CHAPTER 1

CLASSICAL THEOREMS AND RESULTS IN GEOMETRIC FUNCTION THEORY

Geometric function theory has a lengthy history in complex analysis. It all started in the early 1900s. The study of geometric properties of complex-valued functions, known as function theory, uses several analytical tools. This first chapter introduces important terms and outcomes that will be used later in the thesis. These are the results of the analytic and harmonic mapping studies.

1.1 Analytic Univalent Maps

The complex plane is denoted by $\mathbb{C}$ in this thesis. Also, let

$$U_r(z_0) =: \{z \in \mathbb{C}, |z - z_0| < r\} \quad (r > 0),$$

be the neighborhood of point z_0. In particular, when $r = 1$ and $z_0 = 0$, we write $U_1(0) = U$ which denotes the open unit disk centered at the origin. Let $\mathscr{H}(U)$ be the class of analytic functions $f : U \to \mathbb{C}$. Further, let $\mathscr{A}$ represent the class of all normalized analytic functions $f \in \mathscr{H}(U)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Definition 1.1.1. *[1, p. 5] A function $f \in \mathscr{H}(U)$ is said to be univalent (one-to-one) if f takes different points in U to different values, that is, for any two distinct points z_1*

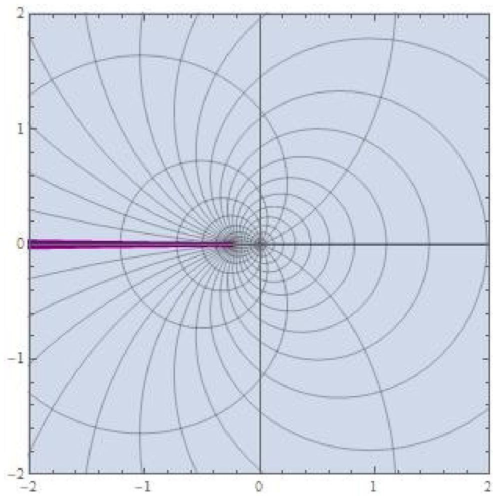


Figure 1.1: Image of U under Koebe map $k(z)$.

and z_2 *in* U *with* $z_1 \neq z_2$, $f(z_1) \neq f(z_2)$.

The property of “locally univalent" in the function $f \in \mathscr{H}(U)$ implies its univalent in some neighborhoods at an arbitrary point in U. The condition that $f'(z) \neq 0$ in U is both necessary and sufficient for local univalence. We are interested to study the following class of functions:

$$\mathscr{S} =: \left\{ f \in \mathscr{A} : f \text{ is analytic and one-to one}, f(0) = f'(0) - 1 = 0 \right\}.$$

The class $\mathscr{S}$ is the main aspect of the classical mathematics branch known as the univalent functions theory. A good example of the functions that belong to the class $\mathscr{S}$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \dots. \tag{1.2}$$

This Koebe function maps U onto $\mathbb{C} \backslash (-\infty, -\frac{1}{4}]$, see figure (1.1). The Koebe function is an extremal function for the class $\mathscr{S}$. The origin of this discipline of mathematics may be traced back to 1914, with Gronwall's demonstration of the Area theorem [2, P. 57-59]. In 1916, the Bieberbach Conjecture served as the subject's cornerstone. Numerous mathematicians attempted to solve this Conjecture somewhat, and the summary of the attempts is given in Table 1.1 D. Horowit [8] yielded the best result by proving that

Table 1.1: Coefficients Bound for $f \in \mathscr{S}$

Bieberbach [3]	$\|a_2\| < 2$
Lowner [4]	$\|a_3\| < 3$
Garabedian and Schiffer[5]	$\|a_4\| < 4$
Pederson and Schiffer [6]	$\|a_5\| < 5$
Ozawa [7]	$\|a_6\| < 6$

$|a_n| < 1.0657n$ using Carl FitzGerald's [9] deep technique. Finally, in 1985, Louis de Branges proved the Bieberbach's conjecture:

Theorem 1.1.1. *[10, Theorem 3] For each $f \in \mathscr{S}$, the coefficients satisfy $|a_n| \leq n$ for all n. Equality holds for any rotated Koebe function.*

1.2 Convex in One Direction

It should be recalled that a function g on U is convex in the horizontal direction CHD ($\mathbb{R}$-convex) if every line parallel to the real axis has a connected intersection with $g(U)$, and a function g is convex in the imaginary direction CID if every line parallel to the imaginary axis has a connected intersection with $g(U)$. Furthermore, a function g is convex in the direction $e^{i\theta}$ if for every z_0 in $\mathbb{C}$ the set $g(U) \cap \{z_0 + se^{i\theta} : s \in \mathbb{R}\}$ is either connected or empty. Figure 1.2 shows how the lines would be if the domain was convex towards the horizontal and imaginary axis, respectively.

