GEOMETRICAL ANALYSIS OF QUINTIC TRIGONOMETRIC BÉZIER SURFACE

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by

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LIST OF ABBREVIATIONS

CAGD	Computer Aided Geometric Design
CAD	Computer Aided Design
CAM	Computer Aided Manufacturing
CG	Computer Graphics
SC	Shape Index-Curvedness
HK	Gaussian and mean curvature
NURBS	Non Uniform Rational B-Spline
3D	Three-dimensional

LIST OF SYMBOLS

$\mathbb{R}^2$	two-dimensional vectors
$\mathbb{R}^3$	three-dimensional vectors
$\sum$	summation
$\cdot$	dot product
$\times$	cross product
$\sqrt{}$	square root
$\|\|\|$	norm
$\vec{T}$	vector T
κ_1	principal curvature
κ_2	principal curvature
H	mean curvature
K	Gaussian curvature
S	shape index
C	curvedness

ANALISIS GEOMETRI PERMUKAAN BÉZIER KUINTIK TRIGONOMETRI

ABSTRAK

Lengkung Bézier kuintik trigonometri dengan dua parameter bentuk telah dikaji secara meluas kerana kefleksibelannya. Lazimnya, lengkung Bézier telah banyak digunapakai sebagai alat reka bentuk lengkung atau permukaan dalam industri pembuatan. Oleh itu, kajian mengenai kelengkungan permukaan diperlukan dalam analisis reka bentuk. Dalam penyelidikan ini, lengkung kuintik trigonometri Bézier telah digunakan untuk menjana pelbagai permukaan boleh laras seperti permukaan hasil darab tensor, tersapu, berayun, bergaris dan permukaan boleh berkembang dengan menggunakan pelbagai nilai parameter bentuk. Kesan parameter bentuk terhadap permukaan-permukaan ditunjukkan. Kelengkungan Gaussian, kelengkungan min, dan indeks bentuk-kelengkungan (kelengkungan SC) akan digunakan untuk memeriksa ciri-ciri geometri sesuatu permukaan. Plot kelengkungan Gaussian dan min bagi setiap permukaan digambarkan dan dinilai. Selain itu, kajian ini membentangkan kaedah alternatif untuk memeriksa sifat-sifat geometri menggunakan algebra tak berubah. Kelengkungan permukaan boleh dibandingkan menggunakan geometri kebezaan dan pendekatan algebra tak berubah yang membawa kepada penemuan menarik. Tambahan pula, data berangka dipersembahkan untuk menyokong analisis geometri permukaan yang dipaparkan dalam plot 3D. Kesimpulannya, permukaan yang berbeza akan menghasilkan nilai kelengkungan yang berbeza, namun parameter bentuk akan mengubah keamatan kelengkungan. Malahan, titik kawalan boleh menghasilkan pelbagai permukaan dengan variasi kelengkungan.

GEOMETRICAL ANALYSIS OF QUINTIC TRIGONOMETRIC BÉZIER SURFACE

ABSTRACT

The quintic trigonometric Bézier curve with two shape parameters has been extensively investigated due to its flexibility. Commonly, the Bézier curve has been widely used as a curve or surface designing tool in manufacturing industries. Hence, the study of surface curvature is required in design analysis. In this research, the quintic trigonometric Bézier curve has been implemented to generate various adjustable surfaces such as tensor product, swept, swung, ruled, and developable surfaces by using various value of shape parameters. The effect of the shape parameters on the surfaces has been demonstrated. Gaussian curvature, mean curvature, and Shape Index-Curvedness (SC Curvature) will be used to examine the geometric characteristics of surfaces. The Gaussian and mean curvature plots for each surface are visualised and evaluated. In addition, this study presents an alternate method for inspecting the geometrical properties of a surface using algebraic invariants. Surface curvature can be compared using differential geometry and algebraic invariants approach, leading to interesting discoveries. Additionally, the numerical data are presented to support the surface's geometrical analysis that has been demonstrated by the 3D plot display. In conclusion, different surfaces will produce different curvature value, however, the shape parameters will alter the curvature's intensity. In fact, control points can generate several types of surfaces, which leads to various surfaces with variations of curvature.

CHAPTER 1

INTRODUCTION

1.1 Introduction

The utilisation of Computer Aided Geometric Design (CAGD) is undeniable in the manufacturing industry. The combination of the underlying idea of Computer Aided Design (CAD) and Computer Graphics (CG) leads to the pioneering and advancement of CAGD. In the discipline of CAGD, the deep discovery of mathematics for Computer Aided Manufacturing (CAM) or CAD and CG was the forerunner (Farin, 2014). The geometry concept from the mathematics field has been adapted to solve various difficulties by emphasising the study of curves and surfaces. In fact, Bézier curves and surfaces are claimed to be reliable methods for generating free-form curves and surfaces in CAD and CAM (Maqsood et al., 2021).

Bézier curve, the robust type of curve devised by Pierre Bézier and Casteljau has been the main focus in the field of CAGD nowadays. As technology evolves, numerous research of the classical Bézier curve have been adapted to create various novel type of curves that can enhance the modification of the curve to sustain its capability in engineering sectors as technology evolves. For instance, the trigonometric Bézier curve which has been the significant curve that comprises the sine and cosine functions in the Bézier basis functions can be an alternative option for rational Bézier. In fact, the relationship between trigonometric and rational polynomial parameterisation is discovered by Sánchez-Reyes (1998). The discovery also proved that every curve that allows trigonometric parameterisation can be represented as rational Bézier.

