RESIDUAL METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS WITH SINGULAR KERNELS

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by

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LIST OF ABBREVIATIONS

FDEs	Fractional differential equations
FS	Fractional Series
FDWE	Fractional Diffusion-Wave Equation
MWR	Method of Weighted Residual
FBS	Fractional B-Splines
FTPFs	Fractional Truncated Power Functions
FFPE	Fractional Fokker Planck equation

LIST OF SYMBOLS

D^α	Caputo derivatives
Γ	Gamma Function
$n!$	n Factorial
$\beta(z,w)$	Beta Function
$E_\alpha(z)$	Mittag-Leffler function of one parameter
$E_{\alpha,\beta}(z)$	Mittag-Leffler function of two parameter
$\mathbb{R}$	Real Number
$\mathbb{N}$	Natural Number
$\mathbb{C}$	Complex Number
$\mathbb{R}(\alpha)$	Real part of complex number
$\lfloor . \rfloor$	Floor function
$\{\alpha\}$	The fractional part of α
$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	The inverse of Laplace transform

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KAEDAH BAKI UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN PECAHAN DENGAN KERNEL TUNGGAL

ABSTRAK

Dalam tesis ini, kami melaksanakan dua kaedah baki untuk menyelesaikan dua masalah sains dan kejuruteraan tertib pecahan yang penting. Terbitan pecahan digunakan dalam pengertian Caputo. Atas sebab ini, kami memulakan kajian ini dengan membincangkan dan menyediakan beberapa sifat kalkulus pecahan. Kajian komprehensif untuk teori kalkulus pecahan telah dibentangkan. Kaedah baki pertama yang digunakan dalam tesis ini ialah gabungan kaedah siri pecahan dan kaedah penjelmaan Laplace pecahan. Gabungan ini digunakan untuk menyelesaikan sebuah aplikasi penting dalam fizik iaitu persamaan gelombang resapan pecahan dengan tindak balas. Tujuan utama kaedah ini adalah untuk menyelesaikan masalah ini secara analitik. Perlu dinyatakan bahawa kami dapat mencari sebuah penyelesaian yang tepat dalam contoh ini. Kaedah baki kedua ialah kaedah kolokasi yang digunakan untuk menyelesaikan aplikasi kedua iaitu persamaan Fokker Plank pecahan. Beberapa contoh telah disiasat menggunakan kaedah kolokasi tersebut. Tiga ukuran ralat digunakan untuk menguji kecekapan pendekatan ini. Ukuran ini ialah ralat maksimum, ralat min kuasa dua, dan ralat baki dan kesemuanya adalah dari tertib 10^{-15}. Oleh kerana domain pembolehubah x dan y are $(0,\infty)$, penjelmaan Laplace digunakan ke atas x dahulu, kemudian dengan y. Pendekatan ini akan menghasilkan masalah nilai awal yang diwakili dalam t sahaja. Dari situ, kaedah siri pecahan digunakan untuk menyelesaikan sistem persamaan ini. Dua contoh dibincangkan untuk menerangkan prosedur. Keputusan menunjukkan bahawa pendekatan ini adalah cekap dan boleh digunakan untuk masalah lain daripada

jenis yang sama. Idea pendekatan ini adalah untuk menulis penyelesaian anggaran dari segi fungsi percubaan bebas linear $\{\varphi_i\}$. Kita harus ambil perhatian bahawa φ_o memenuhi syarat awal manakala fungsi percubaan lain memenuhi syarat awal homogen. Kemudian, baki dibina dan diortogonalkan ke atas fungsi berat. Pendekatan ini akan menghasilkan sama ada sistem linear atau tak linear berdasarkan masalah nilai awal pecahan. Perisian Mathematica 11.1 digunakan untuk menyelesaikan sistem ini untuk mencari pekali bagi fungsi percubaan di mana $\{1, x, x^2, \ldots, x^n\}$ digunakan sebagai asas piawai. Keputusan yang diperolehi menunjukkan bahawa penyelesaian anggaran tidak bertepatan dengan penyelesaian yang tepat. Ini adalah munasabah kerana kita tidak boleh mendapatkan penyelesaian yang tepat untuk semua masalah terutamanya jika penyelesaian mempunyai bentuk x^α atau fungsi trigonometri. Atas sebab ini, pendekatan kolokasi digunakan bersama-sama dengan fungsi pecahan B-spline sebagai fungsi percubaan untuk menyelesaikan persamaan Fokker Planck pecahan secara analitik. Ini bermakna bahawa penyelesaian anggaran adalah sangat hampir dengan penyelesaian yang tepat. Daripada keputusan ini, pendekatan ini dipercayai teritlak untuk menyelesaikan masalah yang serupa.

