held unchanged.

On the other hand, in an output-oriented model, the objective is to maximize the level of outputs without requiring an increase in input resources. In fact, in an output-oriented model a DMU is called inefficient if it is possible to increase any output without decreasing any other output or increasing any input. In an output-oriented model an inefficient DMU will become efficient through the proportional increase of its outputs while the inputs level is held constant.

2.2 Basic DEA models

2.2.1 CCR model

[The CCR model was introduced by Charnes et al. (1978), assumes constant returns to scale.] The DMU under evaluation is designated as DMU_O where O ranges over $1, 2, \dots, n$ and the vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is used to construct a hull that covers all data points. The envelopment form of the input-oriented CCR model for assessing the efficiency of DMU_O is formulated as following mathematical linear programming:

minimize θ_o

Subject to:

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o x_{io}, \qquad i = 1, \dots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \qquad r = 1, \dots, s$$

$$\lambda_j \geq 0, \qquad j = 1, \dots, n \qquad (2.1)$$

where θ_o is the efficiency score for DMU_O .

Let u_r and v_i represent the weights factors related to the r^{th} output and i^{th} input respectively, so the dual form of model (2.1) which is known as the multiplier CCR model is formulated as follows:

maximize $\theta_o = \sum_{r=1}^{s} u_r y_{ro}$

Subject to:

$$\sum_{i=1}^{m} v_i x_{io} = 1,$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \dots, n$$

$$u_r \geq 0, \qquad r = 1, \dots, s$$

$$v_i \geq 0. \qquad i = 1, \dots, m \qquad (2.2)$$

Definition 2.4. Efficient DMU

1. DMU_O is efficient, if and only if the $\theta_o^* = 1$. (θ^* is the optimal value of the objective function in model (2.1))
2. DMU_O is inefficient, if and only if $0 < \theta_o^* < 1$.

The input-oriented models try to minimize the level of inputs to produce at least the same level of outputs. On the other hand, the output-oriented models maximize the level of output by utilizing the given amount of inputs. The output-oriented CCR model is formulated as follows:

$$
\begin{aligned}
&\text{maximize } \varphi_o \\
&\text{Subject to:} \\
&\textstyle\sum_{j=1}^{n} \lambda_j \, x_{ij} \leq x_{io}, && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j \, y_{rj} \geq \varphi_o y_{ro}, && r = 1, \dots, s \\
&\lambda_j \geq 0, && j = 1, \dots, n
\end{aligned}
\qquad (2.3)
$$

where $\varphi_o = 1/\,\theta_o$.

The multiplier output oriented CCR model which is the dual form of model (2.3) is defined as follows:

$$
\begin{aligned}
&\text{minimize } \textstyle\sum_{i=1}^{m} v_i \, x_{io} \\
&\text{Subject to:} \\
&\textstyle\sum_{r=1}^{s} u_r \, y_{ro} = 1, \\
&\textstyle\sum_{i=1}^{m} v_i \, x_{ij} - \sum_{r=1}^{s} u_r \, y_{rj} \geq 0, && j = 1, \dots, n \\
&u_r \geq 0, && r = 1, \dots, s \\
&v_i \geq 0. && i = 1, \dots, m
\end{aligned}
\qquad (2.4)
$$

2.2.2 BCC model

The BCC model was introduced by Banker et al. (1984). In fact, the BBC model is an expansion in the formulation of the CCR model to analyse the variable returns to scale which ignores the proportionality assumption. The envelopment form of input-oriented BBC model is formulated as follows:

minimize θ_o

Subject to:

$\sum_{j=1}^{n} \lambda_j \, x_{ij} \leq \theta_o x_{io}, \qquad i = 1, \dots, m$

$\sum_{j=1}^{n} \lambda_j \, y_{rj} \geq y_{ro}, \qquad r = 1, \dots, s$

$\sum_{j=1}^{n} \lambda_j = 1, \qquad j = 1, \dots, n$

$\lambda_j \geq 0. \qquad j = 1, \dots, n$ (2.5)

The dual form of the model (2.5) which is known as the multiplier form of the input-oriented BCC model is given as follows:

maximize $\theta_o = \sum_{r=1}^{s} u_r \, y_{ro} - v_o$

Subject to:

$\sum_{r=1}^{s} v_i \, x_{io} = 1,$

$\sum_{r=1}^{s} u_r \, y_{rj} - \sum_{i=1}^{m} v_i \, x_{ij} - v_o \leq 0, \qquad j = 1, \dots, n$