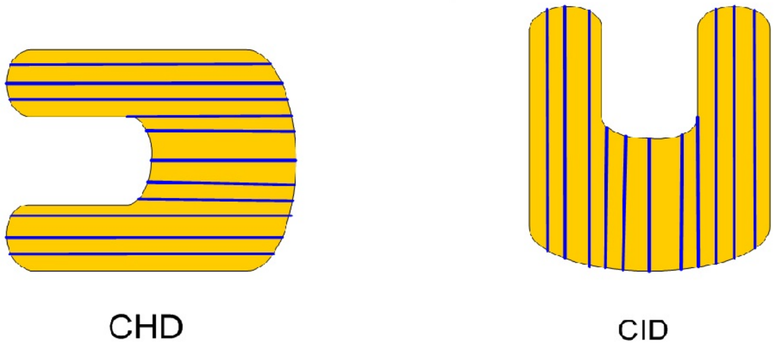


Figure 1.2: Regions convex in the horizontal and imaginary directions, respectively

In [11], Robertson has proved that the function g on U is convex in one direction if and only if $zg'(z)$ is starlike in one direction. An improvement on Robertson's result was done by Royster and Ziegler [12] by referring to [13]:

Theorem 1.2.1. *[12] Let f be a non-constant function analytic in U. The function f maps U univalent onto a domain D convex in the direction of the imaginary axis if and only if there are numbers μ and ν, $0 \le \mu \le 2\pi$ and $0 \le \nu \le \pi$, such that*

$$Re\{-ie^{i\mu}(1-2\cos\nu\, e^{-i\mu}z+e^{-2i\mu}z^2)f'(z)\} \ge 0, \quad z \in U. \tag{1.3}$$

Furthermore, $f(e^{i(\mu-\nu)})$ and $f(e^{i(\mu+\nu)})$ are the right and left extremes, respectively, of D, that is,

$$f(e^{i(\mu-\nu)}) = \sup_{|z|<1} \mathrm{Re} f(z) \quad \text{and} \quad f(e^{i(\mu+\nu)}) = \inf_{|z|<1} \mathrm{Re} f(z).$$

The parameters μ and ν in (1.3) determine the orientation, stretching, and curve of the convex domain. The inequality restriction also further restricts the shape of the mapped domain by requiring it to lie on one side of the imaginary axis. This may reduce the range of μ and ν that give possible mappings satisfying the inequality. It is worth noting that if we set $\mu = \nu = \pi/2$, the inequality (1.3) reduce to

$$\mathrm{Re}\,\{(1-z^2)f'(z)\} \geq 0, \quad z \in U,$$

On the other hand, if g is a $\mathbb{R}$-convex function, then ig is a function convex in the direction of the imaginary axis. By letting $f = ig$ and $\mu = \nu = 0$, the inequality (1.3) reduces to

$$\mathrm{Re}\,\{(1-z)^2 g'(z)\} \geq 0, \quad z \in U, \tag{1.4}$$

where $f(1) = ig(1)$ is both the right and left extreme. Indeed, it is simple to prove that g is $\mathbb{R}$-convex if g satisfies (1.4) by utilizing the ideas from [13]. When equality holds in (1.4) for some $z \in U$, then $g(U)$ is a horizontal strip. Suppose that (1.4) has strict inequality. That strict inequality then corresponds to

$$\mathrm{Im}\left\{(1-z)^2 f'(z)\right\} > 0, \quad \text{for all } z \in U. \tag{1.5}$$

Now, for $t > 1/2$, consider the circle

$$C_t = \left\{ z \in \mathbb{C} : z(s) = 1 - \frac{1}{t+is},\ -\infty < s < \infty \right\},$$

which lies in the unit disk U and tangent to 1. If $z \in C_t$, then

$$z'(s) = \frac{i}{(t+is)^2} = i(1-z(s))^2.$$

Thus, by (1.5),

$$\begin{aligned}\frac{\partial}{\partial s}\text{Im}\,\{g(z(s))\} &= \text{Im}\,\left\{(i(1-z(s))^2)g'(z(s))\right\} \\ &= \text{Im}\,\left\{(1-z(s))^2 f'(z(s))\right\} > 0.\end{aligned}$$

This implies that $g(C_t)$'s are non-intersecting analytic arcs that can be represented as functions of a single variable. Using prime end theory, it can be said $f(U)$ is $\mathbb{R}$-convex (see [13] for details).

Lemma 1.2.1. *[14, Theorem 4.87] Suppose g is a non-constant analytic function in U that satisfies the condition $g(0) = 0$ and $g'(0) \neq 0$, and let*

$$\varphi(z) = \frac{z}{(1+ze^{i\theta_1})(1+ze^{-i\theta_2})}, \quad (\theta_i, i = 1,2 \in \mathbb{R}).$$

Then g is $\mathbb{R}$-convex if

$$\text{Re}\left(\frac{zg'(z)}{\varphi(z)}\right) > 0, \quad (\forall z \in U). \tag{1.6}$$

1.3 Harmonic Univalent Maps

A complex-valued function f defined on a domain $\Omega \subset \mathbb{C}$ is said to be harmonic if

$$\Delta f = 4\left(\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}\right)\right) = 0, \tag{1.7}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.8}$$

The differential operators in (1.8) are known as Wirtinger derivatives, For further details, we refer to [14, Also see [15]].

The following canonical representation of a harmonic function is also important

Theorem 1.3.1. *[15, Canonical Representation] The harmonic mapping f of U has canonical representation $f = h + \overline{g}$, such that h and g are analytic in U. The representation is unique up to an additive constant.*

Note that the canonical representation gives a series representation of f in terms of h and g,

$$f(z) = h(z) + \overline{g}(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \tag{1.9}$$

where h and g are known as the analytic part and co-analytic part of f, respectively.