RESIDUAL METHODS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS WITH SINGULAR KERNELS

ABSTRACT

In this thesis, we implement two residual methods to solve two important fractional applications in science and engineering. The fractional derivative is used in the Caputo sense. For this reason, we start this study by discussing and providing several properties of fractional calculus. Comprehensive study for the theory of fractional calculus are presented. The first residual method used in this thesis is a combination of fractional series method and fractional Laplace transform method. This combination is use to solve an important application in physics which is the fractional diffusion-wave equation with a reaction. The main purpose of this method is to solve this problem analytically. It is worth mentioning that we are able to find the exact solution in our examples. The second residual method is the collocation method. We use it to solve the second application which is the fractional Fokker Plank equation. Several examples are investigated using the collocation method. Three error measures are used to test the efficiency of this approach. These measures are the maximum error, mean-square error, and residual error and all of them are of order 10^{-15}. Since the domains of the variables x and y are $(0,\infty)$, the Laplace transform is applied with respect to x first, then with respect to y. This approach will produce initial value problem represented in t only. From there, the fractional series method is applied to solve this system of equations. Two examples are discussed to explain the procedure. The results show that this approach is efficient and can be used for other problems from the same type. The idea of this approach is to write the approximate solution in terms of linearly independent trial

functions $\{\varphi_i\}$. We should note that φ_o satisfies the initial condition while other trial functions satisfy homogeneous initial conditions. Then, the residual is constructed and orthogonalized with respect to the weight function. This approach will produce either a linear or nonlinear system based on the fractional initial value problem. Mathematica software 11.1 is used to solve this system to find the coefficients of the trial functions whereby $\{1, x, x^2, \ldots, x^n\}$ is used as the standard basis. Obtained results show that the approximate solution is not coincide with the exact solution. This is reasonable since we cannot get accurate solution for all problems especially if the solution has the form x^α or trigonometric function. For this reason, the collocation approach is used along with the fractional B-spline functions as trial functions to solve the fractional Fokker Planck equation analytically. This means that the approximate solution is very close to the exact solution. From these results, this approach is believed to be generalized for solving similar problems.

CHAPTER 1

INTRODUCTION

1.1 Introduction

The purpose of this thesis is to utilize various analytical methods that can be used to solve fractional differential equations. At the beginning, we combined the Laplace transforms and series methods to solve fractional diffusion wave equations with reactions. Then, we used a reliable method of solving the fractional Fokker Planck equation based on the B-Spline method. This chapter discusses the background of the research, problem statement and research question, scope of study, methodology, objectives, research gap, and thesis outline.
It is worth to mention that we used Mathematica version 11.1.0.0 year 2017 to draw all the graph including of this thesis.

1.2 Background

September 30th, 1695 was the born of fractional calculus with reason to a profound problem raised by a French mathematician called L'Hospital in a letter to a German mathematician named Leibniz. The adorable response of Leibniz to that profound problem laminated a major inspiration for all generations of scientists and is incessant to motivate the minds of modern researchers. Fractional calculus has maintained the attention of best-level mathematicians for last three centuries, and over the last decade it has been used to tackle the dynamics of complex systems from different fields of science and engineering.

An important point to note is that it was Abel who proposed the first application of fractional derivatives in 1823 and further discussed by Podlubny et al. (2017). As previously described, Abel utilized fractional derivatives to fix an integral equation that appears as an example of fractional calculus's important applications, the tautochrone problem is formulated. This problem shows the path along which an object must sliding down along the influence of gravity regardless of its initial position. Through fractional calculus, the solution was determined to be a part of inverted cycloid. Abel is commonly referred to as the father of the complete framework for fractional calculus for many different reasons. As a result of the first paper, Abel introduced integrals that are fractionally ordered that are currently known as the Riemann-Liouville fractional integrals, as well as differentiation that are fractionally ordered in the form of the Caputo fractional derivative Abel (1823).

1.3 Motivation

Recently, fractional calculus has been gaining considerable attention. Fractional differential equations (FDEs) have been adapted as mathematical models as discussed in Baleanu et al. (2012). Several applications for FDEs can be found in models of viscoelastic behavior Jha and Dasgupta (2019), anomalous diffusion Metzler and Klafter (2000), compartment models, economics, epidemiology, dynamics of particles, biology, and signal and image processing Matlob and Jamali (2019).

We select the fractional diffusion–wave equation with a reaction since it is defined in the sense of the generalized diffusion, which has fractional derivative. We should note that the diffusion process is asymmetric or stochastic, but it is determined on a molecule scale, so a fractional modification is needed. We should also note that when the derivative is of order one, this equation become a diffusion equation and when the derivative is of order 2, it becomes wave equation. From this understanding, we are interested to see the behavior of the problem when the derivative is fractional and the order is in between 1 and 2.

The Fokker-Planck Equation is used in models of standard diffusion problems involving external fields. Recently, the integer order diffusion equation was generalized to a fractional diffusion equation in which the differentiation with respect to t and x are replaced by differentiation of non-integer order. We notice that this equation becomes a hot topic for researchers. This has sparked our interest to explore this equation further. Alquran et al. (2017) used the fractional series method to solve time fractional phi-4 equation while Zada et al. (2022) used the fractional Laplace transform method to solve the integro-differential equations. These motivate us to combine the two approaches to solve our proposed problems to see if we could generate better results compared to the previous works. The fractional series method with Laplace transform is the most suitable method for solving the fractional diffusion–wave equation with a reaction. First, the domain of the variables x and y are $(0,\infty)$ that allows us to take the Laplace transform and reduce the problem into fractional initial value problem with one variable. Second, the fractional power rule fits properly with the new problem. Third, this approach will produce an analytical solution. Finally, this approach is very accurate and give approximate solution close to the exact solution.

