# MODELING AND DETECTING OF OUTLIER VALUES IN THE SAUDI STOCK EXCHANGE

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**by**

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## LIST OF SYMBOLS AND ABBREVIATION

| | |
|---|---|
| ACC | Accuracy |
| ANN | Artificial neural networks |
| ACF | Autocorrelation Function |
| ARIMA | Autoregressive Integrated Moving Average |
| Bl14 | Best Localized wavelet transform filter of length 14 |
| c4 | Coiflet |
| CFT | Continuous Fourier Transform |
| CWT | Continuous Wavelet Transform |
| d4 | Daubechies |
| DFT | Discrete Fourier Transform |
| DWT | Discrete Wavelet Transform |
| ELM | Extreme Learning Machines |
| FN | False negative |
| FN rate | False negative rate |
| FP | False positive |
| FP rate | False positive rate |
| FFT | Fast Fourier Transform |
| FT | Fourier Transform |
| GARCH | Generalized Auto Regressive Conditional Heteroskedasticity |
| HWT | Haar Wavelet Transform |
| HoF-D | High-Pass Filter |

| | |
|---|---|
| HP | Hodrick-Prescott Filter |
| IIF | Infinite Impulse Response Filters |
| inf. | Inflation rate |
| INF | Inflation rate |
| IDFT | Inverse Discrete Fourier Transform |
| LA8 | Least Symmetric |
| LoF-D | Low-Pass Filter |
| MCC | Matthews correlation coefficient |
| MODWT | Maximum overlapping Discrete Wavelet Transform |
| MAPE | Mean Absolute Percentage |
| MAPE | Mean Absolute Percentage Error |
| MASE | Mena Absolute Scaled Error |
| MLP | Multi-Layer Perceptron |
| Loil | Natural log oil price |
| OWT | Orthogonal Wavelet Transform |
| RBFNDDA | Radial Basis Function Network with Dynamic Decay Adjustment |
| RBFNN | Radial basis function neural network |
| Repo | Repo rate |
| RMSE | Root Mean Squared Error |
| RMSE | Root Means Squared Error |
| SNNS | Stuttgart Neural Network Simulator |
| SSE | Sum square error |
| TN | True negative |
| TP | true positive |

| | |
|---|---|
| TP rate | True positive rate |
| WT | Wavelet Transform |

## PERMODELAN DAN PENGESAN NILAI TERPENCIL DALAM PERTUKARAN SAHAM SAUDI

## ABSTRAK

Tesis ini memfokuskan kepada masalah mengesan dan membetulkan nilai terpencil dalam set data pasaran saham Arab Saudi (Tadawul). Matlamatnya adalah untuk mengenal pasti outliers tersebut dengan menggunakan dua pendekatan. Pendekatan pertama adalah berdasarkan transformasi MODWT dan diterokai dalam dua cara berbeza: pertama transformasi digunakan pada siri asal, kemudian outlier dikesan dalam butiran yang diperoleh menggunakan pagar Tukey dan sama ada digantikan dengan median atau dikeluarkan. Dalam aplikasi kedua MODWT, kami mula-mula memodelkan pulangan dengan model seperti GARCH. Kemudian pertimbangkan pekali wavelet bagi sisa untuk mengesan outlier yang dikenal pasti sebagai nilai yang berada di luar set ambang kuantil wavelet dan yang disimulasikan buat kali pertama. Siri bersih diperoleh melalui songsangan MODWT dan dimodelkan menggunakan kesesuaian model yang sesuai. Prestasi prosedur yang baik dari segi pengesanan outlier yang betul ditunjukkan, dan digunakan pada harga penutupan pasaran Saham Saudi dan indeks S&P 500. Dalam pendekatan kedua kami meneroka rangkaian saraf MLP, dan kemudian RBFNN. MLP pertama kali digunakan, dan outlier dikelaskan berdasarkan kadar repo pembolehubah ekonomi, kadar inflasi, dan logaritma harga minyak. Dengan cara yang sama, RBFNN digunakan untuk mengesan dan mengklasifikasikan outlier berdasarkan pembolehubah ini. Tetapan optimum RBFNN diperoleh dengan bantuan algoritma PSO. Kedua-dua algoritma pengelasan terbukti berjaya dan dinilai oleh pelbagai metrik, yang menunjukkan prestasi yang baik. Prestasi keseluruhan kaedah berasaskan MLP nampaknya sama baik dengan

kaedah MODWT seperti yang ditunjukkan dan disokong oleh kadar positif sebenar ke atas set data ujian sampel.