Hence, the functionality of the curves will be increased and various degrees of trigonometric Bézier curves with shape parameters are emerged such as quadratic trigonometric Bézier curve (Uzma et al., 2012), cubic trigonometric Bézier (Han et al., 2009), quartic trigonometric Bézier (Dube and Sharma, 2013; Zhu et al., 2012) and quintic trigonometric Bézier curve (Misro et al., 2017). Apart from curve, surface also has been established by several types of Bézier curve's basis functions. Through the shape parameters that complement the trigonometric Bézier curve, it can produce an adjustable surface as proposed by Ammad and Misro (2020). The adjustable shape parameters allow the designer to alter the surface without modifying its control points. Then, it would be an interesting topic to implement the concept of CAD generated surface from trigonometric Bézier curve. Advanced surface construction methods that are frequently available in CAD include swept, ruled, and swung surfaces.

In addition, one of the intriguing surfaces to study is a developable surface. It is a remarkable surface that was constructed from the ruled surface and can be flattened out into a plane without breaking and tearing. The evolution of developable surfaces are becoming acknowledged by many scholars with significant results since it has been applied in manufacturing industries that produced aircraft skin, automotive components, ship hulls and clothing (Ammad et al., 2021). The fact that the intrinsic geometry of the developable surface, which is zero Gaussian curvature, allows the surface to be managed easily. On top of that, developable surface also has been claimed as more cost-effective in the manufacturing process (Gavriil et al., 2019). Nevertheless, there are some possibilites of defective products being produced due to machining problems.

Therefore, surface curvature is very crucial element in determining the surface quality and smoothness through surface curvature analysis. This statement is supported by Bartkowiak and Brown (2019), which claimed that the smoothness of the topological surface can be indicated by using curvature. Besides, Garcia et al. (2021) also stated that Gaussian curvature can offer additional information for surface analysis. Surface curvature analysis or surface interrogation terms have been widely used to describe how the surface behaves. From the differential geometry field, Gaussian and mean curvature are the two primary surface curvatures that have been used as measuring tools to quantify and interpret the geometrical characteristics of surfaces. Gaussian and mean curvature have also contributed in analysing the medical image from various radiological imaging methods like X-ray and Magnetic Resonance Imaging (MRI) (Pienaar et al., 2008; Tang et al., 2005).

In contrast, the mean and Gaussian curvature at a surface's point are not highly represent the local surface (Koenderink and Van Doorn, 1992). Thus, the authors proposed two novel surface local shape measures: the shape index (S) and curvedness (C). Based on the research of Cantzler and Fisher (2001), the comparison between Gaussian and mean curvature (HK) with Shape Index-Curvedness (SC) has proved that both approaches gave a nearly identical performance on images considering single surfaces. On top of that, the application of shape index and curvedness have been studied numerously in image or shape recognition (Zhao et al., 2013), medical imaging analysis (Clouchoux et al., 2012; Hu et al., 2013) and product design (Bendjebla et al., 2018).

Invariants are quantities assigned to objects that do not change when the coordinate system is transformed, making them helpful recognition descriptors (Zisserman et al., 1995). Forsyth et al. (1991) briefly discussed the invariant theory and presented a few examples of different variants such as plane translation, differential invariants, projective invariants for pairs of plane conics and others. In 2001, Iri et al. extracted the algebraic invariants from the digital elevation data which correlates to the topographical concepts of troughs and crests to analyse the geomorphological quantities of terrain topography. Then, a broader perspective from Iri et al. (2001) has been adopted by Fujita and Ohsaki (2010b) as shape constraints to generate a high constructability developable surface.

1.2 Motivation

Bézier curve is immensely beneficial in various fields such as CAD, CAGD, computer graphics, engineering and architecture. As technology evolves, the classical Bézier curves have been extended to various advanced curves with distinct basis functions and polynomial degrees. The quintic trigonometric Bézier curve with two shape parameters is believed to be a powerful curve due to its functionality and flexibility which means the desired curve can be obtained by modifying the shape parameters range. Moreover, a higher degree of trigonometric Bézier curve will also gain a greater control over the curves and surfaces compared to lower degree since it has more control points which possessed more degree of freedom.

In addition to these advantages, variety of surfaces can be constructed which inherit similar properties to its curve. Parametric surfaces are commonly used to design shapes and products in manufacturing industries or even in architecture and animation since it is the most promising shape characterisation technique. Thus, surface analysis is crucial to identify any defects and produce better quality products. Then, surface curvature analysis can be an option to improve the overall design due to its intrinsic and extrinsic properties of local geometry. In other words, the desired surface design can be achieved by analysing the suitability of surface curvature according to the designers' preferences.

1.3 Problem Statements

Surface shapes have an essential role in geometric modelling which beneficial to various related fields. The costs associated with rejecting defective products are very high in the manufacturing sector due to machining problems and other external factors. Therefore, it is important to evaluate the product's surface design quality. Surface curvature analysis is one of the techniques that visually aid the designer and engineer to inspect the products and produce fewer defective products.