Pitolli (2018) used the collocation method with B-spline in the Predator-Prey models. Inspired from their work, we modified the method to make it suitable to be used for our model. Collocation method is a very powerful method to solve either integer or fractional initial value problems. It is easy to implement, and easy to program it. We initially tried to use different trial functions such as the standard basis for polynomials of degree less than or equal . However, we were not satisfied with the results until we see the trial function used in Pitolli (2018). This gives the motivation to adopt this approach in our work and it turned out to be successfully applied to solve our problems.

1.4 Problem Statements and Research Questions

A substantial number of methods are available for solving different fractional differential equations. Some of these methods are not self-starters such as multi-step methods and some of them have low order such as one-step methods. While some of them give low accuracy with such problem such as Adomian decomposition method. Computational time and complexity of the method are main disadvantages to several methods. To overcome these problems, we want to propose a combination of a few methods that could possibly generate good results and this will be the main focus of this research.

In this thesis, first, we illustrate a Laplace transform technique to solve fractional diffusion wave equation. Secondly, we illustrate a B-Spline technique to solve the Fractional Fokker Planck equation. Prior to implementing these methods, therefore, main properties will be illustrated and discussed to ensure that they will achieve accurate and reliable results. In our thesis, we aim to answer the following questions:

1. Does the method of B-Spline for solving the Fractional Fokker Planck equation give us accurate and reliable results?

2. Does the method of Laplace transform in solving a class of fractional-time diffusion-wave equation give us accurate and reliable results?

3. Are the proposed methods promising to solve other physical problems?

1.5 Scope of the study

The study focuses on solving fractional differential equations by using different applicable analytical methods. We solved two different fractional differential equations. Firstly, we illustrate a analytical technique for solving a class of fractional-time diffusion-wave equation based on Laplace transform and fractional series method. Then, by utilizing the B-Spline method, we will solve the fractional Fokker Planck equation analytically. We give the prove of the fractional series method. In addition, we proved several results in the fractional derivative and fractional Laplace transform.

1.6 Objectives of the Study

Several objectives are sought by this study:

1. To find an analytical method for solving fractional diffusion-wave equation with a reaction based on the Laplace transform and fractional series method.

2. To find a reliable method for solving the fractional Fokker Planck equation based on the B-Spline method.

3. To investigate the efficiency of proposed methods by solving several proposed problems.

4. To study the properties of fractional calculus and their proof.

1.7 Methodology

With the weighted residual method, solutions to differential equations are approximated by linear combinations of trial functions or shape functions with unknown coefficients. As a result, an error or residual is derived from the approximate solution applied to the governing differential equation. Finally, in order to determine the unknown coefficients, the residual is forced to disappear at average points or made to be as small as possible depending on the weight function. A number of weighting functions are available, and some of the most popular have been named. A system of algebraic equations is produced by making the residual vanish over the entire solution domain. From this perspective, the weighted residual method is promising method in solving our problems.

Several scholars and researchers presented their own ideas on weighted residual methods. Some of the authors have tried to describe the details of the analysis of these methods. Instead of doing detailed analysis, some authors illustrated how these methods were applied. In Lindgren (2009) paper, a weighted residual method provides simple and accurate solutions to boundary value problems with nonlinear differential equations. B. Finlayson and Scriven (1966) provides a table about the history of approximate methods on his research. Several weighted residual analyses were conducted by B. A. Finlayson (2013) and B. Finlayson and Scriven (1966). There was a discussion of weighted residual methods and their steps in Keskin (2019).

In this thesis, we study two problems. The first problem considers solving a class of fractional-time diffusion-wave equation that is defined by the following, Schneider and

Wyss (1989)

$$D_t^{2\alpha}\mu(x,y,t) = a_1\mu_{xx}(x,y,t) + a_2\mu_{yy}(x,y,t) - a_3\mu_x(x,y,t) \tag{1.1}$$

$$-a_4\mu_y(x,y,t) - a_5\mu(x,y,t) + \theta(x,y,t),$$

such that

$$\mu(x,y,0) = g_1(x,y), \quad 0 < x < \infty,\ 0 < y < \infty, \tag{1.2}$$

$$\mu_t^{\alpha}(x,y,0) = g_2(x,y), \quad 0 < x < \infty,\ 0 < y < \infty, \tag{1.3}$$

$$\mu(0,y,t) = h_1(y,t), \quad 0 < y < \infty,\ 0 < t < T, \tag{1.4}$$

$$\mu_x(0,y,t) = h_2(y,t), \quad 0 < y < \infty,\ 0 < t < T, \tag{1.5}$$

$$\mu(x,0,t) = r_1(x,t), \quad 0 < x < \infty,\ 0 < t < T, \tag{1.6}$$

$$\mu_y(x,0,t) = r_2(x,t), \quad 0 < x < \infty,\ 0 < t < T, \tag{1.7}$$

where $\frac{1}{2} < \alpha \leq 1, a_1, a_2 > 0, a_3, a_4, a_5 \geq 0$ are constants, and $\theta(x,y,t), g_1(x,y), g_2(x,y)$, $h_1(y,t), h_2(y,t), r_1(x,t), r_2(x,t)$ are continuous functions, and $D^{2\alpha}$ is the Caputo derivatives. We proposed Laplace-series method for solving this problem, Alquran et al. (2017).