**MODELING AND DETECTING OF OUTLIER VALUES IN THE SAUDI STOCK EXCHANGE**

## ABSTRACT

This thesis focuses on the problem of detecting and correcting outliers in the Saudi Arabia stock market (Tadawul) datasets. The aim is to identify such outliers by adopting two approaches. The first approach is based on the MODWT transform and explored in two different ways: first the transform is applied to the original series, then outliers are detected in the obtained details using the Tukey fences and are either replaced by the median or removed. In the second application of the MODWT, we first model the returns with a GARCH-like model. Then consider the wavelet coefficients of residuals to detect outliers which are identified as values that fall outside a set of wavelet quantile thresholds that are simulated for the first time. The clean series is obtained through the inverse of the MODWT and modelled using an appropriate model fit. The good performance of the procedure in term of correct detection of outliers is demonstrated, and applied to the close price of the Saudi Stock market and the S&P 500 index. In the second approach we explore the MLP neural network, and then the RBFNN. The MLP is first applied, and outliers are classified based on the economic variables repo rate, inflation rate, and logarithm of oil price. In similar manner the RBFNN is applied to detect and classify outliers based on these variables. The optimal setting of the RBFNN is obtained with the help of the PSO algorithm. Both classification algorithms proved to successful and are evaluated by various metrics that show good performance. The overall performance of the MLP based method seems to be as good as the MODWT method as shown and supported by the true positive rates over the sample test dataset.

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

The development of time series analysis focused on two specific areas or applications. The first one was spectral analysis in the frequency domain of communications engineering, and the second one was the analysis of correlation in the time domain of mathematical statistics and finance. This current study being undertaken focuses largely on the decomposition in time and scale of the random behaviour of the stock market data. The analysis will be based on the wavelet transform applied to the stock market data.

According to Elfouly et al. (2006), the discrete wavelet transform, especially the Maximum Overlapping Discrete Wavelet Transform (MODWT), has been a popular and widely used methodology for the past twenty years. It has been used in various fields, such as finance, mathematical statistics, medicine, and engineering. This method systematically analyzes a time series by decomposing the series in time and scale domains. The MODWT has several properties that can perform various functions. For instance, it is able to adapt itself to capture features across a large range of frequencies and hence can be used to extract components of time series such as trend, seasonality, business cycle, and noise. Therefore, the MODWT method can capture events in the stock market data and identify or associate these events with specific time horizons and locations. In the field of trade, finance, and industry, the stock market's behaviour plays a crucial role. Researchers have introduced various methodologies and theories to explain and analyze stock market data. However, the stock market is driven by complex dynamics. This complexity increases the difficulty of making any hypothesis or assumptions about the market data. For instance, the linear time series models used for simplifying stock market data rarely produce satisfactory results.

This study will be about the application of MODWT on stock market data. In doing so, this study hopes to acquire a better understanding of the outlier values detections, removable and predictions. In addition, this study aims to improve forecasting accuracy by exploring the Neural Network models.

## 1.2 Maximum Overlapping Discrete Wavelet Transform (MODWT)

The research related to wavelet transforms (WT) is a challenging research topic since the theory of wavelet transforms was established through numerous overlapping theories in the areas of mathematics and statistics. WT appeared in the literature in the 1980s, with many applications in signal and image processing. While the Fourier transform creates a representation of a signal in the frequency domain, the wavelet transform creates a representation of the signal in both the time and frequency domain, thereby allowing efficient access to localized information about the signal and overcoming the limitations of the Fourier transforms.

The main idea of wavelets is to extend a function of interest in terms of a basic function called the mother wavelet by using dilation and translation process. Often, the scale of the dilation is chosen to be a power of two, and the sample size of the discrete-time signal to be a power of two as well. However, the MODWT, which is a modified form of the traditional DWT, can be used for any sample size Gençay et al. (2001); DWT is a powerful multiresolution tool for time-scale analysis of time series and has been used to break down an original time series into different components, each of which may carry meaningful clues from the original time series Pual Addison (2017) and Bhatnagar (2020).

Any financial market is typically described as a complex dynamic system and consists of a large number of shareholders. These shareholders encounter an unlimited number of economic factors that affect their behaviour. The main objective of the shareholders is to obtain ideal performances by responding to the system based on their past experiences and future expectations.  Because of the complexity of the financial market, the analysis of financial data is very difficult, and the MODWT could be an efficient tool to extract information from such data. The MODWT has attractive properties that are discussed in chapter two and provide great tools to understand how the information is processed in time and scale domains.

## 1.3 Stock Prices

A company’s ownership is divided into identical numbers of shares that investors can purchase are called shareholders. The market price index is a combination of the share prices of a set of companies. New York, Chicago, Frankfurt, London, Bombay, and Saudi Arabia have well-known stock markets. A company that needs funds to build a new factory or develop a new product can acquire capital by issuing its shares to investors. If the company makes a profit, part of this profit may be paid out to shareholders as a dividend per share. Share market values reflect investors’ expectations about the future dividend and capital growth of the company. A stock index is a mathematical measurement of a company's performance or a number of companies as a group. The major stock market index in the American Stock Exchange, which tracks 30% of the most vital industrial shares quoted, is the New York Stock Exchange (Tariq S. Al Shammary et al. 2020). At the same time, the Standard and

Poor’s 500 Index (S&P 500) represents the average price of the biggest 500 shares quoted on the New York Stock Exchange, American Stock Exchange, and the United States over-the-counter market.

Stock closing price refers to the last price at which a stock trades during a regular trading session. For many markets, regular trading sessions run from 9:30 a.m. to 4:00 p.m. However, a number of stock markets offer after-hours trading. Some financial publications and market data vendors use the last trade in these after-hours markets as the closing price for the day. Others, however, publish the 4:00 p.m. price as the closing price and display prices for after-hours trading separately.