The Jacobian of the complex-valued function $f(x,y) = u(x,y) + iv(x,y)$ at any point

$z = x + iy$ in U, is given by

$$J_f(z) =: \begin{vmatrix} u_x(z) & v_x(z) \\ u_y(z) & v_y(z) \end{vmatrix} = u_x(z)v_y(z) - v_x(z)u_y(z),$$

assuming that all of the partial derivatives exist at z. We can express the Jacobian in terms of z and $\bar{z}$ derivatives

$$J_f(z) = |h'|^2 - |g'|^2. \tag{1.10}$$

If $J_f(z) > 0$ for all $z \in U$, then f is said to be sense-preserving, and if $J_f(z) < 0$, it is said to be sense-reversing for all z in U. For more result on $J_f(z)$, one may refer to [16, 17]. Furthermore, the second complex dilatation of the function f at an arbitrary point z in U is an analytic function $\omega_f : U \to \mathbb{C}$ given by

$$\omega_f(z) = \frac{g'(z)}{h'(z)}, \quad z \in U, \tag{1.11}$$

that represents a ratio between the real part of f and the imaginary part of f. This means it measures how much the mapping f stretches or shrinks angles locally around z. In the other direction, it measures how much the conformal function distorts. Recall that the mapping f is said to be conformal if it maps an infinitesimal square in its base domain to an infinitesimal square in its range. If the $|\omega_f|$ lies in the unit disk, then the $J_f > 0$ implies that the $|g'/h'| < 1$ subsequently f is sense-preserving (conformal).

The class $\mathscr{H}(U)$ is closed under the product and composition rule, but this is not true for the class of harmonic functions. We have the following result:

Theorem 1.3.2. *[18, Proposition 2]*

(I) Let $f : \Omega \to \mathbb{C}$ be analytic, and $g : U \to \Omega$ be harmonic. Then $f \circ g : U \to \mathbb{C}$, is harmonic.

(II) Let Let $f : \Omega \to \mathbb{C}$ be harmonic, and $g : U \to \Omega$ be analytic. Then $f \circ g : U \to \mathbb{C}$, need not be harmonic.

Proof. (I) Letting $h = f \circ g$ where g is an analytic function with $\zeta = g(z)$ and f is a harmonic. Since

$$h_z = \frac{dh}{dz} = \frac{df(\zeta)}{dz} = \frac{df(\zeta)}{d\zeta}\frac{dg(z)}{dz},$$

it follows that

$$\begin{aligned}
h_{z\bar{z}} &= \frac{d}{d\bar{z}}\left(\frac{dh}{dz}\right) \\
&= \frac{df(\zeta)}{d\zeta}\frac{d}{d\bar{z}}\left(\frac{dg(z)}{dz}\right) + \frac{d}{d\bar{z}}\left(\frac{df(\zeta)}{dz}\right)\frac{dg(z)}{dz} \\
&= 0 + \frac{d}{d\bar{z}}\left(\frac{df(\zeta)}{dz}\right)\frac{dg(z)}{dz} \quad \text{(since } g \text{ is analytic)} \\
&= \frac{d}{d\bar{z}}\left(\frac{df(\zeta)}{dz}\right)\frac{dg(z)}{dz} = 0 \quad \text{(since } f \text{ is harmonic)}.
\end{aligned}$$

(II) Let $f(z) = z + (1/4)\bar{z}$ be a harmonic function and $g(z) = z^2$ be another analytic function. Thus, it obvious that $f_{z\bar{z}} = 0$, but $(g \circ f)_{z\bar{z}} = 1/2$. Hence, $(g \circ f)$ is not a harmonic function.

□

Suppose that the mapping $f = h + \overline{g}$ is harmonic on domain $\Omega \subset \mathbb{C}$. According to the Riemann mapping theorem, there exists a bijective function $\varphi : \mathscr{D} \to \Omega$. Since f is harmonic and φ is analytic one can infer that the composed $f \circ \varphi$ maps the unit disc U harmonically onto the region Ω of the complex plane $\mathbb{C}$. If $f = h + \overline{g}$, $h'(0) = f_z(0) \neq 0$ is a sense-preserving harmonic in $\mathscr{D}$, then the normalized form of f given by

$$\frac{f(z) - f(0)}{f_z(0)}.$$

Hence, a normalized harmonic function f has the following series representation:

$$f(z) = h(z) + \overline{g}(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}. \tag{1.12}$$

Let $\mathscr{S}_H$ denote the class of all normalized sense-preserving one-to-one harmonic mappings f defined on U of the form (1.12). Similarly, let $\mathscr{S}_H^0$ denote the class of all normalized sense-preserving one-to-one harmonic mappings f of the form (1.12) with $b_1 = 0$. Then we have

$$\mathscr{S} \subset \mathscr{S}_H^0 \subset \mathscr{S}_H.$$

Note that a sense-preserving harmonic function $f \in \mathscr{S}_H$ (or $f \in \mathscr{S}_H^0$) is in class $\mathscr{K}_H$ (or$\mathscr{K}_H^0$) if it maps U onto a convex domain. Likewise, a sense-preserving harmonic function $f \in \mathscr{S}_H$ (or $f \in \mathscr{S}_H^0$) is in class $\mathscr{S}_{CHD}$ (or$\mathscr{S}_{CHD}^0$) if its range is convex in the direction of the real axis, and it will be in the class $\mathscr{S}_{CID}$ (or$\mathscr{S}_{CID}^0$) if its range is convex in the direction of the imaginary axis. More can be found in [19–21].

1.4 Harmonic Shears

Clunie and Sheil-Small developed the “shear construction" for producing harmonic mappings. The procedure “shears" a conformal mapping along parallel lines to construct a harmonic mapping with a predetermined dilation onto a region that is convex in one direction. The following result contains the shear construction.