The second problem consider the following fractional Fokker Plank equation, Risken and Voigtlaender (1984)

$$D_z^{\eta}\Omega(y,z) + D_y^{\eta}\Omega(y,z) - D_y^{2\eta}\Omega(y,z) = 0, \tag{1.8}$$

$$\Omega(y,0) = h(y) \tag{1.9}$$

where $y, z > 0$ and $\eta \in (0, 1]$. Using collocation method, the weight function is selected so that at n distinct points in the domain, the residual is zero. The weight function can be used as the displaced Dirac delta function. We proposed the B-Spline method, Pitolli (2018) to solve the this problem.

The below flowchart to describes the methods that we used to solve our problem. In this thesis, we implement two residual methods to solve two important fractional applications in science and engineering.

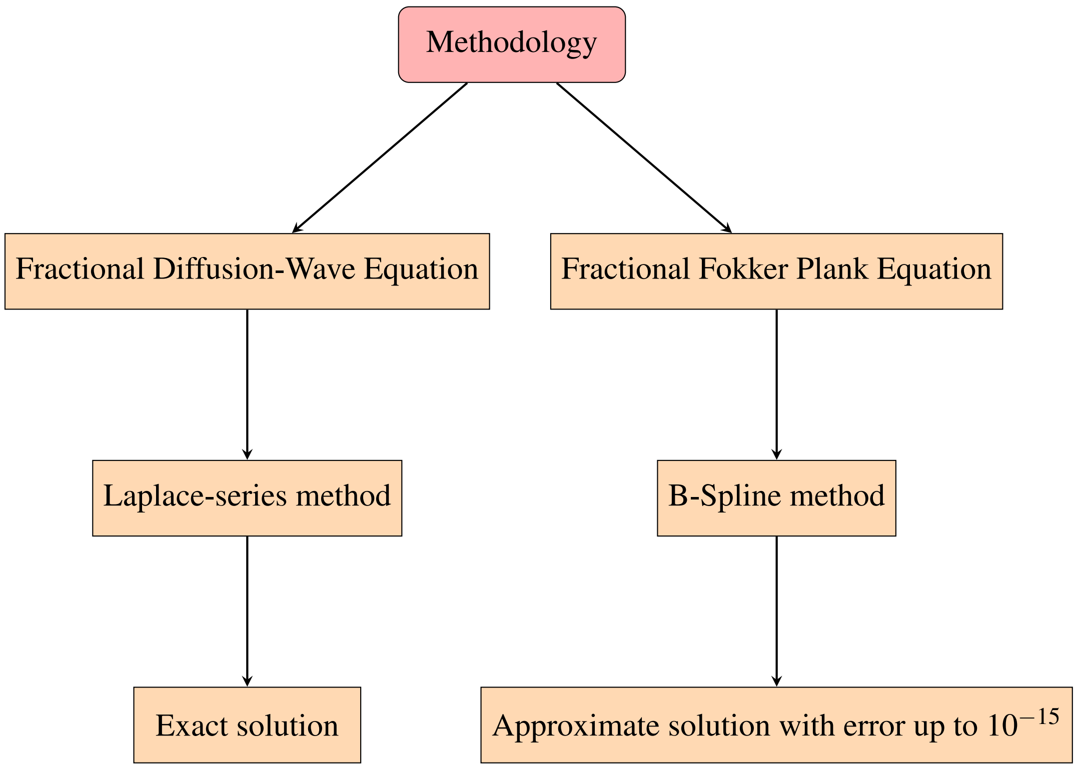

Figure 1.1: The methodologies of the thesis

1.8 Limitations

For the investigation of all types of fractional differential equations, no unique method can be used. In this way, no matter what changes are made to a method at any given time, new solutions are able to be developed which are always useful. All existing implemented methods and solutions have some limitations depending on the issues they face. It is difficult to solve fractional differential equations when the order is high, and in some cases, no solution can be reached. It is imperative to search for new methods for recovering exact solutions for space-time fractional differential equations. Fractional initial value problems are difficult to solve and sometimes not possible, especially with the nonlinear case. For this reason, we try to choose linear problems so that we can know the exact solution and compare with them. In addition, if the solution has some forms such as trigonometric, it needs large computational time and computational cost to find the approximate solutions. For this reason, we try to limit ourselves to some doable cases.

1.9 Research Gap

Caputo derivative is non-local derivative with singular kernel. This make the study of initial and boundary value problems are difficult. Exact solutions for such problems are not easy to compute. This make many researchers use different definitions to avoid this difficulty. Although many researchers discussed the numerical solution of diffusion-wave equation with a reaction and Fokker Planck equation in the integer derivative case, we could not find reference for the fractional case. This gap in this area encourage us to investigate such problems. It was a big challenge to us to check the accuracy of our results, but we use several error measures to overcome this difficulty. By this thesis, we hope the door will be open for the research in similar problems. Many similar problems can be discussed now such as fractional eigenvalue problems, fractional integro-differential problems, and system of fractional initial value problems.