## 1.4 Problem Statement

The Tadawul was established in 2007, and it has been trading ever since. The amount of traded stocks was referred to as unweighted, and it has been developing significantly. However, the market data has been experiencing considerable anomalies, such as outlier values and structural breaks in stock prices. As a result, there is a high level of risk in data modelling misfitting and uncertainties. This may practically affect the decision-makers because of a lack of accuracy in forecasting when using standard forecasting methods. Therefore, in order to attract investors and provide a conducive and stable investment environment, which is important for any stock market to be successful, more efficient methods to detect and predict outlier values are needed. This help to identify uncertainty and volatility patterns, isolate structural breaks, and achieve a high level of accuracy in forecasting. There is no doubt that forecasting stock prices have an important role in making the decision whether to invest or not. The role of forecasting is to reduce uncertainty in the future by giving a scientific vision of what the stock price will look like in the near future.

Time series forecasting is greatly affected by the choice of the appropriate model for the time series data, as it directly affects the accuracy of the predictions. Until obtaining prediction models for the time-series data that have the ability to understand the reality of the stock and provide high accuracy in the forecasting, these models must take into account all considerations related to data, whether linear or non-linear models are applied.

## 1.5 Objectives of the study

This study aims to apply mathematical method to detect and improve the forecasting accuracy of outlier values in stock prices. More specifically, in the Saudi Stock Exchange (Tadawul). The main goals of this study are:

1) To use the MODWT in detecting the outlier values by directly applying the transform to the original series.
2) To use the MODWT in detecting outliers in the residuals from GARCH-like models.
3) To explore the MLP neural network models to predict outlier values.
4) To develop RBFNN neural network model to detect the outlier values.

## 1.6 Significance of the study

Since Tadawul is a rapidly developing financial market, it still experiences difficulties in data analysis which negatively impacts its performance and affects the authorities' efforts to provide an attractive investment environment. Hence, this study suggests a method of detecting, removing, and predicting the outlier values. This will improve data analysis in Tadawul and help to improve its investment environment. One of the main contributions of this study is to demonstrate the use of the MODWT as a mathematical tool that can decompose time series data and, hence, help detect and remove outlier values in the stock market data.

In this study, the exploration of the Artificial Neural Network models is considered as well. These methods have been successfully applied in time series forecasting and are promising in anomaly detection. The significance of this study is furthermore justified by the fact that, as far as known, no research work focuses on the application of the MODWT and MLP in detecting outliers in the Tadawul returns. This study aims to contribute effectively to solving some of the financial problems in Tadawul, particularly in detecting outlier data points, which then help in forecasting accuracy. This, in turn, helps Tadawul to attract more investors.

## 1.7 Scope of the study

The scope of this study is limited to investigating the discrete wavelet methods, particularly the MODWT transform firstly applied directly to the original series without making any assumption on the underlying model and secondly applied to residuals from a class of GARCH-like models. Also, the application of machine learning algorithms is investigated, namely the MLP neural network. The dataset that is used for this study will be limited to data based on the Tadawul, but the proposed methodologies can be easily applied to other datasets.

## 1.8 Organization of the Thesis

This thesis comprises 8 chapters and is organized in the following manner. Chapter 1 introduces the context within which this study is being carried out, followed by the objectives and methodology of the study, and finally, the significance and the main contribution of this study.

In chapter 2, a literature review of the MODWT will be presented together with a detailed discussion of their respective advantages and disadvantages. This chapter highlights where and when this method of the transform will be used in the study and how the concept can be applied to any financial data time series. Also, some of the neural network models and algorithms used to detect outlier values are reviewed. The chapter will also conclude with a brief discussion of the Tadawul dataset.

Chapter 3 introduces the mathematical framework, which presents the mathematical methods and the statistical models, namely the Tukey method, WT, MODWT, Neural Network models, and other related models.

Chapters 4 and 5 are dedicated to the problem of detecting and removal of outlier values in Saudi Stock Market data using mainly the MODWT transform. In chapter 4, the MODWT is applied directly to the original return series, and then the Tukey method is applied to the reconstructed series. In chapter 5, the MODWT is applied to the residuals computed from GARCH-like models, and outliers are detected using simulated wavelets quantiles thresholds.

In chapters 6 and 7, the neural network models are explored in detecting outliers; namely, the MLP neural network algorithm is applied in Chapter 6, and the application of radial basis function neural network is applied in chapter 7.

Finally, chapter 8 concludes this study by highlighting the significance and the main contribution of this study to the analysis of stock market datainTadawul. It shows that these results can be applied to any stock market data in general. Also, it suggests areas for further future research based on the findings of this study.

# CHAPTER 2

# LITERATURE REVIEW OF WAVELET TRANSFORMS AND OTHER FILTERING METHODS

## 2.1 Introduction and Definitions

In this chapter, several important concepts related to the presence of outliers in financial time series data are discussed. In general, outliers data are abnormal values that are not consistent with the overall distribution of the underlying data. The presence of outliers in data reduces data quality and has adverse effects on the performance of the analysis models. The problem of outlier detection can be found in a wide variety of domains, such as machine monitoring, financial markets, and social network analysis.