Theorem 1.4.1. *[19, Theorem 5.3] If h, g are holomorphic functions on U, such that $|g'(z)/h'(z)| < 1$ for all $z \in U$, then the harmonic mapping of the form $f = h + \overline{g}$ is univalent and maps U onto a $\mathbb{R}$-convex domain if and only if the analytic mapping $h - g = \varphi$ is univalent and maps U onto a domain that is convex in the same direction.*

To use the shearing construction, assume φ is a conformal mapping that maps U onto a domain convex in the horizontal direction. The harmonic shear $f = h + \overline{g}$ of φ is given by the solution of differential equations

$$\begin{cases} h' - g' = \varphi' \\ g' = \omega_f h', \end{cases}$$

for a given complex dilatation ω_f. As a result of solving the preceding set of equations, it follows that

$$h(z) = \int_0^z \frac{\varphi'(\zeta)}{1 - \omega_f(\zeta)} d\zeta. \tag{1.13}$$

For the co-analytic part g, we obtain

$$g(z) = \int_0^z \frac{\omega_f(\zeta)\varphi'(\zeta)}{1 - \omega_f(\zeta)} d\zeta. \tag{1.14}$$

Note that

$$
\begin{aligned}
f(z) = h(z) + \overline{g}(z) &= \int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta + \overline{\int_0^z \frac{\omega_f(\zeta)\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta} \\
&= \int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta + \overline{\int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta} - \overline{\int_0^z \varphi'(\zeta) d\zeta} \\
&= \int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta + \overline{\int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta} - \overline{\varphi(z)}.
\end{aligned}
$$

Hence, it is obvious that harmonic mapping can be expressed as

$$
f(z) = 2\mathrm{Re}\left\{ \int_0^z \frac{\varphi'(\zeta)}{1-\omega_f(\zeta)} d\zeta \right\} - \overline{\varphi(z)}. \tag{1.15}
$$

The last equation is only applicable if the conformal mapping φ is known. Next, we have examples for some univalent harmonic functions mapped onto the $\mathbb{R}-$convex domain with the given second complex dilatation ω_f.

Example 1. *Let $\varphi = z/(1-z)$ be a univalent analytic map of the unit disk U onto a convex domain; see Figure 1.3(a). We will build a univalent harmonic function $f = h + \overline{g}$ with second complex dilatation $\omega_f(z) = z$ ($|\omega(z)| < 1$). Using* (1.15)*, it follows that*

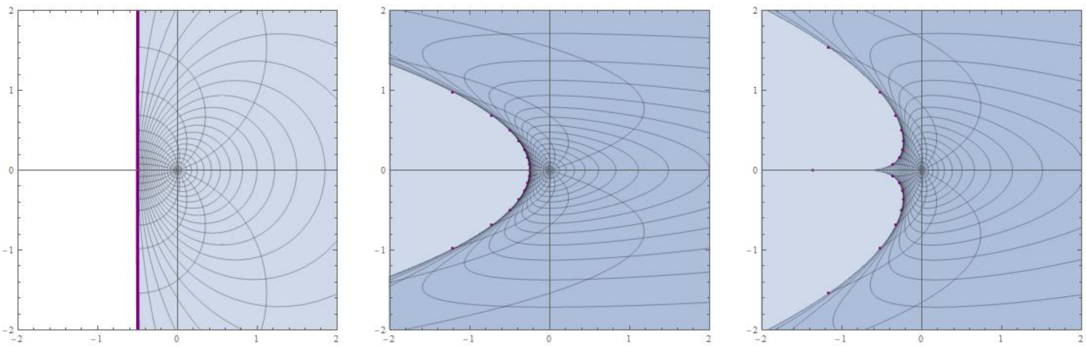
$$
\begin{aligned}
f(z) &= 2\mathrm{Re}\left\{ \int_0^z \frac{1}{(1-\zeta)^2(1-\zeta)} d\zeta \right\} - \overline{\left(\frac{z}{1-z}\right)} \\
&= \mathrm{Re}\left\{ \frac{2z-z^2}{(1-z)^2} \right\} - \overline{\left(\frac{z}{1-z}\right)}.
\end{aligned}
$$

Example 2. *Let $\varphi = z/(1-z)$ be a univalent convex function. We are going to construct a univalent harmonic function $f = h + \overline{g}$ with second complex dilatation $\omega_f(z) = z^2$. Then f will be univalent harmonic and sense-preserving with $|\omega(z)| < 1$.*

Using (1.15)*, it follows that*

$$f(z) = 2\text{Re}\left\{\int_0^z \frac{1}{(1-\zeta)^2(1-\zeta^2)}\, d\zeta\right\} - \overline{\left(\frac{z}{1-z^2}\right)}$$
$$= \frac{1}{4}\text{Re}\left\{\left(\frac{6z-4z^2}{(1-z)^2} + \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right)\right\} - \overline{\left(\frac{z}{1-z}\right)}$$

Figures (b) and (c) in Figure 1.3 show that applying different dilatations to the same



(a) Image of $\varphi(U)$ (b) Image of $f(U)$ when $\omega_f = z$ (c) Image of $f(U)$ when $\omega_f = z^2$

Figure 1.3: Image of U under φ, and f that is shear of $z/(1-z)$ with diffrent dilatations.

harmonic function can lead to very different ranges.