$u_r \geq 0, \qquad r = 1, \dots, s$

$v_i \geq 0, \qquad i = 1, \dots, m$

v_o is free in sign, (2.6)

where u_r and v_i are given weights to the r^{th} output and i^{th} input respectively and v_o is scalar indicator for returns to scale. The envelopment form of output-oriented BBC model is defined as follows:

maximize φ_o

Subject to:

$\sum_{j=1}^{n} \lambda_j \, x_{ij} \leq x_{io}, \qquad i = 1, \dots, m$

$\sum_{j=1}^{n} \lambda_j \, y_{rj} \geq \varphi_o y_{ro}, \qquad r = 1, \dots, s$

$\sum_{j=1}^{n} \lambda_j = 1, \qquad j = 1, \dots, n$

$\lambda_j \geq 0.$ (2.7)

And the following model is defined as the multiplier form of output-oriented BCC model:

minimize $\sum_{i=1}^{m} v_i \, x_{io} + v_o$

Subject to:

$\sum_{r=1}^{s} u_r \, y_{ro} = 1,$

$\sum_{i=1}^{m} v_i \, x_{ij} - \sum_{r=1}^{s} u_r \, y_{rj} + v_o \geq 0, \qquad j = 1, \dots, n$

$u_r \geq 0, \qquad r = 1, \dots, s$

$v_i \geq 0, \qquad i = 1, \dots, m$

v_o is free in sign. (2.8)

The definition of efficiency in the BBC model is the same as the CCR model, however for an inefficient DMU, the BCC efficiency is less than or equal to the CCR efficiency due to the restriction on PPS caused by the additional restriction ($\sum_{j=1}^{n} \lambda_j = 1$) in the BBC model.

2.2.3 Additive model

Additive DEA model was developed by Charnes et al. (1985) as a non-radial DEA model that unlike the CCR model and BBC model which are different in the input-oriented form and output-oriented form, combines input and output orientations in a single model and measure all the inefficiency scores. This model directly deals with the input excesses and output shortfalls. The envelopment form of Additive model is formulated as the follows:

maximize $\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+$

Subject to:

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, && i = 1, \dots, m \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, && r = 1, \dots, s \\
&\textstyle\sum_{j=1}^{n} \lambda_j = 1, && j = 1, \dots, n \\
&\lambda_j \geq 0, && j = 1, \dots, n \\
&s_i^- \geq 0, && i = 1, \dots, m \\
&s_r^+ \geq 0, && r = 1, \dots, s
\end{aligned}
\tag{2.9}
$$

where s_i^- is the slack variable for the i^{th} input and s_r^+ is the slack variable for the r^{th} output.

Definition 2.5. Efficiency in additive model

A DMU is additive efficient, if and only if the optimal value of objective function is equal to zero which means all slacks are zero. $\forall\, i, j : ({s_i^-}^*, {s_r^+}^*) = 0$.

The multiplier form of model (2.9) is indicated in the following formulation:

Minimize $\sum_{i=1}^{m} v_i\, x_{io} - \sum_{r=1}^{s} u_r\, y_{ro} + v_o$

Subject to:

$\sum_{i=1}^{m} v_i\, x_{ij} - \sum_{r=1}^{s} u_r\, y_{rj} + v_o \geq 0, \qquad j = 1, \dots, n$

$u_r \geq 1, \qquad r = 1, \dots, s$

$v_i \geq 1, \qquad i = 1, \dots, m$

$v_o \; is\; free\; in\; sign. \qquad (2.10)$

2.2.4 Russell measure model

Russell measure (RM) model is another non-radial DEA model that was proposed by Fare and Lovell (1978) when they observed some difficulties with the Farrell's (1957) measure of technical efficiency. [Russell measure model to measure the efficiency of DMU_O is formulated as the following non-linear mathematical programming:]

minimize $z = \frac{1}{m+s}\left(\sum_{i=1}^{m} \theta_i + \sum_{r=1}^{s} \frac{1}{\varphi_r}\right)$

Subject to:

$\sum_{j=1}^{n} \lambda_j\, x_{ij} \leq \theta_i x_{io}, \qquad i = 1, \dots, m$

$\sum_{j=1}^{n} \lambda_j\, y_{rj} \geq \varphi_r y_{ro}, \qquad r = 1, \dots, s$

$\varphi_r \geq 1, \qquad r = 1, \dots, s$

$\theta_i \leq 1, \qquad i = 1, \dots, m$

$\lambda_j \geq 0, \qquad j = 1, \dots, n \qquad (2.11)$

where θ_i is the contraction variable for i^{th} input and φ_r is the expansion variable for r^{th} output.