1.10 Outline of the thesis

In this thesis, six chapters are presented. This first chapter describes introduction, background, motivation, problem statements and research questions, scope of the study, objective of the study, methodology, limitation, research gap and outline of the thesis. In chapter 2, we will study the existing research and debates related to our study. In chapter 3, there will be a review of some basic concepts that are needed for this study. First, some special functions are discussed, such as gamma functions and beta functions, as well as Mittag-Leffler functions. Second, we define some important definitions of fractional calculus and their properties. In the end, we discuss Laplace transform and the Fokker-Plank equation.

In chapter 4, we solve a fractional Diffusion-Wave equation with reaction by using Laplace-Series method. In chapter 5, we study a fractional Fokker-Plank equation by using B-Spline method. The conclusion of the thesis is in chapter 6 along with possible future research projects.

CHAPTER 2

LITERATURE REVIEW

The purpose of this chapter is to review the existing research and debates related to our study. In section 2.1 and 2.2 the historical development of fractional calculus will be discussed and how fractional calculus can be applied. In section 2.3, the history of Fokker Plank equation in its one dimension and several dimensions are going to be studied. In section 2.4, present the history of B-spline method in solving differential equation. In section 2.5, the method of solving Fractional Diffusion-Wave Equation is going to be reviewed. In Section 2.6, the method of fractional series is discussed.

2.1 Historical Development

During 1695, L'Hopital, a French mathematician, asked a German mathematician called Leibnitz to solve the following question, "If the order $n = \frac{1}{2}$, how can I find the derivative for this function,

$$f(x) = x."$$

Leibnitz's response was "This is an apparent paradox from which, one day, useful consequences will be drawn" Kilbas et al. (2006). By the question of L'hopital, the fractional calculus started appearing in the world. Fractional Calculus is considered to have been born on September 30, 1695. Many mathematicians have studied the question of L'hopital in the following decades: Euler in 1738, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1826, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. There are many concepts of functional integrals and derivatives found by the mathematicians using their own notation and methodology Aygören (2014).

In this regard, the most noteworthy achievements can be found in De Oliveira and Tenreiro Machado (2014), based on the following:

1. To determine the derivative of positive order, Fourier proposed an integral representation in 1822.

2. Fractional Calculus was applied for the first time in 1826 when Abel solved an integral equation related to the tautochrone problem.

3. A formula for differentiating the exponential function was proposed by Liouville in 1832. As far as Liouville is concerned, this is the first definition he gave. According to Liouville's second definition, an integral is used instead of an integer in order to describe the non-integer integration.

4. With the definition of derivatives, Weyl resolved a problem related with periodic functions.

2.2 Fractional Calculus

There are more than 300 years of history behind fractional derivatives and integrals. Scientists and engineers in the modern era realized that fractional derivatives and integrals provided better processes for describing nature's complicated phenomena. This includes non-Brownian motion, systems identification, control, and viscoelastic materials Matlob and Jamali (2019). We can use the non-local property of the fractional derivative to describe those complex systems which involve long-memory in time in a better way. To analyze the experimental data described in a fractional way, the numerical process has become an essential method Li et al. (2011). Furthermore, fractional derivatives and integrals have a wide range of applications in engineering and science. Electronics (Tapadar et al. (2022)), viscoelasticity (Matlob and Jamali (2019)), fluid mechanics, electrochemistry (Magin (2004)), models of biological populations (Ionescu et al. (2017)), optics (Yilmaz (2021)), signal processing (Assaleh and Ahmad (2007)), quantum mechanics (Al-Raeei (2021)), electricity (Tapadar et al. (2022)), and ecological systems (Ray et al. (2014)).

Literature reviews are used to become familiar with contemporary thinking and research on a given topic. Research on the fractional calculus has been conducted over time. Several research articles have been examined for review.

Vo and Ekpenyong (2022) described some determinations of integer-order mechanical models in capturing features of compliance data from macrophages under various clinical conditions. Using both integer order models and fractional calculus versions in Mittag-Leffler form, Vo and Ekpenyong found that the viscoelastic parameters from fractional Kelvin-Voigt model quantify the pharmacological interventions and maturation of macrophages more strictly than integer-order models.

Yilmaz (2021) created a new type of geometric phase model with fractional derivatives. Additionaly, Yilmaz introduced a magnetic curve whose direction is determined by the electric field. For various values of the conformable fractional derivative, models that are consistent with the theory are examined and analyzed.

Furthermore, Valentim Jr et al. (2020) asked them-self if the fractional calculus can help to improve tumor growth model, and the answer was an important article where it published in 2020. The article shows how to utilize a fractional approach, and studied derives analytical solutions for five of these models, whose parameters are best matched to existing clinical data. In terms of tumor growth prediction, results show that fractional models not only have better performance, which is mostly wanted for decision-making in oncology, but also reveal interesting characteristics to be further explored.

Rusyaman et al. (2020) presented a topic on the fractional relationship using a fractional differential equation model applied to empirical data on surface tension and viscosity measured in laboratory experiments on lubricating oil. The output of this articles is "there is an empirical relationship can be expressed between surface tension and viscosity".