Theorem 1.4.2. *[22, Theorem 1.2] Let $\theta \in [0, \pi)$. A harmonic $f = h + \bar{g}$ locally univalent in U is a univalent mapping of U onto a domain convex in the direction of θ for some $0 \leq \theta < 2\pi$ if and only if $h - e^{2i\theta} g = \varphi$ is a conformal univalent mapping of U onto a domain convex in the direction of θ.*

Clearly, when $\theta = 0$, $h = \varphi + g$ as well as leads to $f = \varphi + 2\text{Re}\{g\}$. As a result, one can assert that f is not equal to φ by the addition $2\text{Re}\{g\}$. Geometrically, this can be represented by cutting φ into horizontal slices and continuously translating or scaling each slice to obtain f. This matches the “shear" name in the theorem. As can be seen from [23–45], the harmonic shear is a focus of the latest researches.

1.5 Harmonic Convolutions

The convolution of harmonic mappings can be regarded as a generalization of the convolution of conformal mappings. This has piqued the interest of complex researchers, and many interesting theorems have been published [46–66]. However, there is no equivalent consequence to the harmonic convolution results. For example, Ruschewey and Sheil show that convexity is preserved under analytic convolution but not preserved under harmonic convolution.

Definition 1.5.1. *[14, Convolutions] Let f and F be analytic function on U given by*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \textit{and} \quad F(z) = \sum_{n=1}^{\infty} A_n z^n.$$

Then the Hadamard product (convolution) of f and F is defined as

$$f(z) * F(z) = \sum_{n=1}^{\infty} a_n A_n z^n.$$

For any analytic function $F(z) = \sum_{n=1}^{\infty} a_n z^n$ with $F(0) = 0$, we have

$$\begin{aligned} F(z) * \frac{z}{(1-z)} &= \sum_{n=1}^{\infty} a_n z^n * \sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} a_n z^n = F(z), \\ F(z) * \frac{z}{(1-z)^2} &= \sum_{n=1}^{\infty} a_n z^n * \sum_{n=1}^{\infty} n z^n = z \sum_{n=1}^{\infty} n a_n z^{n-1} = zF'(z). \end{aligned} \tag{1.16}$$

Also,

$$F(z) * \log\left(\frac{1+z}{1-z}\right) = \int_0^z \frac{F(\zeta) - F(-\zeta)}{\zeta} d\zeta. \tag{1.17}$$

To see how (1.17) constructed, note that

$$\log(1+z) = \int \frac{1}{1+z} dz = \int \left(\sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1}\right) dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n,$$

and

$$\log(1-z) = \int \frac{-1}{1-z} dz = -\int \left(\sum_{n=1}^{\infty} z^{n-1}\right) dz = -\sum_{n=1}^{\infty} \frac{1}{n} z^n.$$

Summing both equations will then give

$$\log\left(\frac{1+z}{1-z}\right) = \log(1+z) - \log(1-z) = \sum_{n=1}^{\infty} \frac{(1+(-1)^{n-1})}{n} z^n.$$

Therefore,

$$\begin{aligned} F(z) * \log\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty} a_n z^n * \sum_{n=1}^{\infty} \frac{(1+(-1)^{n-1})}{n} z^n = \sum_{n=1}^{\infty} \frac{(1+(-1)^{n-1})}{n} a_n z^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n} a_n z^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a_n z^n = \sum_{n=1}^{\infty} \frac{1}{n} a_n z^n - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_n z^n \\ &= \int_0^z \frac{F(\zeta)}{\zeta} d\zeta - \int_0^z \frac{F(-\zeta)}{\zeta} d\zeta. \end{aligned}$$

In the end, we will obtain (1.17). Using the same approach, we obtain

$$F(z) * \log\left(\frac{1+iz}{1-iz}\right) = \int_0^z \frac{F(-i\zeta) - F(i\zeta)}{\zeta} d\zeta. \tag{1.18}$$

The concept of convolution can be generalized to planar harmonic mappings.

Definition 1.5.2. *[14, Definition 4.120] For harmonic univalent functions*

$$f = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$$
$$F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n,$$

their harmonic convolution is given by

$$(f \widetilde{*} F)(z) = (h * H)(z) + \overline{(g * G)}(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{B}_n \overline{z}^n \tag{1.19}$$

Example 3. *Let the mapping* $f_1 = h_1 + \overline{g}_1$ *where* $h_1 + g_1 = z$ *and* $\omega_1 = z$ *. Then*

$$h_1 = \log(1+z) \quad \textit{and} \quad g_1 = z - \log(1+z).$$

Also, let the mapping $f_2 = h_2 + \overline{g}_2$ *where* $h_2 + g_2 = 1/2\log((1+z)/(1-z))$ *and* $\omega_2 = z^2$*. Then*

$$h_2(z) = \frac{1}{4}\left(\log\left(\frac{1+z}{1-z}\right) - i\log\left(\frac{1+iz}{1-iz}\right)\right),$$
$$g_2(z) = \frac{1}{4}\left(\log\left(\frac{1+z}{1-z}\right) + i\log\left(\frac{1+iz}{1-iz}\right)\right).$$