Time series can be defined in econometrics and mathematical finance as a set of observations or random numbers that have been collected and arranged sequentially in time. Robert and David (2000) refer to these collections of observations indexed by time as a stochastic process. Discrete-time series is a sequence of observations taken at specific time periods, such as hourly, daily, or weekly. When the data can be recorded continuously, this time series is said to be a continuous-time series, such as electronic signals. Schuster (1906) was one of the earliest researchers to record time-series data, and that was monthly time-series data, as mentioned by Brockwell and Davis (2002).

Stationary time series is defined as a series which fluctuates around a constant mean, constant variance and time-independent covariances. Conversely, the non-stationary time series is defined as a series that fluctuates around a non-constant mean and\or non-constant variance over time Brockwell and Davis (2002). Most of the probability theories focus on the stationary time series, and often non-stationary time series are subject to different kinds of transformations, such as considering the first or second difference and eliminating trends. Transformation is widely used in financial time series. There are several methods to distinguish between stationary and non-stationary time series, such as examining the data graphically, running a unit root test (Dickey-Fuller test), and examining the autocorrelation function (ACF). However, it should be noted that most of the financial data are non-stationary time series. Thus, the series needs to be transformed before any model fits Mills and Markelloo (2008).

Time series analysis, including data model fitting, aims to extract essential characteristics and obtain meaningful statistical information that can be used along with past values to predict future data values before they are measured. Many models, such as the ARIMA model, GARCH models, Artificial Neural Network algorithms and fuzzy models, were proposed in the forecasting techniques by Brockwell and Davis (2002), Qu et al. (2006), Chen et al. (2006).

Nobre and Neves (2019) explored in the financial data a combination of the Principal Component Analysis (PCA), DWT and XGBoost in an attempt to create a safer system that allows achieving high returns with lower risks. The PCA is used to reduce dimensions, whereas the DWT is used to reduce noise without losing the main structure of the data. Their results show that they were able to identify buying and selling strategies and achieved an average return (49.26%).

### 2.1.1 Outliers in Time Series

Outliers in time series are data points where unexpected values arise and are associated with an important change in the data that can cause problems in applying standard statistical procedures Chandola et al. (2009) and Hawkings (1980). Outliers occur in many real-life data. For example, Vishwakarma et al. (2020) considered in their study the detection of outliers in two real-time series data of TCS stock price and

Aluminum trading prices for the period January 2, 2006, to April 12, 2015. Also, Van de Wiel et al. (2019) considered the outlier detection problem in real-time series data of water sensors from the Dutch water authority “Aa en Maas” where they used the water heights on the upper part of the weirs for the analysis.

The problem of detecting outliers is considered in a variety of fields, including personalised marketing, credit card fraud detection, and financial applications (loan approval, stock market data). The existence of outliers may be due to several causes, such as poor data quality and low-quality measurements Hoaglin et al.(1986). Detecting outlier values is useful because these values might hold important information in many applications, see Bruno and Garza (2010), Fileto et al. (2015) Giacometti and Soulet(2016), Rasheed and Alhajj (2014), but they may as well negatively bias the entire result of an analysis. In financial data such as price index, outliers are defined as extreme observations that are far away from the average value. Barnet and Lewis (1994) define outlier observations as the observations that deviate significantly from other members of the sample. Similarly, Johnson (1992) defines an outlier as an observation in a data set that seems to be inconsistent with other observations of that set of data.

Outlier detection methods can be divided into univariate methods proposed by earlier works in this field, and multivariate methods, that form most of the current research. The detection methods can also be divided into parametric and nonparametric methods Williams et al. (2002). Parametric methods require a known underlying distribution of the observations Hawkins (1980), Rousseeuw and Leory (1987), Barnett and Lewis (1994). The nonparametric methods are based on statistical estimates of unknown distribution parameters Hadi (1992), Caussinus and Roiz (1990), and these methods are usually unsuitable for data in high-dimension.

Data from financial periods are often disturbed by outliers due to the impact of rare and recurring events. Therefore, before modelling such data, cleaning the data and checking for the existence of outlier values is often started. Financial data often contain volatility which can be broadly defined as anything that is highly variable. It can also be defined as the movements of the variable under certain circumstances.; the more intense the volatility, the more the variable fluctuates over time. Volatility is connected to unpredictability, uncertainty and high risk. Generally speaking, the term volatility is associated with risk; hence high volatility is linked to poor market and disruption, thus, unfair security pricing.

Within the class of non-parametric, outlier detection methods can set apart the data mining methods, also called distance-based methods. These methods are usually based on local distance measures and are capable of handling large databases Knorr and Ng (1997), Williams and Huang (1997), DuMouchel and Schonlau(1998), Jin et al. (2001), Williams et al. (2002), Bay and Schwabacher (2003).