Definition 2.6. Efficiency in RM model

A DMU is RM efficient, if and only if the optimal value of objective function is equal to one which means the optimal value of $\theta_i = 1, \forall\, i$ and $\varphi_r = 1, \forall j$.

2.2.5 Slack based measure of efficiency (SBM)

The slack based measure of efficiency is a non-radial model that deals with slacks (input excess and output shortfall) directly which was proposed by Tone (2001). The SBM model is invariant with respect to the units of data, and it is monotone decreasing with respect to input excess and output shortfall. This model is not translation invariant, which means if the original data are translated, the efficient frontier and the position of the DMUs relative to the efficient frontier will be changed. Also, this measure is reference set dependent which means it should be determined only by consulting the reference set of the DMU concerned. In order to measure the efficiency of DMU_O, the following fractional programming for SBM model can be applied:

$$\text{minimize } \rho = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}}$$

Subject to:

$$\begin{array}{ll}
\sum_{j=1}^{n}\lambda_j\, x_{ij} + s_i^- = x_{io}, & i = 1, \dots, m \\
\sum_{j=1}^{n}\lambda_j\, y_{rj} - s_r^+ = y_{ro}, & r = 1, \dots, s \\
\lambda_j \geq 0, & j = 1, \dots, n \\
s_i^- \geq 0, & i = 1, \dots, m \\
s_r^+ \geq 0, & r = 1, \dots, s
\end{array} \qquad (2.12)$$

where θ_i is the contraction variable for i^{th} input and φ_r is the expansion variable for r^{th} output.

Note if $x_{io} = 0$, then $\frac{s_i^-}{x_{io}}$ will be eliminated from objective function and if $y_{ro} \leq 0$, then it is possible to change it with a small positive scalar and so $\frac{s_r^+}{y_{ro}}$ plays a penalty role. It can be observed that by increasing s_i^-, s_r^+ while the other variables are supposed to be fixed, the value of objective function will be decreased, and it shows that the slack base measure of efficiency is monotone decreasing. Moreover, $0 \leq \rho \leq 1$.

Definition 2.6. Efficiency in SBM model

A DMU is SBM efficient, if and only if the optimal value of objective function is equal to one ($\rho^* = 1$) which means the optimal value of the input and output slacks are equal to zero; $(s_i^{-*}, s_r^{+*}) = 0$.

SBM model can be transformed to a linear mathematical programming by introducing a positive scalar variable (t) as follows:

minimize $\tau = t - \frac{1}{m}\sum_{i=1}^{m}\frac{ts_i^-}{x_{io}}$

Subject to:

$t + \frac{1}{r}\sum_{r=1}^{s}\frac{ts_r^+}{y_{ro}} = 1,$

$\sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}, \qquad i = 1, \dots, m$

$\sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro}, \qquad r = 1, \dots, s$

$\lambda_j \geq 0, \qquad j = 1, \dots, n$

$s_i^- \geq 0, \qquad i = 1, \dots, m$

$s_r^+ \geq 0. \qquad r = 1, \dots, s \qquad (2.13)$

Model (2.13) can be transformed into a linear model by defining $S^- = ts^-$, $S^+ = ts^+$, $\Lambda = t\lambda$ as shown in the following mathematical programming:

minimize $\tau = t - \frac{1}{m}\sum_{i=1}^{m}\frac{SS_i^-}{x_{io}}$

Subject to:

$t + \frac{1}{r}\sum_{r=1}^{s}\frac{S_r^+}{y_{ro}} = 1,$

$\sum_{j=1}^{n}\Lambda_j\, x_{ij} + S_i^- = tx_{io},$ $\quad i = 1, \ldots, m$

$\sum_{j=1}^{n}\Lambda_j\, y_{rj} - S_r^+ = ty_{ro},$ $\quad r = 1, \ldots, s$

$\Lambda_j \geq 0,$ $\quad j = 1, \ldots, n$

$S_i^- \geq 0,$ $\quad i = 1, \ldots, m$

$S_r^+ \geq 0,$ $\quad r = 1, \ldots, s$

$t > 0.$ (2.14)

Let $(\tau^*, t^*$,Λ^*,$S^{-^*}, S^{+^*})$ be an optimal solution of model (2.14). Hence the optimal solution of SBM model is defined by $\rho^* = \tau^*, \lambda^* = \frac{\Lambda^*}{t^*}$, $s^{-^*} = \frac{S^{-^*}}{t^*}$ and $s^{+^*} = \frac{S^{+^*}}{t^*}$.