Previous studies of surface analysis are limited to Bézier, triangular Bézier, rational Bézier and biquadratic Bézier surface patch. Considering a higher degree of trigonometric Bézier basis function and the lack of research in surface curvature and surface analysis of trigonometric Bézier surface, this research will focus on determining Gaussian and mean curvatures on different types of surfaces with arbitrarily assigned shape parameters. Furthermore, shape index-curvedness of curvature (SC curvature) and algebraic invariants will be another alternative measure to evaluate curvature analysis of surfaces.

1.4 Objectives

This thesis focuses on the geometrical analysis and characterisation of various surfaces that are constructed by quintic trigonometric Bézier curve with shape parameters using several curvature analysis approaches. The following objectives would assist this research:

1. To construct various adjustable surfaces using quintic trigonometric Bézier curve and characterise its geometrical properties.
2. To inspect the effect of shape parameters of the generated surfaces.
3. To analyse and compare the curvature and geometric appearance of quintic trigonometric Bézier surfaces using different methods.

1.5 Scope and Limitations

This research offers the general idea of surface curvature analysis described from the mathematical perspective that can be extended in various advanced applications. Advance research implementation can be done using powerful technology from the engineering approach. The primary purpose of this research is to explore and compare a few surface curvature techniques on the various type of generated surfaces using quintic trigonometric Bézier curve. The scope of the study is constrained to five basic CAD surfaces: tensor product, swept, swung, ruled and enveloping developable surface which are not as complex as manufacturing design. In addition, the shape parameters of each surface are arbitrarily assigned to ease the result comparison. Meanwhile, the demonstrated three-dimensional surface curvature plot are restricted to the differential geometry approach.

1.6 Outline of Thesis

The following are the brief descriptions of every chapter in this thesis:

Chapter 2 reviews on CAGD, including the idea of quintic trigonometric Bézier curve and surface. Besides, this chapter also gives an overview regarding the surface curvature methods and their importance. Then, the comprehensive literature reviews and linked works are delivered.

In Chapter 3, the quintic trigonometric Bézier basis function and its properties are presented. The construction of different types of quintic trigonometric Bézier surfaces such as symmetric surface, tensor-product, swung, swept, ruled and enveloping developable surfaces are also demonstrated. Gaussian curvature and mean curvature of differential geometry, SC curvature and algebraic invariants methods are also described in this chapter.

Besides, Chapter 4 demonstrated the effect of arbitrarily chosen shape parameters for each generated surface with their Gaussian and mean curvature plots. The comparisons between the three proposed methods are made for better findings. The curvature analysis of surfaces are discussed comprehensively in this chapter.

Lastly, Chapter 5 concluded this thesis with some recommendations for future works and highlighted some general limitations in this research based on the findings which can motivate other scientific research.

CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

2.1 Computer Aided Geometric Design

Computer-Aided Geometric Design CAGD is a branch knowledge area that contributes to computer graphics, engineering, architecture, and animations. Curves, surfaces and their properties are significant interest in this field. Nowadays, research is becoming more advanced as technology evolves. From the primitive curves and surfaces, their type becoming more diverse. Curves and surfaces are extended to various types, allowing researchers and designers to overcome some of the current curve and surface limitations. Bézier and spline curves are two famous curves that typically capture the curiosity of researchers. With the use of shape parameters, the Bézier curve has been broadened to other unique curves, allowing the designer to adjust the curve's shape according to their preference.

On top of that, curves like the Non Uniform Rational B-Spline (NURBS), B-spline curve and rational Bézier curve are the example of different types of curves. These curves have their advantages. In comparison to the classical Bézier curve, rational Bézier can accurately describe conics curves and surfaces with the aid of its weights. However, the classical Bézier curve has also evolved to increase its application and functionality. Therefore, several researches of the Bézier curve comprising the new creation of the basis functions have emerged, attracting various scholars' attention. From the A-class of Bézier-like (Chen and Wang, 2003) to trigonometric Bézier (Tan and Zhu, 2019) to hybrid Bézier (Bibi et al., 2021) to fractional Bézier curve (Said

Mad Zain et al., 2021), the invention of novel basis functions have been diverse.

2.2 B-spline Curve

Basis spline (B-spline) curve is also another versatile curve in the CAGD consist of piecewise polynomial basis functions. B-spline is believed to be discovered in the early 19$^{\text{th}}$ century to solve the convolution of probability distributions (Farin, 2014). The development of the contemporary theory of spline approximation was pioneered by Schoenberg (1946) which B-spline was applied in statistical data smoothing. Generally, as mentioned in Farin (2014), the B-spline has been a beneficial curve in morphing and designing a font. For instance, Zhenyu et al. (2020) manipulated and modelled hairy-brush Japanese characters using the uniform B-splines. Another example of B-spline application in morphing can be seen from Gaun et al. (2014), which B-spline morphing had become the CAD extension for automated hot-and-cold geometry transformation of aero engine components.

Over the past few decades, the attention of B-spline research have been shifted to trigonometric B-splines. In the research of Walz (1997), the properties of trigonometric B-spline is studied and being utilised for curve design. The variation of trigonometric B-spline also can be described in distinct degree such as quadratic trigonometric B-spline which is implemented in image interpolation using genetic algorithm (GA) (Hussain et al., 2017) and aircraft design (Majeed et al., 2021). Additionally, there are B-spline with a higher degree of polynomials such as cubic trigonometric B-spline (Abbas et al., 2014; Majeed and Qayyum, 2020). Whereas, B-spline functions are predominantly used in numerical methods for partial differential equation (PDE) problems (Chandrasekharan Nair and Awasthi, 2019; Jiwari et al., 2019; Roul and Kumari, 2022).