Moreover, in Ibrahim et al. (2022) paper, a mathematical model based on fractional partial differential equations was presented. The class is formulated by the proportional-Caputo hybrid operator. In addition, some features of the geometric functions in the unit disk are applied to define the upper bound solutions for this class of fractional partial differential equations. The result of this article is "the model strongly develops the details of the given data sets, and could probably help the medical staff in the meantime the diagnosis process."

In addition, Assaleh and Ahmad (2007) presented an original methodology for speech signal modeling utilizing fractional calculus. This method differs from the celebrated Linear Predictive Coding method, that uses integer order models. Shown by means of mathematical reenactments by involving a couple of integrals of partial orders as basis function, the speech signal can be modeled precisely. The modern methodology has the value of requiring fewer model boundaries, and is shown to be better than the

Linear Predictive Codin method in capturing the subtleties of the modeled signal.

The previous applications give us impression that fractional calculus is a hot research topic and motivate us to look for more applications. During our search, we selected two main problems which are fractional Fokker-Planck Equation and fractional diffusion–wave equation with a reaction. These two problems will be investigated and solved using the methods that will be introduced in the coming sections.

2.3 Fokker Plank Equation

There is a long history of studies on the theory equation of Fokker–Planck, dating back to Einstein, Langevin, Fokker, and Planck, among many others Shahmorad et al. (2020), Risken and Voigtlaender (1984), Hoover et al. (1980). Because of impressive advancements in analysis and computation, plus a wide range of available applications, the theory of the Fokker-Planck equation is exceedingly rich in content.

A wide range of scientific phenomena can be described by it, including the relationship between random force and fluctuations, and non linearity in pattern formation. The Fokker–Planck equation can be used to describe many physical, chemical, biological, and economic systems. Numerous algorithms have been explored for numerical solutions of the Fokker–Planck equation in modern years, see Leonenko and Phillips (2015), Shizgal (2016).

The Fokker–Planck equation has received considerable theoretical attention in recent years. Each of the proposed methods has its advantages and limitations, as might be expected. A number of authors have used path integral methods Bao and Shizgal (2019), Zhang et al. (2020).

A numerical evaluation of the Onsager-Machlup-functions using a mathematical formalism provided by Wehner and Wolfer, Leonenko and Phillips (2015), has been presented. Eigenvalue expansion is applicable to a wide class of Fokker–Planck operators Shizgal (2016). The eigenvalue of Fokker–Planck equation can be solved very accurately by using various spectral methods and pseudo spectral methods. When the Fokker–Planck equation deals with a Lorentz gas system, the complete set of eigenvalues and eigenfunctions determine the dynamics completely.

The Fokker Planck Equation was named after the famous Physicists Fokker and Planck who studied Brownian motion in a radiation field and proposed a theory of fluctuations based on it. When small pollen grains are suspended in liquid or gas, they exhibit a highly animated and irregular motion. Robert Brown first observed this phenomenon in 1827 Brown (1827). Brownian motion is named after this phenomenon.

For a one-dimensional Brownian motion with a single variable x and a distribution function $W(v,t)$, Fokker-Planck's equations take the form

$$\frac{dW}{dt} = \left[-\frac{d}{dx}D^{(1)}(x) + \frac{d^2}{dx^2}D^{(2)}(x)\right] W \, . \tag{2.1}$$

The diffusion coefficient is $D^{(2)}(x) > 0$, while the drift coefficient is $D^{(1)}(x)$. It is also possible that drift and diffusion are time-dependent. Drift coefficients are sometimes linear while diffusion coefficients are constant. According to mathematical literature, equation (2.1) is a parabolic partial differential equation of order 2 and it is commonly known as the forward Kolmogorov equation.

By using the equation (2.1), we will write it in a general form to N variables $x_1, x_2, \ldots, x_n$ as

$$\frac{dW}{dt} = \left[-\sum_{i=1}^{N} \frac{d}{dx_i} D_i^{(1)}(\{x\}) + \sum_{i,j=1}^{N} \frac{d^2}{dx_i\, dx_j} D_{i\,j}^{(2)}(\{x\})\right] W \, . \tag{2.2}$$

Drift vector $D_i^{(1)}$ and diffusion tensor $D_{ij}^{(2)}$ are generally dependent on N variables $x_1, x_2, \ldots, x_n = \{x\}$. For N macroscopic variables $\{x\}$, such that $\{x\}$ can be different variables types such as position and velocity, the distribution function (2.2) can be defined as $W(\{x\}, t)$

To understand a macroscopic system, as with Brownian motion, you would have to solve all of its microscopic equations. Fokker-Planck equations are simply equations of motion for fluctuations in macroscopic variables. When we use a deterministic treatment, we ignore the fluctuations of the macroscopic variables. Therefore, we would neglect the diffusion term in the Fokker-Planck equation (2.61). As a result, the differential equations (2.2), equate to the system of differential equation $(i = 1, 2, \ldots, N)$

$$\frac{dx_i}{dt} = D_i^{(1)}(x_1, \ldots, x_n) = D_i^{(1)}(\{x\}), \quad \text{for the } N \text{ macrovariables } \{x\}. \tag{2.3}$$

There are other equations of motion for distribution functions besides the Fokker-Planck Equation. Also, there are Boltzmann equations and Master equations. For continuous macroscopic variables, Fokker-Planck equation is the simplest equation. These variables usually describe macroscopic but small subsystems, such as the position and velocity of a Brownian particle in motion, the current in an electric circuit, and the electrical field in a laser.