The optimal value of SBM model (ρ^*) is not greater than the optimal value of CCR model (θ^*) because the CCR model does not account slacks in efficiency measure and accounts only for purely technical inefficiency whereas the SBM model accounts the input and output slacks into efficiency measurement and therefore SBM model account for all inefficiencies. If the SBM model is assumed in variable returns to scale, it can be easily expressed by adding the convexity constraint ($\sum_{j=1}^{n}\lambda_j = 1$) into the formulation (2.12) (Cooper et al., 2007).

The following mathematical programming is the dual program of SBM model:

maximize $\sum_{r=1}^{s} u_r \, y_{ro} - \sum_{r=1}^{s} v_i \, x_{io}$

Subject to:

$\sum_{r=1}^{s} u_r \, y_{ro} - \sum_{i=1}^{m} v_i \, x_{io} + 1 \leq s y_{ro},$

$\sum_{r=1}^{s} u_r \, y_{rj} - \sum_{i=1}^{m} v_i \, x_{ij} \leq 0, \qquad j = 1, \dots, n$

$(m x_{io}) v_i \geq 1, \qquad i = 1, \dots, m$

$u_r \geq 0, \qquad r = 1, \dots, s$

$v_i \geq 0. \qquad i = 1, \dots, m \qquad (2.15)$

DMU_O is efficient in the multiplier form of SBM model when the optimal value of objective function in the optimal solution is equal to zero (Tone, 2001).

2.3 Weakness in weight distribution and discrimination power in DEA models

Applying DEA models for a set of DMUs when the number of DMUs is less than the number of inputs and outputs may result in evaluating many of these DMUs as an efficient unit. The high number of efficient DMUs will not give an acceptable and reliable ranking of DMUs which is necessary in many real-world applications of DEA models. This is one of the main drawbacks of the conventional DEA models. On the other hand, in the multiplier DEA models a variable weight is associated with each input and output. By varying these weights, the efficiency of DMU_O can be determined by maximizing the ratio of weighted sum of its outputs to the weighted sum of its inputs. The optimal weight assessed by the conventional multiplier model such as CCR and BCC, put DMU_O in the best light compared to all other DMUs, because the model is free to set its weight to attempt to reach the efficient frontier. The flexibility in selecting weights permits the DMUs to choose very small weights and even zero for some inputs and outputs which is not acceptable, because it means some of the variables were not used in the efficiency assessment. To overcome this problem and restrict the flexibility of inputs and outputs weights, several models have been presented which are called weight restriction models. These models improve the discrimination power of DEA models.

Generally, weight restrictions will be added into the DEA models in the form of additional constraints on the weights of inputs and outputs in the multiplier model, which leads to the expansion of the production technology (Allen et al., 1997; Roll et al., 1991). In order to incorporate weight restrictions into DEA models various methods and models have been presented in the literature of DEA, a very detailed classification is given in Allen et al. (1997) and Thanassoulis et al. (2004). Absolute weight restriction model by Dyson and Thanassoulis (1988), Assurance region model of type I by Thompson et al. (1986), Assurance region type II Thompson et al. (1990), common weights model by Roll et al. (1991), super efficiency model by Andersen and Petersen (1993), cross efficiency assessment by Green et al. (1996), multi objective programs by Li and Reeves (1999) and weight restriction based on production trade-offs by Podinovski (2004), are some of the well-known weight restriction approaches.

2.4 Approaches for handling lack of discrimination in DEA models

2.4.1 Absolute weight restriction

Dyson and Thanassoulis (1988) were the first who introduced absolute weight restrictions which the inputs and outputs weight can only vary in a specific range. In other words, this method defines lower and upper bounds for weight factors related to inputs or outputs. The absolute weight restrictions are defined as follows:

$$\rho_r \le u_r \le \eta_r. \qquad\qquad \delta_i \le v_i \le \tau_i.$$