Another class of outlier detection methods is founded on clustering techniques, where a cluster of small sizes can be considered a clustered outliers Kaufman and Rousseeuw (1990), Ng and Han (1994), Barbara and Chen (2000), Acuna and Rodriguez (2004). Hu and Sung (2003) proposed a method to identify both high and low-density pattern clustering, further partitioning this class into hard and soft classifiers. The former partition the data into two non-overlapping sets: outliers and non-outliers. The latter offers a data outlyingness ranking Schiffman et al. (1981), Ng and Han (1994), Shekhar et al. (2001), Shekhar and Chawla (2002), Lu et al. (2003).

Yuan (2018) applied machine learning methods to financial data and aimed to predict stock by comparing the long short-term memory (LSTM), gated recurrent units (GRU), support vector machine (SVM), and extreme gradient boosting (XGBoost). The results show that the recurrent neural network outperforms in time-series prediction. Both LSTM and GRU show higher accuracy rates than the other two classification methods. Especially for gated recurrent units, its accuracy rate is around 5% higher than SVM and XGBoost.

### 2.1.2 Time Series Filtering

The concept of filtering was developed in engineering, and it aims to eliminate and reduce the random variations in low and high-frequency data without changing the true pattern of the data. The filtering method has been applied in the decomposition and forecasting of time series. In forecasting, the filtering methods involve one or more parameters that are used to weigh historical values or residuals in the series. More generally, filters are used in economics and financial series to eliminate and extract a time series component such as business cycle, noises, trends, and seasonality Makridakis et al. (1998); Hamliton (1994). A linear time series filter converts the input time series into another time series (output). The series is regarded as the output of the convolution of the linear filter used by Gencay et al. (2002).

There are many types of filtering methods that have been introduced in finance and economic time series. Gencay et al. (2002) introduce the Hodrick-Prescott Filter (HP). This filter is widely used to identify the business cycle component of a macroeconomic time series. The most common filtering techniques used are the Infinite Impulse Response Filters (IIF) and the Non-Causal Infinite Impulse Response Filters (IIF), which are also linear filters commonly used in the analysis of prices in the financial market. There are as well the Low-Pass, High-Pass and Band Pass Filters that are used. The Fourier Transform Filter is one of the most prominent filters in the frequency domain and time series analysis. The Wavelet Transform is another powerful filtering method that focuses on the time-frequency domain and is widely used to decompose discrete-time series data.

## 2.2 Spectral Analysis

Spectral analysis can be defined as a transformation function that transforms the time series data into a components series, i.e. new time series. In particular, it is a variance analysis tool that shows the time series as a sum of sines and cosines of different frequencies and amplitudes. The spectral density function has several applications in applied mathematics and engineering Ding li (2003).

Percival and Walden (1993) explored the time series analysis in the spectral domain on the basis of the spectral density function and statistical properties of the autocorrelation function. The spectral analysis aims to estimate and study the spectrum. Under the stationarity assumption, the Fourier transform is the best example among spectral analysis techniques since it has the property of time-shift invariant. Since (2008) several research works have been published on spectral analysis. Empirical research has been examined for the foreign exchange market based on the multidimensional time series and spectral distances Sato (2008). Moreover, using German data, Uebele and Ritschl (2009) used the spectral analysis method to examine the movement of the national income and financial markets. In addition, Jouini (2009) applied the spectral density method to detect structural breaks or regime shifts with their locations using Tunisian financial time series data and found that the unconditional volatility of the series does not appear to be constant.

## 2.3 Fourier Transform

Joseph Fourier showed that any $2\pi$ -periodic function could be written as a sum of sinusoidal components with suitable coefficients. This space of square-integrable functions on the interval $[0, 2\pi]$, denoted by $L_2[0,2\pi]$, was later introduced by Henri Lebesgue. For example

$$\frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}...$$

is an orthonormal basis for this space Gencay et al., (2002); Briggs and Henson, (1995) and Vuorenmaa(2004).

The expression $f$ means a function $f : R \to R$, $L_2^r[-\pi,\pi]$ and $L_1^r[-\pi,\pi]$ denote the subspaces of real-valued elements of $L_2^r[-\pi,\pi]$ and $L_1^r[-\pi,\pi]$ respectively.

Suppose that $f \in L_1^r[-\pi,\pi]$ then, it can be approximated by its trigonometric form of the Fourier series, (the superscript $r$ refers to real-valued elements of $L_1^r[-\pi,\pi]$ , Bachman et al., (2000).

$$\frac{a_0}{2} + \sum_{n\in N} a_n \cos nt + \sum_{n\in N} b_n \sin nt, \tag{2.1}$$

Where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt.$$

$a_n$ and $b_n$ are the sine and cosine Fourier coefficients respectively, $n \in Z, t \in R.$

Fourier transform is a special kind of filtering method that approximate functions defined for all real numbers $t$ instead of the closed interval $[-\pi,\pi]$ for any

$f \in L_1(R)$. Bachman et al. (2004) explored some of the main properties of the Fourier transform, such as energy preservation. Moreover, it is a linear isometric function from $L_2(R)$ to $L_2(R)$. Isometric function is a linear map $A: X \to Y$ (where $X$ and $Y$ are normed space) satisfying the condition of Bachman et al.(2004):

$$\|AX\| = \|X\| \text{ For every } x \in X. \tag{2.2}$$

Because it contains imaginary numbers, $i = \sqrt{-1}$, Fourier transforms have a strong relationship with complex numbers.