A larger subsystem, however, may allow the fluctuations to be neglected, resulting in deterministic equations. A stochastic description is required for large systems in these cases when the deterministic equations cannot be solved Jordan et al. (1998).

2.4 B-spline

I will begin by introducing the history of the spline interpolation method. In the late 1960s, Bickley (1968) utilized the spline interpolation method to solve differential equations. He studied second-order linear boundary value problems with arbitrary spline functions on domains. Based on the domain chosen for the problem, the spline function is constructed by utilizing the boundary conditions. It was believed at the time that the results were very promising.

In 1969, Fyfe (1969) examined and discussed Bickley (1968) and Curtis and Powell (1967) cubic spline methods and error estimations. With Fyfe's correction spline, the solution can be adjusted more effectively. In addition, Fyfe demonstrated that the method is less efficient for non-equal intervals using a correction spline. The same interval spline was used to test a problem, and minimal computations were required. Deferred corrections were the least effective when applied to intervals that were not equal. Although Fyfe claimed that this method was more efficient than Finite-difference Methods because the spline could give approximate solutions at any point in the interval.

In 1975, Ahlberg and Ito (1975) began utilizing B-splines to solve ordinary differential equations. Several researchers have utilized B-splines to solve linear and nonlinear partial differential equations because of their simplicity. For years, researchers have developed and studied the use of B-splines in designing curves and interpolating points. Recent studies involving B-splines for solving partial differential equations will be discussed.

Pitolli (2018), solved fractional differential problems utilizing fractional B-splines collocation method. Essentially, the idea is to utilized noninteger degree piecewise polynomials generated by fractional B-splines as approximating space. Next, in the collocation step, an exact differentiation rule involving the generalized finite difference operator is applied to approximate accurately and efficiently the fractional derivative of the approximating function. Several fractional Lotka-Volterra and predator-pray models with variable coefficients were solved to demonstrate the effectiveness of the technique in solving nonlinear dynamical systems of fractional order. Based on the numerical tests, they showed that the method they proposed is accurate and low-cost at the same time.

Lakestani et al. (2012), established the operational matrix by utilizing B-spline functions of fractional derivative of degree α in the Caputo sense. Utilizing this technique enables them to solved different problems directly since it reduces them to a system of algebraic equations. By using this method, linear and nonlinear fractional differential equations are solved. The new technique presented in this paper is illustrated with problems to demonstrate its validity and applicability. In their paper, the authors show that the new approach is effective at solving the problem.

Yaseen et al. (2017), presented an approach for solving fractional sub-diffusion equations numerically based on cubic trigonometric B-spline collocations. For discretizing the time derivative, the usual finite difference scheme is utilized. With the help of the Grünwald–Letnikov discretization of the Riemann–Louville derivative, the cubic trigonometric B-spline functions are utilized to approximate the second-order derivative with respect to space. They demonstrate the stability of the scheme by utilizing the Fourier method, and they test its accuracy by applying it to a test example. Numerical tests have verified the accuracy and efficiency of the proposed method.

Akram et al. (2022) proposed generalization of the collocation method to solve the time fractional Black-Scholes European option pricing model by extending cubic B-splines. One of the key features of the strategy they used it turns problems of this kind into algebraic equations that can be used to program computers. As a result they found, not only are the problems simplified, but computations are also sped up. In this article, they examined whether the scheme is stable and convergent in terms of Fourier analysis. It is also proposed that a numerical scheme can be constructed with second-order accuracy in the spatial direction. As can be seen from their numerical and graphical results, the suggested approach for the European option prices is in good agreement with analytical solutions.

2.5 Fractional Diffusion-Wave Equation

Two basic examples of fractional differential equations are the diffusion equation and the wave equation. In physics, Nigmatullin (1986) introduced the fractional diffusion equation for describing diffusion in porous media with fractal geometry. It has been shown that the fractional diffusion-wave equation can more accurately model many electromagnetic, acoustic, and mechanical universal responses. The fractional diffusion-wave equation has been numerically analyzed by little authors, when compared to considerable theoretical analysis.

Agrawal (2002) showed that the Fractional diffusion-wave equations defined in bounded space domains that gave a general solution. In the Caputo sense, fractional time derivatives are defined. An equation in the space domain is converted to a wavenumber domain utilized a finite sine transform. In order to reduce the equation to an algebraic equation, they used the Laplace transform. Solutions are obtained by utilizing the inverse Laplace and inverse finite sine transforms. While, A Mittag-Leffler function is used to express the response expressions. The solutions to the first and second derivatives are ordinary diffusion and wave expressions. In order to demonstrate how they presented the technique, two problems are provided. A fractional time derivative of order $\frac{1}{2}$ exhibits slow diffusion, while a fractional time derivative of order $\frac{3}{2}$ exhibits mixed diffusion-wave behavior.