2.2.1 Bézier Curve

Curve is widely used in differential geometry, robotics, kinematics, and engineering (Baydas and Karakas, 2019). In CAGD, Bézier curve is the most common curve that have been applied in many research. It is a unique subset of curve which their control points can be arbitrary defined (Berry and Patterson, 1997). This curve has been first developed by Paul de Casteljau and Pierre Bézier. The research on Bézier curve was first investigated by Paul de Casteljau by applying the de Casteljau's algorithm to evaluate the curve. Then, 1962, a French engineer at Renault company named Pierre Bézier has started to implement the curve to design a car body. The Bézier curve is constructed using the Bernstein polynomials with given control points.

In 2013, Qin et al. mentioned that the usage of control parameters is one of the most prevailing research areas in CAGD. Nowadays, there are many types of Bézier curve with different basis function have been derived. For example, Bézier-like curves (Chen and Wang, 2003), C-Bézier curves (Zhang, 1999), H-Bézier curve (Hu et al., 2018b), trigonometric Bézier curve (Han et al., 2009) and λ-Bézier curve (Hu et al., 2016). Each of these curves have their own shape parameters which enable designer to construct variety design of curves without changing its original control points compared to classical Bézier curve that do not have the shape parameters. Moreover, the polynomial degree of the curves also plays the important role in determining the flexibility of curve shape. Thus, there are plenty research on various Bézier curve basis functions that have been enhanced by variations of degree polynomial.

Bézier curve has given many benefits in many others field. For instance, Bézier curve and surface had been established as the modelling techniques for Volkswagen car body structure (Hochfeld and Ahlers, 1990). According to Fitter et al. (2014), Bézier curve has been implemented and enhanced through different approaches to fit the CAD application such as profile approximation, image extraction, modelisation, smoothing and fairing. In recent years, Bézier curve also has been the utilised in vehicle path generation due to its major advantage which are continuous curvature and adjustable control points that makes the shape modification easier (Kawabata et al., 2015). Apart from that, Bézier curve has been used for data interpolation (Harada and Nakamae, 1982), shape optimization of transonic airfoils (Sohn and Lee, 2013), 3D Valencia orange model (Tinoco et al., 2020) and fingerprint restoration (Tu et al., 2020).

2.3 Trigonometric Bézier Curve

By adjusting the weight, number of control points, shape parameters, knots, as well as lowering and raising the order of the curve within the given constraints, which contribute to the new mathematical basis of the curve representation without affecting its properties (Fitter et al., 2014). Trigonometric Bézier curve has been proposed by many researchers as the alternative basis functions that allow the shape to be alter according to their desired preference. These trigonometric Bézier basis functions have gained much interest among researchers because their implementation can be seen with different degrees of polynomials, which can work with cylinders, cones and circular arcs precisely compared to the ordinary Bézier curve (Han et al., 2009).

From the point of view of degree polynomials, quadratic has been the lowest degree choice of interest. The idea of quadratic trigonometric polynomial curve with a shape parameter have been proposed by Han (2002) which has been studied for B-spline. Then, Uzma et al. (2012) extended the research to quadratic trigonometric Bézier curve that is similar to quadratic Bézier curve. In 2009, Han et al. presented cubic trigonometric Bézier curve with two shape parameters meanwhile Zhu et al. (2012) has introduced quartic trigonometric Bézier curves and surfaces. Trigonometric Bézier curve also have been introduced to quintic degree by Misro et al. (2017). In addition to that, quasi-quintic trigonometric Bézier curve with two shape parameters also has been studied (Bashir et al., 2013; Tan and Zhu, 2019). Other examples of advance trigonometric Bézier curve are sextic trigonometric Bézier curve (Naseer et al., 2021), generalised hybrid trigonometric Bézier (BiBi et al., 2020) and generalised trigonometric Bézier (Ammad et al., 2022).

2.3.1 Quintic Trigonometric Bézier Curve

Quintic trigonometric Bézier curves with two shape parameters are presented by Misro et al. (2017) to show that the shape parameters can permit flexibility on the shape of the curve besides providing the curve's geometrical features. In addition, geometric characteristics of the curve can be maintained by adjusting the value of shape parameters. Hence, this curve has been applied numerously for curve fitting (Adnan et al., 2020) and path planning (Bulut, 2021).

Quintic trigonometric Bézier curve with two shape parameters with six control points $P_i, (i = 0,1,2,3,4,5)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is defined as:

$$z(t) = \sum_{i=0}^{5} P_i f_i(t), \quad t \in [0,1], \quad \alpha, \zeta \in [-4,1]. \tag{2.1}$$

where f_i is the basis functions and P_i is the control points for the quintic trigonometric Bézier as shown in Figure 2.1.