This filter consists of Fourier transform equation of $x_t$ and inverse Fourier transform equation. These two equations constitute the Fourier representation of the sequence $x_t$ , and, together, are called the Fourier transform pair.

When non-stationary series are used, as with data from financial time series, the data are divided into short sections and then treated as quasi-stationary data. In this case, the spectral power cannot be accessed directly, and the Fourier transform estimations through spectral analysis can be applied, with Periodograms being a widely used approach. This technique relies on the data while not returning to the first estimates of the autocovariance sequence. According to Kokoszkaa and Mikosch (2000), the periodograms for a finite sequence $(x_t)_{t=1,\ldots,n}$ can be defined as follows:

$$I_{n,x}(\lambda) = C_{n,x}^2(\lambda) + S_{n,x}^2(\lambda), \tag{2.3}$$

where

$$C_{n,x}(\lambda) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \cos(\lambda t) x_t, \quad \text{and} \quad S_{n,x}(\lambda) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sin(\lambda t) x_{t.}$$

represent respectively cosine and sine transforms of $(x_t)_{t=1,\ldots,n}$ .

In practice, much non-stationary time series can be transformed into stationary series through mathematical transformations such as differencing and eliminating trends. There are significant issues that arise when unsuitable transformation methods are applied. Serious problems may occur if an inappropriate method is used to detrend data, as described in depth by Enders (1995).

One drawback of the Fourier transform particularly lies in its unsuitability for use in particular decomposing characteristics of stock market data, including discontinuity and sudden change, as well as various locally occurring time series abnormalities. Mainly because the Fourier transform assumes that the signal is stationary and that the signal in the sample continues into infinity. The Fourier transform performs poorly when this is not the case.

When a time series is not completely stationary, this means that applying Fourier transform as an analytical tool forces localized components with irregularities to change frequency, meaning that energy from sudden change will be partially redistributed to other frequencies, and in the process of synthesis, decomposing the time series cannot identify localized volatility from the original signal. For financial time series, this is highly relevant due to the non-stationarity of the majority of time series in this area. For this reason, the researchers are looking into other methods, such as the wavelet transform.

## 2.4 Wavelet Transform

In the previous century, research across several disciplines attempted to address the constraints and drawbacks of Fourier transformation. Approaches to resolve these issues and achieving a more flexible approach included the introduction of an innovative idea combining information on time and frequency, known as wavelets.

Unlike Fourier analysis, in which signals are analyzed using sines and cosines. When a signal is analyzed in time for its frequency content, a wavelet function is used. This function is fundamental in any wavelet transform, defined as a mathematical tool when applied in the analysis of a time series that produces multiple series over various resolutions Rioul and Vetterli (1991). The wavelet analysis is based on the concept of analyzing time-series data to form other series using a collection of wave-like functions. Applications have been found for wavelet transforms across various disciplines, including the computer sciences, compressed images/speech and applied mathematics, with more recent applications in economics and finance. In general, wavelet transforms are divided into either discrete or continuous wavelet transforms.

Grossmann and Morlet (1984) were the first to put forward the “Wavelet” notion in (1983), describing the way in which an arbitrary square-integrable real-valued function of any type is suitable for decomposition to form easily managed families containing square integrable wavelets with a constant form. This means that it is possible to capture square-integrable real-valued functions through dilation and translation of the wavelet. For a function to be a wavelet, it must fulfil the admission condition: be self-reciprocal and isometric. For a wavelet function to satisfy the admission condition, it must meet two conditions: integrate to zero and must have a unit energy condition. A function’s energy is found by integrating the squared function over its domain.

A Wavelet is a wave-like function that oscillates and is localized in time. Wavelets have two basic parameters: scale (or dilation) and location. Scale defines how “stretched” or “squished” a wavelet is. This property is related to frequency as defined for waves. Location defines where the wavelet is positioned in time (or space).

Dilation and translation are used in signal analysis and can manage discrete/discontinuous signals Cascio (2007). Wavelet transform is implemented as an efficient algorithm allowing extremely rapid and simple transform calculation.

Wavelet transform is based on the so-called mother wavelet $\psi(t) \in L_2(R)$ as a basic function. Dilations and translations of this function are defined as an orthogonal basis for the space of real values square-integrable functions $L_2(R)$:

$$\psi_{s,l}(x) = 2^{\frac{-s}{2}}\psi(2^{-s}x - l)$$

The variables s and $l$ are integers that scale and dilate the mother function $\psi(t)$ to generate wavelets, such as a Daubechies wavelet family. The mother wavelet is required to meet a set of mathematical conditions Gencayet al.(2002):

a) Zero mean and square-integrable to one.
b) Regularity, which is vital for smooth reconstructions.
c) Related filters exist
d) Orthogonality
e) Compact support.