An extension of the two-dimensional differential transform method by Momani et al. (2007) will allow the method to be applied to diffusion-wave equations with space- and time-fractional derivatives. Using Taylor's generalized formula and Caputo fractional derivatives, the new generalization is presented. With their proofs, they introduced new theorems that were never known before. These results are illustrated with several problems to demonstrate their effectiveness. They proved that the technique is very effective and convenient for solving fractional order partial differential equations.

Al-Khaled and Momani (2005) generalized the partial differential equation of diffusion by replacing the first order time derivative with a fractional derivative of order $\alpha, 0 < \alpha \leq 2$. A generalized fractional diffusion equation (diffusion-wave) is approximated using the decomposition method. Caputo sense was used in their paper to define fractional derivatives. They found that, the decomposition methodology can be utilized to find precise solutions to a wide range of examples. Specifically, for $\alpha = 1$ and $\alpha = 2$, the general solution is reduced to the diffusion and wave solutions. Diffusion and wave propagation are related by numerical results (when $\alpha = \frac{3}{4}$).

2.6 Fractional Series method

As can be seen in any book of analysis, power series have become fundamental tools for studying elementary functions and other types of function. They are widely used in computational science to obtain approximations of functions Apostol (1991). It has enabled scientists in Mathematics, Chemistry, and other disciplines to make approximate solutions of a wide variety of systems, while neglecting higher order terms about equilibrium points. Utilizing this method, one can linearize a problem and analyze it easily.

Sezer and Akyüz-Daşcıogˇlu (2007) discuss a generalization of the pantograph equation, a functional differential equation that has a linear functional argument. This paper presents an approximate solution of the pantograph equation using Taylor polynomials for retarded and advanced cases. When the known functions in equation can be expanded into Taylor series, the method has the best advantage. It is important to choose a large truncation limit N in order to get the best approximation from the Taylor expansion of functions.

Yalçinbaş and Sezer (2000) develops a Taylor method for solving high-order linear Volterra-Fredholm integrodifferential equations under mixed conditions in terms of Taylor polynomials. This method has the advantage of expressing the solution as a truncated Taylor series, and therefore, as a Taylor polynomial when $x = c$. As a result, the solution $y(x)$ at low computation effort can be calculated after the coefficients of the series are calculated. In the paper, they prove that the method is efficiency by finding an analytical solution in many examples.

Abu Arqub, Abo-Hammour, et al. (2013), present a novel analytical approach which has been developed to solve higher-order initial value problems of ordinary differential equations. A rapidly convergent series with easily commutable components was implemented utilizing symbolic computation software to construct a series solution for higher-order initial value problems. In the suggested method, a polynomial is utilized to construct an analytical solution, thereby reproducing the exact solution when the solution is polynomial. To demonstrate the precision, and qualification of this technique, they present different examples. Methods found to be effective, straightforward, and easy were found to be very active.

Due to their complex integration-differential definition, fractional derivatives are difficult to study due to the complexity of their integro-differential formula, requiring careful manipulation with standard integer operators. In many papers, authors utilizing series solutions to solve fractional differential equations based on fractional power series.

Liaqat et al. (2022), modify the power series solution method to fractional order by using conformable derivatives for the solution of coupled systems of nonlinear fractional partial differential equations. The method is known as conformable fractional power series. The absolute errors of three examples are considered numerically to evaluate the efficiency and consistency of the method. The recommendations have proven to be unpretentious, accurate, valid, and capable. Compared to homotopy analysis and Adomian decomposition, it has a powerful advantage in solving nonlinear complications. Further, the residual power series method requires calculating fractional derivatives every time the coefficients are generated, whereas this technique only requires equating coefficients. Series solutions are also analyzed for convergence and error.

Cheng et al. (2022), examined Keller-Segel type time fractional diffusion equations and their solutions. Based on symmetry analysis, this fractional system admits Lie symmetries in Riemann–Liouville sense. The Erdélyi–Kober differential operator is used to derive the power series solution based on similarity reductions. A generalized Noether theorem is used to discuss conservation laws based on the symmetries above.

Additionally, the invariant subspace method and the q-homotopy analysis method are used to calculate the initial values of time fractional Keller-Segel equations in Caputo sense.

Bayrak et al. (2022), proposed an enhanced residual power series technique for approximate solution of linear and nonlinear space-time fractional problems with Dirichlet boundary conditions by utilizing new parameter λ. Based on this parameter, the numerical solutions for space-time fractional differential equations can be established. Because Dirichlet boundary conditions vary for each example, the best option of parameter λ depends on the example. As a result of this research, they have made a major contribution. Additionally, the illustrated examples demonstrate that the best approximate solutions are constructed for distinct values of parameter λ. Furthermore, numerical examples demonstrate the activity and accuracy of this method.

Kumar et al. (2022), presented a method for solving non-local boundary value problems arising in chemical reactor theory based on fractional-order Lagrange polynomials. They began by determining the operational matrix for integer and fractional derivatives in the proposed numerical method. As a result of the operational matrix and collocation at the nodal points, they obtained a system of algebraic equations that can be readily resolved for unbeknown coefficients. A convergence analysis of the proposed technique has also been conducted. Various numerical problems showed the naivety of this technique and its high accuracy, even while utilizing fractional order Lagrange polynomials.