The quintic trigonometric Bézier basis functions for arbitrarily real values of shape parameters α and ζ, where $-4 \leq \alpha, \zeta \leq 1$, and for $t \in [0,1]$ are defined as:

$$\begin{aligned}
f_0(t) &= (1-\sin\frac{\pi t}{2})^4(1-\alpha\sin\frac{\pi t}{2}),\\
f_1(t) &= \sin\frac{\pi t}{2}(1-\sin\frac{\pi t}{2})^3(4+\alpha-\alpha\sin\frac{\pi t}{2}),\\
f_2(t) &= (1-\sin\frac{\pi t}{2})^2(1-\cos\frac{\pi t}{2})(8\sin\frac{\pi t}{2}+3\cos\frac{\pi t}{2}+9),\\
f_3(t) &= (1-\cos\frac{\pi t}{2})^2(1-\sin\frac{\pi t}{2})(8\cos\frac{\pi t}{2}+3\sin\frac{\pi t}{2}+9),\\
f_4(t) &= \cos\frac{\pi t}{2}(1-\cos\frac{\pi t}{2})^3(4+\zeta-\zeta\cos\frac{\pi t}{2}),\\
f_5(t) &= (1-\cos\frac{\pi t}{2})^4(1-\zeta\cos\frac{\pi t}{2}).
\end{aligned} \tag{2.2}$$

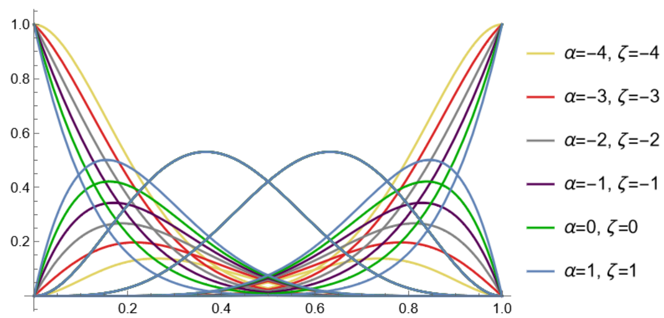


Figure 2.1: The quintic trigonometric Bézier basis functions.

2.3.1(i) Properties

i) Endpoint terminal

The properties of the quintic trigonometric Bézier curve are endpoint terminal, convex hull, symmetry, and geometric invariance. The first and last control points are the endpoints of the curve. Both endpoints interpolate and connect with the Bézier curve. In other words,

$$
\begin{aligned}
z(0) &= P_0, \\
z(1) &= P_5, \\
z'(0) &= -\frac{\pi}{2}(P_0 - P_1)(4+\alpha), \\
z'(1) &= -\frac{\pi}{2}(P_4 - P_5)(4+\zeta), \\
z''(0) &= \pi^2(3P_2 - 2P_1)(3+\alpha) + P_0(3+2\alpha), \\
z''(1) &= \pi^2(3P_3 - 2P_4)(3+\zeta) + P_5(3+2\zeta).
\end{aligned}
\tag{2.3}
$$

ii) Convex Hull

Convex hull property of the quintic trigonometric Bézier curve is satisfied when the curve segment of point P_i, where $i = 0,1,2,3,4,5$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ lies inside its control point polygon. The curves' projection lies inside the convex hull for $\alpha, \zeta \in [-4,1]$ is demonstrated in Figure 2.2.

There are several efficient algorithms for computing the convex hull of a set of points (Cormen et al., 1990). According to the above definitions, the convex hull of a Bézier curve is the boundary of the intersection of all the convex sets containing all vertices.

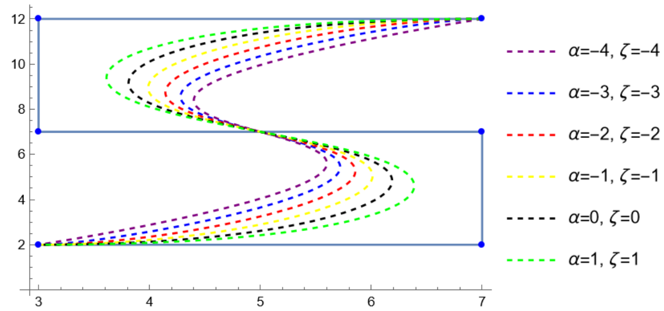


Figure 2.2: The projection of quintic trigonometric Bézier curve inside the convex hull

iii) Symmetry

If the control points are defined in the opposite order, the same Bézier curve shape will be produced. To be specific, $P_0, P_1, P_2, P_3, P_4, P_5$ and $P_5, P_4, P_3, P_2, P_1, P_0$ are defined as the same quintic trigonometric Bézier curve in different parameterisation, i.e.,

$$z(t;\alpha,\zeta : P_0,P_1,P_2,P_3,P_4,P_5) = z(1-t;\zeta,\alpha : P_5,P_4,P_3,P_2,P_1,P_0), \qquad (2.4)$$

where -4 $\leq \alpha, \zeta \leq 1$ and $0 \leq t \leq 1$.

iv) Geometric Invariance

Geometric invariance is also another property of the quintic trigonometric Bézier curve. The partition of unity of quintic trigonometric Bézier basis functions provides the form of the Bézier curve that remains unchanged when its control points are rotated and translated. The curve's shape is independent of the coordinate system used and it fulfils the equations:

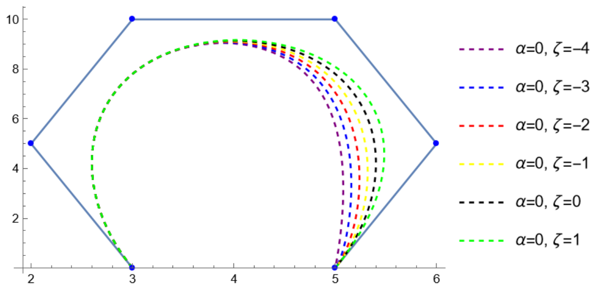
$$\begin{aligned} z(t;\alpha,\zeta : P_0+m,P_1+m,P_2+m,P_3+m,P_4+m,P_5+m) \\ = z(t;\alpha,\zeta : P_0,P_1,P_2,P_3,P_4,P_5)+m, \end{aligned} \qquad (2.5)$$

$$z(t;\alpha,\zeta:P_0\times T,P_1\times T,P_2\times T,P_3\times T,P_4\times T,P_5\times T)$$
$$=z(t;\alpha,\zeta:P_0,P_1,P_2,P_3,P_4,P_5)\times T, \tag{2.6}$$

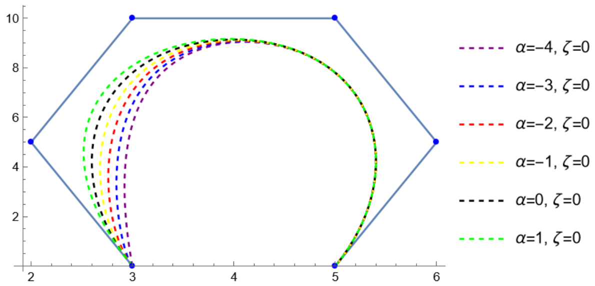
where $0\leq t\leq 1, -4\leq\alpha,\zeta\leq 1$, m is an arbitrary vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, while T is an arbitrary $d\times d$ matrix with $d=2$ or 3.

2.3.2 Effect of Shape Parameters on Curve

Recently, the notions of shape parameters of Bézier basis functions have been the centre of attention in CAGD studies. The effects of shape parameters on the quintic trigonometric Bézier curve are demonstrated in Figure 2.3, where the control points are $P_0=(3,0), P_1=(2,5), P_2=(3,10), P_3=(5,10), P_4=(6,5)$ and $P_5=(5,0)$. For simplification, the α and ζ parameter values are fixed in Figure 2.3(a) and 2.3(b) respectively.



(a) α=0 with variation of ζ values



(b) ζ=0 with variation of α values

Figure 2.3: The effect of shape parameters on quintic trigonometric Bézier curve

Generally, as can be seen in Figure 2.3(a) and 2.3(b), modifying the value of ζ only affects the right side of the curve meanwhile changing the α values will affect the left side of the curve. In other words, control points $P_5 - P_4$ and $P_4 - P_3$ will be impacted by the shape parameter ζ whereas α will give the influence on $P_2 - P_1$ and $P_1 - P_0$. Furthermore, decreasing the shape parameters will make the curve moves away from the control polygon.

2.4 Surface Modelling

In modelling surface or shape, CAD is considered as an efficient design tools. Then, a mathematical and topological descriptions of the geometry that can be incorporated into a computer would make the modelling process easier. In recent years, the study of surface modelling has been focused on the adjustable engineering or CAD surfaces based on their mathematical representation. An example of surface types which contribute in the advancement of the study is Bézier surface. In fact, Bézier surfaces are claimed to be essential method to represent surfaces in computer based engineering design and software engineering (Rababah and Mann, 2011).

Commonly, there are a few basic types of surface tools available in CAD software which are sweep and revolve. In 2018, Hu et al. have constructed several adjustable engineering surfaces such as cylinder, bilinear, ruled, swung, swept and rotation surface using shape adjustable generalised Bézier (SG)-curve with shape parameters. Later in 2020, Ammad and Misro have implement the quintic trigonometric Bézier curve to generate swung and swept surface. Additionally, the ruled, cylindrical, swept and swung surface are also created using generalised hybrid trigonometric Bézier basis functions (Bibi et al., 2021). Recently, by the study of Zain and Misro (2023), the flexible and adjustable fractional Bézier curve is extended to construct tensor product and ruled surface.

2.5 Tensor-product Bézier Surface

In 3D space, a parametric surface that consists of two Bernstein polynomials with u-direction and v-direction is called tensor-product Bézier surface of degree (m,n). Since the range of u-direction and v-direction are in the range between 0 and 1, the surface has a limited rectangular boundary term as a patch. The simplest form of this surface is the bilinear Bézier surface, or the other name is the bilinear Bézier patch. It is named bilinear because the degree of m and n are linear in each parametric direction. In addition, there are also other common Bézier patches with different and extended degrees, such as biquadratic Bézier surface and bicubic Bézier surface.

Furthermore, Bézier surfaces have been implemented in many large car manufacturers such as Renault and Volkswagen company. In Volkswagen, they started using computational methods to design the body of their car in the mid-1960s. To ensure that the surface modelling is applied in the car-body design successfully, this company made a few standard requirements for the software and mathematical methods: high approximation quality, high surface smoothness, efficient shape manipulation, low computational cost, fast data processing and easy data exchange. Although the Bézier curve and surface's fundamental theory was introduced earlier in 1969, the research is quite limited since the company preferred to use Coon's patch interpolation. The Coons approach has been applied until 1972 before they considered using the

Bézier patch due to its advantages in the iterative variation of the blending functions and superposition of well-shaped small 'corrective' patches as ultimate help (Hochfeld and Ahlers, 1990).