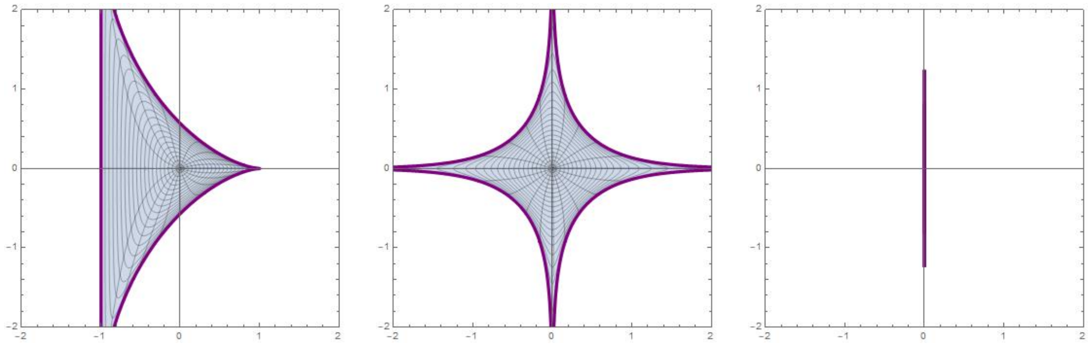
By definition (1.5.1) *and equation* (1.19)*, we have* $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$ *where*

$$h_1 * h_2 = \log(1+z) * \frac{1}{4}\left(\log\left(\frac{1+z}{1-z}\right) - i\log\left(\frac{1+iz}{1-iz}\right)\right), \tag{1.20}$$
$$g_1 * g_2 = (z - \log(1+z)) * \frac{1}{4}\left(\log\left(\frac{1+z}{1-z}\right) + i\log\left(\frac{1+iz}{1-iz}\right)\right). \tag{1.21}$$

By making use of equations (1.17) *and* (1.18)*, respectively, we obtain*

$$h_1 * h_2 = \frac{1}{4}\left(\int_0^z \frac{\log\left(\frac{1+\zeta}{1-\zeta}\right)}{\zeta}\,d\zeta + i\int_0^z \frac{\log\left(\frac{1+iz}{1-iz}\right)}{\zeta}\,d\zeta\right)$$

$$g_1 * g_2 = \frac{1}{4}\left(\int_0^z \frac{2\zeta-\log\left(\frac{1+\zeta}{1-\zeta}\right)}{\zeta}\,d\zeta + i\int_0^z \frac{2i\zeta-\log\left(\frac{1+i\zeta}{1-i\zeta}\right)}{\zeta}\,d\zeta\right).$$



(a) Image of $f_1(U)$ (b) Image of $f_2(U)$ (c) Image of $(f_1 * f_2)(U)$

Figure 1.4: Image of U under f_1, f_2, and $f_1 * f_2$.

1.6 Minimal Surfaces and Harmonic Maps

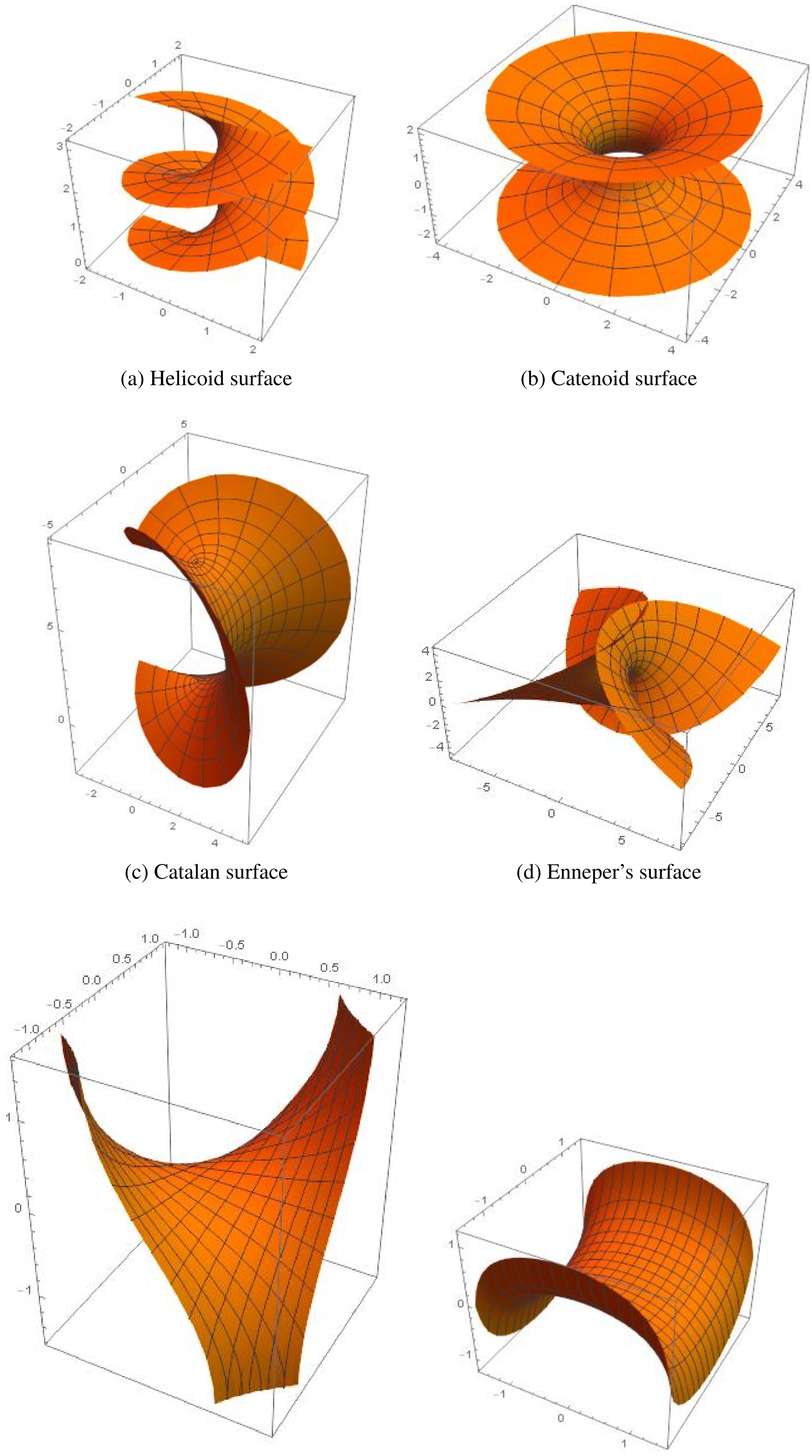
A fundamental connection between the harmonic maps and the minimal surfaces exists because the Euclidean coordinates of the minimal surface are harmonic. The projection of a minimal surface onto its base plane is known as harmonic mapping, and there is an exact formula for each harmonic mapping that may be used to obtain the surface associated with it. Due to that fact, the harmonic mappings can be used to study minimal surfaces and vice versa.