There are many possible mother wavelets Meyer (1993), which generate such a representation. Most of them cannot be written in terms of explicit functions but can be defined by the equation

$$\psi(x) = \sqrt{2}\sum_{l=-\infty}^{\infty} c_l\, \varphi(2x - l) \tag{2.4}$$

where $\varphi(t)$ is the scaling function Daubechies (1992) and is the solution of the dilation equation, e.g., Kaiser (1994); Strang and Nguyen (1995):

$$\varphi(x) = \sqrt{2}\sum_{l=-\infty}^{\infty} g_l\, \varphi(2x - l) \tag{2.5}$$

In contrast to the short-time Fourier transform, which requires chopping up a signal into segments and performing a Fourier Transform over each segment, the wavelet transform can extract the local spectral and temporal information simultaneously. In addition to that, if the characteristic shape that is being tried to extract from a signal is known, then there is a wide variety of wavelets to choose from to best match that shape.

It should be noted that the frequency and scale are inversely related in wavelet transform. As the scale reduces, the frequency increases. This means that moving to a higher frequency with lower time support, while an increase in the scale parameter leads to moving toward lower frequencies Gencay et al. (2002)

The DWT diagram below illustrates the process involved in the discrete wavelet transform decomposition up to level 2 Aggarwal et al. (2008):

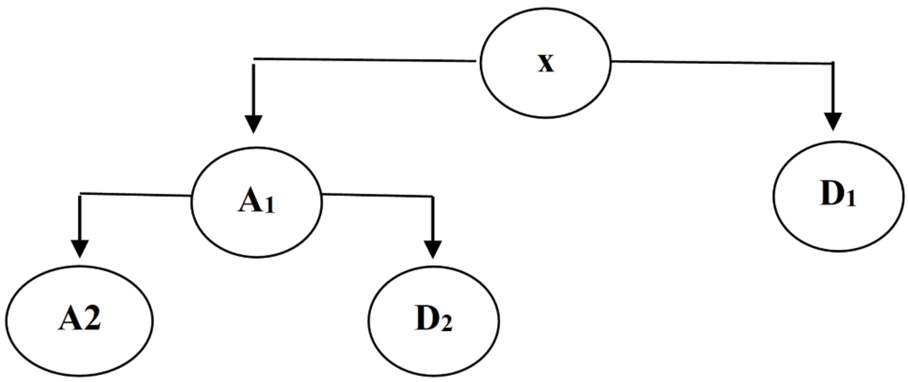


Figure 2.1 DWT diagram Wavelet transform decomposition up to level 2

In the first stage or level, one of the wavelet transforms, approximation $A_1$ and detail $D_1$ of the series are obtained X such that $X = D_1 + A_1$. The approximations are the high-scale, low-frequency components of signal X. The details are the low-scale, high-frequency components. For many signals, the low-frequency content is the most important part. In the subsequent level, the approximation $A_1$ is decomposed into $A_2$, and $D_2$ to obtain the additive decomposition up to level 2 Aggarwal et al.(2008):

$$X = D_1 + A_1 = D_1 + D_2 + A_2$$

This latest equation defines a simple multiresolution analysis of X. The $D_j$ are called the jth level detail and the $A_j$ is the jth level wavelet smooth for X.

After decomposition, the reconstruction of the series X can be done through the inverse wavelet transform to form the initial signal in the time domain with no loss of information. Figure 2.2 shows how the series X is decomposed Yu et al. (2000):

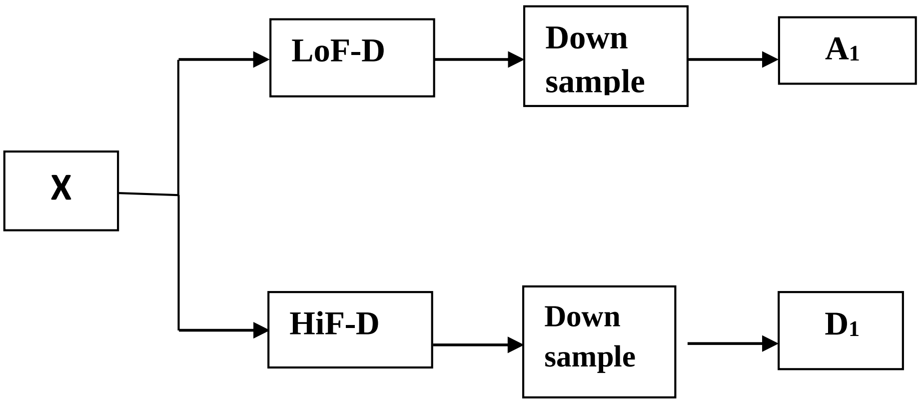


Figure 2.2 First-level wavelet signal decomposition.

First level or scale one wavelet signal decomposition of $X$ into $A_1$, and $D_1$ is given in Figure 2.2. The reconstruction of the original signal is possible through approximation and detailed coefficients as specified below

$$A_{J-1} = D_J + A_J, \qquad A_{J-2} = D_{J-1} + A_{J-1}, \ldots.$$

$$X = D_1 + A_1 \tag{2.6}$$

### 2.4.1 DWT and MODWT

There are two types of wavelet transforms: the continuous wavelet transform (CWT) and the discrete wavelet transform (DWT). Orthogonal wavelet transforms, such as the standard DWT, require that the sample size N to be a power of 2. The limitation here is focused on frequency location and time bands. A modification of the standard DWT called the Maximum Overlap Discrete Wavelet Transform (MODWT) forms another wavelet transform, capable of processing any sample size N, and this does not need to be a power of 2. It should be noted that the MODWT is a redundant nonorthogonal transform that can be used as an alternative to the standard DWT. In fact, as shown in Percival and Walden (2000), the MODWT forms the same multiresolution analysis as the DWT and allows the processing of a series of any sample size. As is true for the DWT, the MODWT can be used to form an analysis of variance based on the wavelet and scaling coefficients but cannot be used to form such an analysis based on the details and smooths as in the DWT transform.

In general, wavelet transforms have numerous attractive properties, and these are essential when analyzing time series. These properties can be simplified and summarized as given below Ding li. (2003):

a) Effective algorithm.

b) Adequate filter.

c) Localized in frequency and time domain

d) Excellent reconstruction.

e) Decorrelate signals

Other advantages of wavelet transform can be summarized as follows by Gencayet al. (2002):

a) Wavelet transforms can be effectively applied to study data from non-linear, non-stationary time series.

b) They can capture numerous features or behaviours, including transients, sharp features, and spikey or discontinuous features.

c) They are ideal for modelling compressed observation, with this type of analysis producing sparsely represented observations.

d) They can be used in noise reduction and de-trending based on the multi-resolution application of the wavelet analysis.

In addition, wavelet transforms can be effective in studying almost all-time series, including financial time series data. Due to the non-stationarity in the majority of financial time series, wavelet transforms to offer a convenient tool for studying localized and time-limited features of financial data.

Various types of wavelet analysis have been put forward by authors in different fields of applications. For example, Mallat (1989) considered pyramidal algorithms in studying how multiple resolution representations of compressed data could be applied when processing images. He also describes the features of a number of wavelets.

### 2.4.2 Wavelet Filters

The discrete wavelet transform can be regarded as a special linear filtering operation with wavelet filters of finite length $h_l, l = 0,1, \dots L$ Percival and Walden (2000), pages 69-71. These wavelets filters must satisfy three basic conditions:

1. The wavelet filter must sum the zero $\sum_{l=0}^{L} h_l = 0$

2. Their square must sum to one $\sum_{l=0}^{L} h_l^2 = 1$

3. The filter is orthogonal to its even shifts $\sum h_l h_{l+2n} = 0$.

The first level wavelets coefficients of a signal X are obtained by circularly filtering the series X with the filter $h_l, l = 0, \dots, L$. The corresponding scaling filter to the wavelet filter, known as the quadrature mirror filter, is used to obtain the details and satisfies similar conditions, except that it must not sum to zero.

### 2.4.3 Continuous Wavelet Transform

The continuous wavelet transform (CWT) is a very useful tool for many applications and is well-established for engineering applications. Let denote by $W(u,s)$ the CWT transform of a signal x(t) at a scale (u>0) and translational value s. The $W(u,s)$ is obtained by computing a convolution of the signal x(t) with the scaled wavelet. $W(u,s)$ is the time and frequency representation of the signal and the parameters u and s of the wavelet are allowed to vary continuously. Thus the CWT maps the signal under study into a two-dimensional function of time and frequency and therefore provides true time-frequency representations

The major distinctions separating Fourier transform from the continuous wavelet transform include, firstly, the 2-parameter characteristic (time and frequency) of CWT, in comparison to the single-parameter indexing of Fourier transform (frequency only). Second, for CWT, the window is used to widen or narrow depending on the frequency used, but Fourier transform uses multiplication of the windowed signal by window function and integral in a continuous manner over time. An important limitation of the FFT is its inability to provide the time dependence of the signal spectrum.

The continuous wavelet transformation of a function of one variable is a function of two variables. Clearly, the transformation is redundant; there is extra information above what is needed for ideal reconstruction. To minimize the redundancy of the transformation, one can select discrete values of u and s and still have a transformation that is invertible. Discretizing in CWT presents a highly effective approach to minimizing wavelet coefficient numbers and preserving every part of the function information. Discretization is achieved by applying critical sampling, which for CWT can be presented as follows:

$s = 2^{-j}$ and $u = k2^{-j}$.

Discrete Wavelet Transform (DWT) is then achieved Gencay et al. (2002), with the determination of the critical sample through time and frequency resolution. The CWT is reversible such that signal reconstruction can occur after filtering or manipulating wavelet coefficients.

### 2.4.4 Discrete Wavelet Transform

Discrete wavelet transform offers a number of benefits in signal reconstruction localized at various levels in time and frequency. DWT reconstruction makes use of high- and low-pass filters, known as wavelet and scaling filters. These filters are derived using the two-scale equation and the mother wavelet function and are mainly used in the first instance to run the DWT.

The current work focuses particularly on discrete rather than continuous wavelet transform due to the multiple resolutions offered for time-frequency analysis and the excessive computation time and memory required by CWT Percival and Walden, (2000). Another factor which influences the selection of DWT over CWT is the choice of values for the parameters u and s, in which case the choice is constrained in the DWT