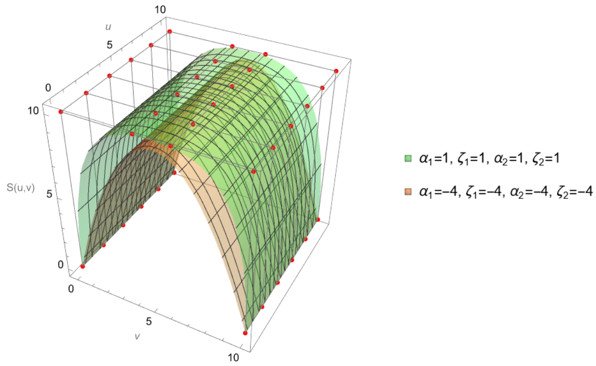
2.5.1 Quintic Trigonometric Bézier Surface

A biquintic trigonometric Bézier surface can be generated by extending the concept of a quintic trigonometric Bézier curve. The term biquintic itself means that the both u and v-direction of surface have the same polynomial degree. Then, this tensor-product surface is constructed by the degree $m = 5$ and $n = 5$, with shape parameters α_1, ζ_1, and α_2, ζ_2 as their shape parameters. The adjustable surface can be achieved with the help of four shape parameter values without modifying its 36 control points as suggested by Ammad and Misro (2020). Additionally, the boundary property, convex hull, symmetry and invariance of Bézier surface are all satisfied. In this research, the Gaussian and mean curvatures of variety biquintic trigonometric Bézier surfaces such as swept, swung and ruled surfaces will be studied.

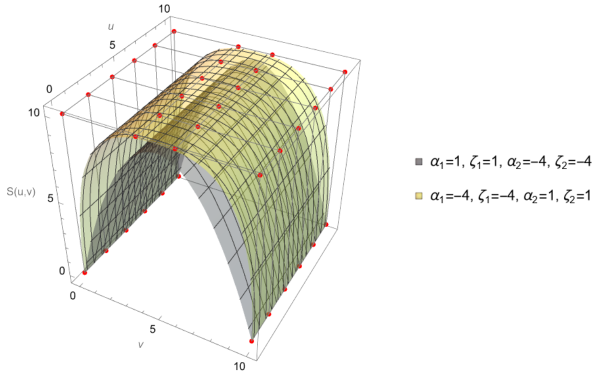
2.5.2 Effect of Shape Parameters on Surface

Other than curves, surfaces are also can be generated from quintic trigonometric Bézier, including tensor product surfaces which are constructed from the product of two parametric curves in two directions by using Equation (3.1). The construction of tensor product surfaces in Figure 2.4 and Figure 2.5 is based on these control points: (0, 0, 0), (0, 0, 10), (4, 0, 10), (6, 0, 10), (10, 0, 10), (10, 0, 0), (0, 2, 0), (0, 2, 10), (4, 2, 10), (6, 2, 10), (10, 2, 10), (10, 2, 0), (0, 4, 0), (0, 4, 10), (4, 4, 10), (6, 4, 10), (10, 4, 10), (10, 4, 0), (0, 6, 0), (0, 6, 10), (4, 6, 10), (6, 6, 10), (10, 6, 10), (10, 6, 0), (0, 8, 0), (0, 8, 10), (4, 8, 10), (6, 8, 10), (10, 8, 10), (10, 8, 0), (0, 10, 0), (0, 10, 10), (4, 10, 10), (6, 10, 10), (10, 10, 10), (10, 10, 0).

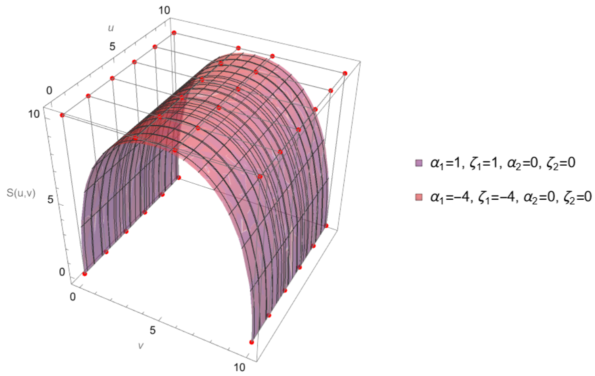
From Figure 2.4(a), it can be seen that as the value of α and ζ increases, the closer the surface towards the control polygons. Meanwhile, the changes of shape parameters of u and v direction are exemplified in Figure 2.4(b). Generally, increasing the shape parameters will make the surface moves towards the control polygon. Note that, for this particular surface, the shape parameters α_1 and ζ_1 do not give significant changes to the surface which can be observed in Figure 2.4(c). Thus, from Figure 2.4(b) it can be seen that the yellow surface is closer to the control polygon due to the value of α_2 and ζ_2 equal to one.



(a) Variation of $\alpha_1 = \zeta_1 = \alpha_2 = \zeta_2$ values



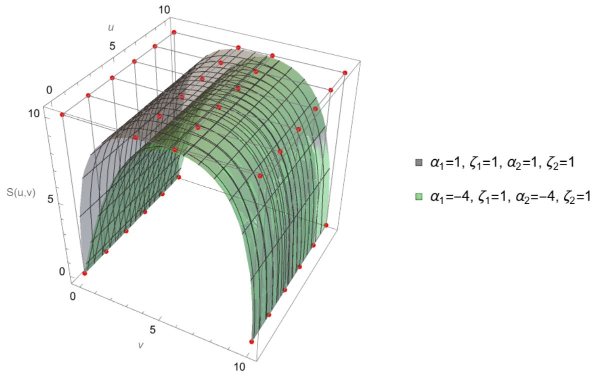
(b) Variation of $\alpha_1 = \zeta_1$ and $\alpha_2 = \zeta_2$ values



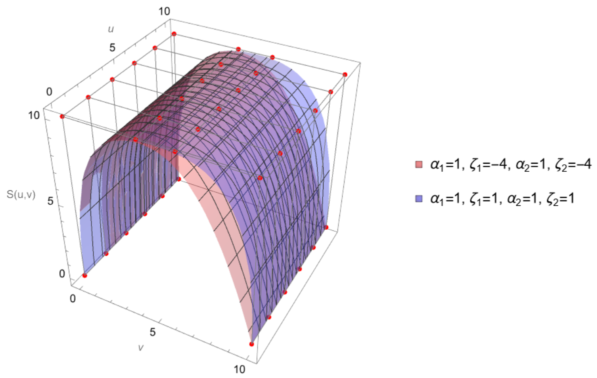
(c) Change of α_1 and ζ_1 values

Figure 2.4: The effect of shape parameters on quintic trigonometric Bézier surface (cont.)

In addition, the changes of α and ζ parameter values have been demonstrated in Figure 2.5. A change of α_1 does not give any significant modification on the surfaces. On the other hand, as the value of α_2 and ζ_2 decreases, the surface shift to right and left respectively based on Figure 2.5(a) and 2.5(b).



(a) Changes of α_1 and α_2 values



(b) Changes of ζ_1 and ζ_2 values

Figure 2.5: The effect of shape parameters on quintic trigonometric Bézier surface

2.6 Developable Surface

Developable surface is one of the interesting surfaces to be studied. It is a unique surface developed from the ruled surface, which can be extended onto a plane without being shattered and stretched. According to Chu and Chen (2004), a developable surface is mathematically expressed as the envelope of a single-family plane with a fixed tangent plane at all points along its ruling. Due to its advantage, which can construct complex surfaces by splicing numerous developable patches together, developable surfaces are extensively implemented in graphic designs, animations, automotive industry, aeroplane bodies, pipelines, clothes, hulls of ship and other manufacturing industries that are unadaptable to surface stretching (Hu et al., 2018a). Maekawa and Patrikalakis (1994) also claimed that developable surface is important to plate-metal-based industries such as shipbuilding.

The evolution of developable surfaces is becoming acknowledged by many scholars with significant results. The two types of construction techniques of developable surfaces are point geometric representation and line and plane geometric representation. A developable surface is considered a tensor-product surface that satisfies certain boundaries in Euclidean space if the method used is geometric point representation. Meanwhile, using the line and plane geometric representation, the developable surface is considered a curve in 3D projective space that can construct the duality between points and planes. The concept of duality between points and planes of Bézier, rational Bézier and B-spline has been introduced (Bodduluri and Ravani, 1993; Pottmann and Farin, 1995).

In order to counter the limitations of the methods presented by Bodduluri and Ravani (1994) and Pottmann and Farin (1995), some academicians proposed a new type of developable surface with the aid of shape parameters. The additional shape parameters on the developable surface will enable the designer to modify the surface without adjusting the control points and planes. The research regarding developable surfaces with shape parameters has been implemented with different basis functions and degrees of polynomials. For instance, Li and Zhu (2020) constructed C-Bézier developable surface with shape parameters. The construction of the quartic developable surface also has been applied using the H-Bézier basis function with parameters by Hu and Wu (2019).

2.7 Surface Analysis

Surface analysis or interrogation is very crucial in industrial field to analyse the quality of the products' design. The meaning of interrogation method is also the extraction of the geometric information from a model (Patrikalakis and Maekawa, 2002). The purpose of surface interrogation is to extract surface's geometric properties that support the programming and planning of complex surface machining (Lee and Ji, 1997). This statement also supported by Hahmann (1999) who stated that surface interrogation contribute crucial tools to analyse the curvature characteristics and intrinsic shape of parametric surface.

Obviously, the surface analysis is incorporated with analysing the surface features. In manufacturing and engineering fields, the words features have variety definitions as stated in Bendjebla et al. (2018) which are functional, design, and manufacturing feature. Collectively, feature composed a physical entity that are made up of physical part which have patterns of interest in a component model's geometry and topology that serve as high-level entities for analysis. Generally, the feature recognition or the extraction and identification process of manufacturing features are made through the geometry approach.

The utilisation of surface interrogation methods can be categorised into two: visualising surface features and detecting surface characteristics (Hahmann, 1999). The example of surface analysis tools used in Barnhill (1989) are reflection lines and curvatures. Curvature analysis is one of the approaches that can be taken to analyse a