1.6.1 Minimal Surface

The minimal surfaces in three-dimensional Euclidean space have their early history in the calculus of variations developed by Euler and Lagrange in the 18th century, as well as in later studies by Enneper, Scherk, Schwarz, Riemann, and Weierstrass in the 19th century. Besides the 19th-century contributors, other mathematicians like Osserman, W Hengartner, G Schober, Peter Duren, Michael Dorff, Stacey Muir, S Ponnusamy, and A Rasila have contributed to the minimal surface theory. By setting $0 \leq u \leq 2\pi$ and $-2\pi/3 \leq v \leq 2\pi/3$ for the equation given in the Table 1.2 and using the Wolfram Mathematica software, we are able to draw the minimal surfaces shown in Figure 1.5.

Table 1.2: Numerous parametrizations of well-known minimal surfaces.

The surface's name	The surface's parametrization $X(u,v)$
Helicoid	$(a\sinh v\cos u, a\sinh v\sin u, au)$
Catenoid	$(a\cosh v\cos u, a\cosh v\sin u, av)$
Catalan	$\left(1-\cos u\cosh v, \sin\frac{u}{2}\sinh\frac{v}{2}, u-\sin u\cosh v\right)$
Enneper	$\left(u-\frac{1}{3}u^3+uv^2, v^2-\frac{1}{3}v^3+u^2v, u^2-v^2\right)$
Scherk's singly periodic	$\left(\sinh^{-1}u, \sinh^{-1}v, \sin uv\right)$
Scherk's doubly periodic	$\left(u, v, \ln\frac{\cos u}{\cos v}\right)$



(a) Helicoid surface
(b) Catenoid surface
(c) Catalan surface
(d) Enneper's surface
(e) Scherk's singly periodic surface
(f) Scherk's doubly periodic surface

Figure 1.5: Examples of minimal surfaces

Definition 1.6.1. *[67, Definition 2] The minimal surface S is one that has a mean curvature that is zero at any arbitrary point on it.*

Observe that the vanishing mean curvature on a minimal surface $X = \Phi(U)$, where $U = (u, v)$ and $X = (x, y, z)$ are points in $\mathbb{R}^2$ and $\mathbb{R}^3$ is based on the idea that a surface is a saddle surface with positive curvature in one direction being matched by negative curvature in the opposite direction. For the study of minimal surfaces, we must choose parameters that reflect the surface geometry. Thus, parameterizations of isothermal are therefore required. These parametrizations send small squares to small squares. Geometrically, isothermal parametrization requires that $X_u \perp X_v$. This condition can be expressed using the coefficients of the surface's first fundamental form:

$$E = X_u \cdot X_u = X_v \cdot X_v = G = \eta^2, \quad \text{and} \quad F = X_u \cdot X_v = 0 \quad \eta = \eta(u, v) > 0. \quad (1.22)$$

Also, main curvature H and Gauss curvature K are given by

$$H = \frac{1}{2}\left(\frac{N+L}{\eta^2}\right), \qquad K = \frac{LN - M^2}{\eta^4}. \quad (1.23)$$

Accordingly, the Gaussian curvature K can be expressed as follows:

$$K = -\frac{1}{\eta^2}\Delta(\log(\eta)). \quad (1.24)$$

This is known as Gauss's theorema egregium. Further details are contained in a book by Peter Duren [15, P. 173-175]. From (1.23), it is clear that if S is the regular minimal surface in isothermal parameters, then $2\eta^2 H = L + N$. However, minimal surfaces have vanishing mean curvature. Given the relation $\pm(L+N) = \| \Delta X \|$, we have

$$\Delta X = 0 \quad \Longleftrightarrow \quad H = 0.$$

Based on the above procedures, one can derive the following theorem:

Theorem 1.6.1. *[15, Page 165] Let $X = (x_1, x_2, x_3)$ be a regular surface equipped with the first fundamental form*

$$\| dX \|^2 = \eta^2(du^2 + dv^2), \quad \eta > 0.$$

Then x_1, x_2, x_3 are harmonic functions if and only if X is a minimal.

1.6.2 Enneper-Weierstrass Representation

In the 19th century, German mathematicians Alfred Enneper and Karl Weierstrass introduced the Enneper-Weierstrass representation of a minimal surface in $\mathbb{R}^3$, which presented an intriguing bridge between geometry and complex analysis. The Enneper-Weierstrass representation has been compulsory to the study of minimal surfaces in $\mathbb{R}^3$. The information in this section comes from the books [15] and [14].

Let S be a surface in $\mathbb{R}^3$ parametrized by $X(u,v) = (x(u,v), y(u,v), z(u,v))$. For $k = 1,2,3$, let φ_k be a real valued function define on U given by

$$\varphi_k^2 = 4\left(\frac{\partial x_k}{\partial w}\right)^2.$$

Make use of squaring the complex differential operator given in (1.8), we obtain

$$\varphi_k^2 = \left(\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\left(\frac{\partial x_k}{\partial u}\right)\left(\frac{\partial x_k}{\partial v}\right)\right),$$

which gives

$$|\varphi_k|^2 = 4\left|\frac{\partial x_k}{\partial w}\right|^2 = \left(\left(\frac{\partial x_k}{\partial u}\right)^2 + \left(\frac{\partial x_k}{\partial v}\right)^2\right).$$

Furthermore, it is obvious

$$\begin{aligned}
E = X_u \cdot X_u &= \left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial x_2}{\partial u}\right)^2 + \left(\frac{\partial x_3}{\partial u}\right)^2 = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2, \\
G = X_v \cdot X_v &= \left(\frac{\partial x_1}{\partial v}\right)^2 + \left(\frac{\partial x_2}{\partial v}\right)^2 + \left(\frac{\partial x_3}{\partial v}\right)^2 = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2, \\
F = X_u \cdot X_v &= \left(\frac{\partial x_1}{\partial u}\right)\left(\frac{\partial x_1}{\partial v}\right) + \left(\frac{\partial x_2}{\partial u}\right)\left(\frac{\partial x_2}{\partial v}\right) + \left(\frac{\partial x_3}{\partial u}\right)\left(\frac{\partial x_3}{\partial v}\right) = \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\frac{\partial x_k}{\partial v}\right),
\end{aligned}$$

and,

$$\begin{aligned}
\sum_{k=1}^{3}(\varphi_k)^2 &= \sum_{k=1}^{3}\left(\left(\frac{\partial x_k}{\partial u}\right)^2 - \left(\frac{\partial x_k}{\partial v}\right)^2 - 2i\left(\frac{\partial x_k}{\partial u}\right)\left(\frac{\partial x_k}{\partial v}\right)\right) \\
&= \left(X_u \cdot X_u - X_v \cdot X_v - 2i(X_u \cdot X_v)\right).
\end{aligned}$$

Hence,

$$\sum_{k=1}^{3}(\varphi_k)^2 = (E - G - 2iF).$$

Suppose X is given by isothermal parameters, it follows from (1.22) that

$$\sum_{k=1}^{3}(\varphi_k)^2 = 0.$$

Also,

$$\sum_{k=1}^{3}|\varphi_k|^2 = (X_u \cdot X_u + X_v \cdot X_v) = (E+G) = 2\eta^2 > 0. \tag{1.25}$$

If S is a regular minimal surface, according to Theorem 1.6.1, the coordinate x_k is harmonic, thus

$$\Delta x_k = 4\frac{\partial}{\partial \overline{w}}\left(\frac{\partial x_k}{\partial w}\right) = 4\left(\frac{\partial \varphi_k}{\partial \overline{w}}\right) = 0.$$

Hence, the functions φ_k are analytic. It results in the following theorem, which we can now describe:

Theorem 1.6.2. *Let $X =: (x_1(u,v), x_2(u,v), x_3(u,v))$ be an isothermal parametrization of a regular minimal surface S and let $w = u + iv$. Then the functions*

$$\varphi_k = 2\frac{\partial x_k}{\partial w}, \quad k = 1,2,3,$$

satisfy the conditions

$$\sum_{k=1}^{3}\varphi_k^2 = 0, \qquad \sum_{k=1}^{3}|\varphi_1|^2 > 0, \tag{1.26}$$

and analytic.

Remember that $x_k, k = 1,2,3$ is a parametrization of S given in Theorem 1.6.3, so determining x_k will be useful. To do so, let $w = u + iv$, $\overline{w} = u - iv$, then $dw = du + idv$ and $d\overline{w} = du - idv$. it follows

$$\begin{aligned}\varphi_k dw &= \frac{1}{2}\left(\frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v}\right)(du + idv) \\ &= \frac{1}{2}\left(\left(\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv\right) + i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right),\end{aligned}$$

similarly,

$$\begin{aligned}\overline{\varphi}_k d\overline{w} &= \frac{1}{2}\left(\frac{\partial x_k}{\partial u} + i\frac{\partial x_k}{\partial v}\right)(du - idv) \\ &= \frac{1}{2}\left(\left(\frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv\right) - i\left(\frac{\partial x_k}{\partial u}dv - \frac{\partial x_k}{\partial v}du\right)\right).\end{aligned}$$

This brings us to the next step

$$dx_k = \frac{\partial x_k}{\partial u}du + \frac{\partial x_k}{\partial v}dv = \varphi_k dw + \overline{\varphi_k}d\overline{w} = 2\mathrm{Re}(\varphi_k\, dw).$$

Therefore, integrating both sides yields to

$$x_k = 2\mathrm{Re}\int \varphi_k(\zeta)\, d\zeta + c_k. \tag{1.27}$$

Since the scaling and shift do not affect the geometric shape of the surface. The constants c_k and 2 could be omitted, which leads to

$$x_k = \mathrm{Re}\int \varphi_k(\zeta)\, d\zeta. \tag{1.28}$$

Consequently, we have the following theorem: