LAPLACE TRANSFORM WITH MODIFIED ANALYTICAL APPROXIMATE METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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by

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LIST OF ABBREVIATIONS

ADM	Adomian decomposition method
BVPs	Boundary value problems
FDEs	fractional differential equations
HPM	Homotopy perturbation method
IVPs	Initial value problems
LT-ADM	Laplace transform Adomian decomposition method
LT	Laplace transform
LTM	Laplace transformation method
LT-HPM	Laplace transform homotopy perturbation method
LT-VIM	Laplace transform variational iteration method
LT-MADM	Laplace transform with modified Adomian decomposition method
LT-MHPM	Laplace transform with modified homotopy perturbation method
LT-MVIM	Laplace transform with modified variational iteration method
ODEs	Ordinary differential equations
VIM	Variational iteration method

LIST OF SYMBOLS

A_n	Adomian polynomials
α	Alpha
β	Beta
D_*^α	Caputo fractional differential operator
p	Embedding parameter (HPM) Euler's
$\Gamma(.)$	Gamma function
λ	General Lagrange multiplier (VIM)
$\mathscr{L}$	Laplace transform
N	Nonlinear operator
D^α	Riemann-Liouville fractional differential operator
J^α	Riemann-Liouville fractional integral operator
τ	Tau
z	Trial function
ξ	Xi

KAEDAH PENGHAMPIRAN ANALITIK PENJELMAAN TERUBAHSUAI DENGAN LAPLACE BAGI PERSAMAAN PEMBEZAAN PECAHAN

ABSTRAK

Sejak kebelakangan ini, persamaan pembezaan pecahan (PPP) telah meraih kepentingan dan kepopularitian secara meluas. Ini adalah kerana kepentingannya dalam memodelkan banyak fenomena dalam sains dan kejuruteraan dengan lebih berkesan dan berpraktikal tinggi daripada fenomena sepadan dalam kalkulus klasik. Sebagai contoh, PPP telah berjaya diaplikasikan ke dalam masalah kimia, biologi, fizik, kewangan, teori kawalan, perubatan dan ekonomi.Jadi, pencarian teknik penyelesaian yang tepat dan berkesan bagi persamaan pembezaan tersebut adalah diperlukan. Walaubagaimanapun, kebanyakan PPP tidak mempunyai penyelesaian tepat berdasarkan fakta di mana mereka menggambarkan masalah dunia sebenar dengan kerumitan yang tinggi. Akibatnya, pencarian penyelesaian tepat menjadi bermasalah. Oleh yang demikian, untuk mendapatkan penyelesaian hampiran bagi persamaan tersebut, berlaku perubahan kitaran ke atas kaedah analitik dan berangka. Dalam kajian ini,tumpuan telah diberikan ke atas kaedah hampiran analitik. Kaedah-kaedah ini adalah; gabungan transformasi Laplace (TL) bersama dengan kaedah usikan homotopi (KUH) yang juga dikenali sebagai transformasi Laplace-Kaedah Usikan Homotopi (TL-KUH). kaedah penguraian Adomian (KPA) juga dikenali sebagai transformasi Laplace-kaedah penguraian Adomian (TL-KPA), kaedah lelaran ubahan (KLU), juga dikenali sebagai transformasi Laplace-kaedah lelaran ubahan (TL-KLU). Walaupun kaedah ini telah selalu digunakan dalam menyelesaikan beberapa jenis PPP, namun masih terdapat cabaran di dalam pemilihan anggaran awal; jika tidak, lelaran tak terhingga adalah diperlukan di mana keberkesanan penyelesaian menjadi terhad. Sasaran utama tesis ini adalah untuk menambah

baik dan menggunakan kaedah-kaedah ini untuk mengelakkan kelemahan dan mencari penyelesaian hampiran analitik untuk beberapa kes PPP biasa linear dan tak linear. Kes-kes ini ialah persamaan Bagley Torvik, persamaan jenis Bratu pecahan dan juga persamaan integro-pembezaan pecahan berperingkat tinggi. Untuk kaedah-kaedah yang telah dicadangkan, fungsi percubaan dalam bentuk siri kuasa disarankan sebagai anggaran awal kepada beberapa contoh berangka untuk memberikan penyelesaian anggaran yang lebih tepat. Kaedah-kaedah yang dicadangkan ini telah diperiksa ke atas beberapa contoh. Penyelesaian hampiran yang diperolehi oleh kaedah-kaedah ini telah dibandingkan dengan penyelesaian hampiran yang diperolehi daripada TL-KUH, TL-KPA dan TL-KLU. Keputusan yang diperoleh menunjukkan kecekapan kaedah ini untuk memberikan penyelesaian hampiran dengan cara yang lebih mudah daripada kaedah-kaedah TL-KUH, TL-KPA dan TL-KLU yang biasa dengan ketepatan penyelesaian hampiran yang baik dan megurangkan kerja pengiraan berbanding dengan kaedah analisis yang sebelumnya.

LAPLACE TRANSFORM WITH MODIFIED ANALYTICAL APPROXIMATE METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT

Recently, fractional differential equations (FDEs) have obtained massive importance and popularity. This is because there are many phenomena in the fields of engineering and science that can be modeled using linear and nonlinear this type with associated supplementary conditions. For examples, FDEs have been successfully applied to problems in chemistry , biology, physics, finance , control theory, medicine and economics. Thus, finding accurate and effective solution techniques for such differential equations is needed. However, most FDEs have no exact solutions because of the fact that they represent real-world problems with high complexity. As a result, obtaining an exact solutions is problematic. Thus, to obtain approximate solutions for these equations, the analytical and numerical methods were developed. In this study, approximate analytical methods are highlighted. These methods are: the combination of the Laplace transformation with homotopy perturbation method (HPM) namely Laplace transform homotopy perturbation method (LT-HPM), Adomian decomposition method (ADM) namely Laplace transform Adomian decomposition method (LT-ADM), variational iteration method (VIM) namely Laplace transform variational iteration method (LT-VIM). Despite these methods have been widely applied in solving several types of FDEs, only it still need to be more suitable in choosing the initial approximation; otherwise infinite iterations are required, which restrict the effectiveness of the solutions. The main target of this thesis is to improve and apply these methods to avoid the drawbacks and find the analytical approximate solutions for some cases of linear and nonlinear ordinary FDEs. These cases are Bagley Torvik equation, fractional Bratu type equations and higher order fractional integro-differential equations as well. For the proposed methods, the trial function in the form of a power series is suggested as an initial approximation to give more accurate approximate solution for the numerical examples. The proposed methods are tested on several examples. The approximate solutions obtained by these methods are compared with the approximate solutions obtained by LT-HPM, LT-ADM and LT-VIM. The obtained results showed the efficiency of these methods to provide the approximate solutions in an easier way than the standards LT-HPM, LT-ADM and LT-VIM with good accuracy and require less computational work as compared to previous analytical methods.

CHAPTER 1

INTRODUCTION

1.1 Research Introduction

Fractional calculus deals with derivative and integrals of non-integer order, which was introduced by Leibniz in 1695 (Derakhshan et al., 2021). Fractional calculus generalizes the notions of ordinary calculus. The main reasons for the renewed interest in the field of fractional calculus are the exploration of solutions using fractional differential equations (FDEs), which are also the generalizations of classical differential equations. There is a revival of FDEs during last decades due to the exact description of complex phenomena in widely different disciplines from science, engineering, physics and other areas of science (Islam et al., 2021). Because of the non-local property of the fractional derivative, the solutions of FDEs are able to describe real-life situations better compared to the solutions generated from the corresponding integer order differential equations, which have a local operator that depends on finding the right and left limits (Li et al., 2011).

Different from derivative of integer order, there are various definitions associated with the fractional derivatives. These definitions are commonly not equivalent with each other. The two most usually applied are Riemann-Liouville and Caputo derivative. One of the main advantages of Caputo fractional derivative is that it allows integer order initial and boundary conditions to be included in the formulation of the problems, which have clear physical interpretation as mentioned by Oliveira et al. (2014).

The exact analytical solutions for most FDEs are not known because it is usually difficult to model the real world problem. Hence, the exact solutions for these problem are normally hard to find. Thus, we often opt for approximate analytical or numerical solutions for the equations representing the said problems. Recently several studies focusing on the numerical and approximate analytical solutions of FDEs have been actively developed (Okundalaye & Othman, 2022). Although the numerical methods are usable in a huge number of practical cases, many authors prefer to use the analytical approximate methods because it provides an analytical representation of the solution. This presents better informational solution over interest intervals, while the numerical methods presents solutions in discretized form, which makes it slightly complicated in realizing a continuous representation. The aim of this thesis is to study and improve approximate analytical methods for the solution of boundary value problems (BVPs) as well as the system of BVP of ordinary FDEs.

1.2 Motivation

The main motivation of this thesis is to improve the known approximate methods that provide solutions to FDEs. In most cases, the FDEs do not have exact solutions. Therefore, several methods for the analytical approximate solutions were applied to solve the equations, such as homotopy perturbation method (HPM) (Alipour et al., 2019; He, 2006), Adomian decomposition method (ADM) (Kumar et al., 2022; Wazwaz & Abdul-Majid, 2005) and variational iteration method (VIM) (Li et al., 2020; Wang et al., 2013). Furthermore, some of this analytic methods have been developed to solve linear as well as nonlinear FDEs as discussed in (Hamoud et al., 2018), (Sandoval-Hernandez et al., 2018), and (Johansyah et al., 2021).

Despite the fact that the approximate analytical methods have been extensively applied in solving various types of FDEs, in some applications when series solution is sought for approximate analytical methods, this may result in some disadvantages which limit the effectiveness of the method. One of the important drawbacks is a suitable selection of the initial approximation satisfying the boundary value problem conditions, or infinite number of iterations that are required to get the satisfying approximate solutions (we will present more drawbacks further in detail in Chapter 2). The thesis is motivated by improving existing methods of obtaining new techniques to avoid these drawbacks and reduce the computational work.

1.3 Research Gap

Based on the information collected through the literature search and discussions, a review was conducted of the available knowledge and research related to analytical approximate methods, focusing on the advantages and disadvantages of these methods. We identified a gap in choosing the appropriate initial approximation to obtain high accuracy of approximate solution. Some researchers have adopted the approach of selecting the exact solution on the first approximation (Irandoust-Pakchin et al., 2013); the main drawback of this choice is when the problem does not have an exact solution, while some have selected the initial approximation as the source function or part of it (Jafari et al., 2014); the prime weakness of this choice when is the problem is homogeneous, whereas other choices depended on the initial conditions. However, there is a drawback when the initial condition is equal to zero. In the current study, we will propose a standard to select the initial approximation that satisfies boundary conditions to overcome previous disadvantages.

1.4 Problem Statement

There are no exact analytical solutions for most FDEs because of their complexity, especially for nonlinear equations. Thus, the analytical approximates were often extensively applied to provide analytical approximate solutions for these differential equations such as HPM, ADM and VIM. However, these methods have drawback in the selection of the initial approximation which is still randomly chosen. Moreover, they require an infinite number of iterations which need massive calculations in each iteration. As a result, our aim is to introduce new techniques based on these methods to remove the random choice of initial guess by setting a specific rule depends on unknown parameters, where these parameters contributed to the increase in the number of terms of the polynomial approximation and its degree, which in turn, accelerated the convergence and increased the accuracy from one iteration.

1.5 Objective

The objectives of this study are:

1. To introduce a new modification depending on the Laplace transform (LT) with HPM, ADM and VIM which are called the Laplace transform modified homotopy perturbation method (LT-MHPM), Laplace transform modified Adomian decomposition method (LT-MADM) and Laplace transform modified variational iteration method (LT-MVIM) respectively to solve various kind of BVPs of linear and nonlinear FDEs.

2. To derive LT-MHPM, LT-MVIM and LT-MADM to solve system of BVPs of linear and nonlinear FDEs .

3. To show the accuracy of LT-MHPM, LT-MADM and LT-MVIM by comparing the exact solutions with the approximate solutions obtained by the standard Laplace transform homotopy perturbation method (LT-HPM), Laplace transform Adomian decomposition method (LT-ADM), and Laplace transform variational iteration method (LT-VIM).

1.6 Methodology

The methodology of this study is presented and discussed in this section. The focus will be on the LT-HPM, LT-ADM and LT-VIM to solve various kind of BVPs of linear and nonlinear FDEs. The general framework of these methods will be studied. Then, these methods will be formulated by choosing an initial approximation as a power series with unknown parameters coefficients to solve linear and nonlinear BVP as well as systems of BVP of FDEs, where these parameters contributed to the increase in the number of terms of the polynomial approximation and its degree, which in turn, accelerated the convergence and increased the accuracy.. This step will provide a basis for the research to follow. New modifications of the LT-HPM, LT-ADM and LT-VIM will be suggested and utilized to solve the Bagley Torvik equation, fractional Bratu type equations and higher-order fractional integro-differential equations as well as the systems of BVPs of ordinary FDEs. The accuracy of LT-MHPM, LT-MADM and LT-MVIM will be demonstrated by comparing the exact solutions with the approximate solutions obtained by the standard LT-HPM, LT-ADM, and LT-ADM methods. Numerical experiments will be achieved to show the efficiency of these modifications. All the numerical examples in this study will be conducted using Maple 2016.

1.7 Thesis Outline

The thesis is presented with chapters 2-8 as follows: Chapter 2 reviews the basic concepts and techniques, which are helpful in the study of fractional calculus. Furthermore, the chapter displays a summary description of the basic principles of the approximate analytical methods that will be studied in this thesis. Chapter 3 presents the recent studies that were studied by many authors to find the approximate solutions of various kinds of ordinary FDEs. In Chapter 4, a new modification is suggested, and several numerical examples are tested. A comparison of approximate solutions obtained by LT-HPM and LT-MHPM with the exact solutions is also presented. Moreover, the convergence of LT-MHPM is discussed. In Chapter 5, the LT-ADM and LT-MADM are utilized to solve special kinds of ordinary FDEs such as the Bagley Torvik equation, fractional Bratu type equations and higher-order fractional integro-differential equations. A comparison of approximate solutions obtained by these methods with the exact solutions is also provided. In Chapter 6, the LT-VIM and its modification are applied to solve the same kinds of ordinary FDEs that were solved in the previous chapters. A comparison of approximate solutions obtained by these methods with exact solutions is also carried out. In Chapter 7, systems of linear and nonlinear of ordinary FDEs are solved using LT-HPM, LT-ADM, and LT-VIM and their modifications. A comparisons of the obtained results by them with exact solutions are performed. Finally, Chapter 8 presents the main results of the study and recommendations are forwarded for further research.

CHAPTER 2

BASIC CONCEPTS AND TECHNIQUES

2.1 Introduction

In this chapter, some basic definitions and concepts which has an important role in the theory of FDEs will be introduced. Furthermore, a brief description of the analytical methods needed in this study will also be given. These methods are LT-HPM, LT-VIM and LT-ADM.

2.2 Fractional Calculus

Fractional calculus is a branch of mathematical analysis that studies the integrals and derivatives of non-integer order. It began in 1695, when Leibniz wrote to L'Hopital asking him "Can we generalize integer derivatives to arbitrary order?". L'Hospital's response took a form of question, "What about if the order is 1/2?" "It will lead to a paradox from which one day many useful consequences will be drawn." replied Leibniz (Anastassiou, 2009). The subject lasted more than 300 years. Many mathematicians have been interested in this topic over the years.

The derivative of arbitrary order was firstly mentioned in a text In 1819. Then Lacroix developed a formula for fractional differentiation, where he started with a positive integer order (i.e. $u = x^n$), from which he found the m^{th} derivative by induction as:

$$\frac{d^m u}{dx^m} = \frac{n!}{(n-m)!} x^{n-m}. \tag{2.1}$$

By replacing m by $\frac{1}{2}$ and the fact $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, he obtained when $u = x$,

$$\frac{d^{\frac{1}{2}} u}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

The derivatives of arbitrary order was developed by Euler and Fourier but they did not mention any application or example. The first application was made in 1823 by Niels Henrik Abel, where he exploited the fractional calculus in the solution of an integral equation. Subsequent references to fractional derivatives were made by several famous mathematicians who had many contributions on this topic such as: Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, and others (Lazarević et al., 2014). At present, fractional calculus witnesses many development, and its applications have entered many fields such as: applied mathematics, physics, engineering, economics, elc. Understanding of definitions and use of fractional calculus requires the deep knowledge of Gamma function (Wang et al., 2022).

2.3 Gamma Function

The gamma function denoted Γ is one type of special functions; it was first introduced by Leonhard Euler (1707-1783) (Radford & David, 2021; Wang et al., 2022). It is also considered as an extension of the factorial to non integer values. Moreover, it is one of the fundamental functions of the fractional calculus.

Definition 2.1 *(Wang et al., 2022) For Re(r)>0, the gamma function is defined as :*

$$\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du.$$

There are many important properties of gamma function which help us to calculate its values. Some of them are mentioned here,

Proposition 2.1 *(Wang et al., 2022) For Re(r)>0,*

- $\Gamma(1) = 1$,
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,
- $\Gamma(a) = (a-1)\Gamma(a-1)$,
- $\Gamma(a) = \frac{\Gamma(a+1)}{a}$.

In addition to the previous properties, the gamma function also shows up in an important relation with the Riemann–Liouville Integral and Derivatives. Furthermore, Caputo's Fractional Derivatives.

2.4 Riemann–Liouville Integral

The Riemann–Liouville integral of arbitrary order α ($\alpha > 0$) for function $f(t)$ is a manner of generalization of the well-known Cauchy formula for the *n*-integral

(Mainardi & Francesco, 2010; Morales et al., 2020).

In our notation, if $f(t)$ is Riemann integrable, then for $t > 0$ and $n \in N$, the Cauchy formula for calculating iterated integrals:

$$J^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} f(x) dx, \tag{2.2}$$

where n is the set of positive integers. To generalize the above formula from positive integer values to any positive real values, the positive integer n is replaced by arbitrary positive real number α, and the gamma function is used instead of factorial because it introduces the arbitrary positive real number α. So the integration of the arbitrary order α is defined as follows:

Definition 2.2 *(Samei et al., 2021) For $\alpha \in R^+$, the Riemann–Liouville integral is defined as:*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \tag{2.3}$$

where $\Gamma(\alpha)$ is gamma function, and for $\alpha = 0$, we set $J^0 = I$, the identity operator.

There are some important properties of this formulation of the fractional integral, which help when solving equations involving integrals of fractional order. Some of them are mentioned here,

Proposition 2.2 *(Weilbeer & Marc, 2005) Let α, $\beta \in R$, the Riemann-Liouville fractional integral that satisfies:*

- $J^0 f(t) = f(t)$,

- $J^\alpha + J^\beta f(t) = J^{\alpha+\beta} f(t)$.

Derivatives and integrations are interrelated with each other. After having defined the fractional integral, it is necessary to study the fractional derivative . There are two important types of fractional Derivatives: Riemann–Liouville Derivatives and Caputo's Fractional Derivatives.

2.5 Riemann–Liouville Derivatives

After mainstreaming the concept of fractional integration, the fractional derivative of order α becomes a natural requirement. Several attempts were made to define the fractional derivative. An integral form for the fractional derivative was used in most of these attempts from which Riemann–Liouville Derivative's is considered one of the most popular.

Definition 2.3 *(Mainardi & Francesco, 2010) Let $\alpha \in R^+$, then the Riemann–Liouville fractional derivative of order α is defined as:*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau & , m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & , \alpha = m, \end{cases}$$

for $\alpha = 0$, we set $D^0 = I$, the identity operator.

After having defined the Riemann–Liouville fractional derivative, it is reasonable to shed light on some of its important properties, which help when solving equations involving derivative of fractional order.

Proposition 2.3 *(Weilbeer & Marc, 2005) Let* $\alpha > 0$, $c_1, c_2 \in R$ *and let* f *and* g *be functions such that their fractional derivatives are:*

$$D^{\alpha}(c_1 f(t) + c_2 g(t)) = c_1 D^{\alpha} f(t) + c_2 D^{\alpha} g(t).$$

Proposition 2.4 *(Weilbeer & Marc, 2005) Let* $\alpha > 0$, *and* $f(t) = t^r$ *for some* $r > 0$, *then*

$$D^{\alpha} f(t) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}.$$

It can be noted that the fractional derivative of a constant using the above formula is not zero if α is not integer, and in fact it is:

$$D^{\alpha} c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \alpha > 0, \quad t > 0.$$

2.6 Caputo's Fractional Derivatives

In 1967, Caputo proposed another definition of fractional derivative that can be used,

Definition 2.4 *(Sontakke & Shaikh, 2015) Let* $\alpha \in R^{+}$. *Then the Caputo fractional derivative of order* α *is defined as:*

$$D_*^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} f(\tau) d\tau & , m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & , \alpha = m. \end{cases}$$

There are two main reasons that make Caputo derivative more important than Riemann-Liouville fractional derivative. Firstly, the Riemann-Liouville fractional derivative leads to initial conditions in the form of limit values, for example, $\lim_{t\to 0} D^{\alpha-n} y(t) = c_n$. Such initial condition can be solved by the Riemann-Liouville fractional derivative, but without any clear physical interpretation, leading to little use of their solutions. A typical feature of differential equations (both classical and fractional) is the need to specify additional conditions in order to produce a unique solution. For the case of Caputo FDEs, these additional conditions are just the static initial conditions, which are akin to those of classical ODEs, and are therefore familiar to us. In contrast, for Riemann-Liouville FDEs, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0 \ldots$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. To overcome this disadvantage of Riemann-Liouville derivative, the Caputo fractional derivative is used, where the initial conditions in terms of derivatives of integer order are involved in order to make these initial conditions have clear physical interpretation. Secondly, in contrary to Riemann-Liouville derivative, the Caputo fractional derivative of a constant is zero. Both of these properties are important to understand real-world phenomena. The Caputo fractional derivative has some important properties, such as:

Proposition 2.5 *(Weilbeer & Marc, 2005) Let $\alpha > 0$, $c_1, c_2 \in R$ and let f and g be functions such that their fractional derivatives, then,*

$$D_*^\alpha(c_1 f(t) + c_2 g(t)) = c_1 D_*^\alpha f(t) + c_2 D_*^\alpha g(t).$$

Proposition 2.6 *(Weilbeer & Marc, 2005) Let $\alpha > 0$, $m-1 < \alpha < m$ and let f be function, then,*

$$J^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{D^k f(0)}{k!} t^k.$$

Proposition 2.7 *(Kilbas et al., 2006) Let $\alpha > 0$, $m-1 < \alpha < m$, then the following relation between the Riemann Liouville and the Caputo operator holds:*

$$D_*^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0).$$

It can be noticed from Proposition 2.7 that there is no coincidence between the Riemann-Liouville and Caputo fractional derivatives except when the function is homogeneous. Furthermore, The LT of the Caputo fractional derivative is a generalization of the LT of integer order derivative, where n is replaced by α (Jarad et al., 2020). The same does not hold for the Riemann-Liouville case, as we will see in the next section. Table 2.1 shows a summary of the derivatives

2.7 Laplace Transform

The LT plays an important role in solving integer and fractional order differential equations. The goal of this section is to introduce the notion of the LT and some of its

Table 2.1: Summary of derivatives.

Caputo derivative	$D_*^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} f(\tau) d\tau,$ $m-1 < \alpha < m.$
Riemann–Liouville derivative	$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau,$ $m-1 < \alpha < m.$
$f(t) = t^r$, $r > 0$	$D^{\alpha} f(t) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}.$
$f(t) = constant$	$D^{\alpha} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0, \quad t > 0.$

properties.

Definition 2.5 *(Saif et al., 2020; Schiff & Joel, 2013) For $R(s) > 0$, the LT is defined as:*

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt, \tag{2.4}$$

it is denoted by $\mathscr{L}\{f(t), s\}$.

There are important properties for LT which help use to calculate its values. Some of them are mentioned here.

Proposition 2.8 *(Saif et al., 2020; Schiff & Joel, 2013) For $Re(s) > 0$, $c_1, c_2 \in R$, and $f(t)$, $g(t)$ are any functions, then:*

- $\mathscr{L}\{c_1 f(t) + c_2 g(t), s\} = c_1 \mathscr{L}\{f, s\} + c_2 \mathscr{L}\{g, s\},$
- $\mathscr{L}\{t^{\alpha}, s\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1,$
- $\mathscr{L}\{\int_0^t f(\tau) d\tau, s\} = \frac{F(s)}{s},$
- $\mathscr{L}\{\frac{d^m f(t)}{dt^m}, s\} = s^m F(s) - \sum_{k=1}^{m} s^{m-k} f^{(k-1)}(0).$

Proposition 2.9 *(Sontakke & Shaikh, 2015) The LT of Riemann-Liouville derivative where $m-1<\alpha<m$, is given by:*

$$\mathscr{L}\{D^{\alpha}f(t),s\}=s^{\alpha}F(s)-\sum_{k=0}^{m-1}[D^{\alpha-k-1}f(t)]_{t=0}s^{k}. \tag{2.5}$$

Proposition 2.10 *(Sontakke & Shaikh, 2015) The LT of Caputo fractional derivative where $m-1<\alpha<m$, is given by*

$$\mathscr{L}\{D_{*}^{\alpha}f(t),s\}=s^{\alpha}F(s)-\sum_{k=0}^{m-1}f^{(k)}(0^{+})s^{\alpha-k-1}. \tag{2.6}$$

It can be observed from the last two properties that the LT of Riemann-Liouville fractional derivative requires initial condition with non-integer order derivative $D^{\alpha-k-1}f(t)$ at $t=0$. This limits their practical application due to the absence of physical interpretation of the initial conditions with fractional derivative where as the LT of Caputo fractional derivative requires the values of integer order derivative $f^{k}(0^{+})$, for which a certain physical interpretation exists.

2.8 Analytical Approximate Methods

In the last decades, analytical approximate methods have focused on the comprehensive study of FDEs. These methods can increase our insights into the natural behavior of complex systems by providing a solution of differential equations in the form of infinite series, such as the HPM (He, 1999a, 2006), the ADM (Adomian & George, 1983, 1988), and the VIM (He, 1998, 1999b). One of the main advantages of analytical approximate methods is its ability in providing a continuous representation of the approximate solution, which allows better information of the solution over the time interval (Ghorbani et al., 2011). In this section, the HPM, ADM and VIM are going to be described in order to solving the general nonlinear FDEs. Some important advantages and disadvantages of these methods are going to be discussed, too. The main reasons to choose HPM, ADM, and VIM are that they give excellent flexibility to the expression of the solution and how the solution is explicitly obtained, and provide great freedom in choosing the base functions of the desired solution. Furthermore, the methods consider a very effective approach for solving broad classes of nonlinear partial and ordinary differential equations.

2.8.1 Homotopy Perturbation Method (HPM)

The HPM was first introduced by He (1999a) and was developed by him in He (2003) and He (2006) by providing a new interpretation of the concept of constant expansion in the HPM. The HPM was also studied by many authors to treat nonlinear equations established in science and engineering (Atangana & Secer, 2013; Kayum et al., 2021). The method is a coupling of the traditional perturbation method and homotopy in topology.

The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equations usually is reduced to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the problem goes through a sequence of deformation, the solution of each of which is close to that at the previous stage of the deformation. Eventually at $p = 1$, the problem takes the original form and the final stage of deformation gives the desired solution (Saadatmandi et al., 2009).

Because the HPM depends on homotopy in topology, we will mention the definition of homotopy in topology first. According to Liao and Shijun (2012), a homotopy between two continuous functions $h(t)$ and $f(t)$ from a topological space X to a topological space Y is formally defined to be a continuous function $H : X \times [0,1] \longrightarrow Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then:

$$H(x,0) = h(x) \quad and \quad H(x,1) = f(x). \tag{2.7}$$

To figure out how HPM works let a nonlinear FDE be:

$$D_*^{\alpha} u(t) = f(t) - Lu(t) - Nu(t), \tag{2.8}$$

where $m-1 < \alpha \leq m$, $m \in N$, $t > 0$,

with the following boundary conditions,

$$\beta(u, \frac{du}{dt}), \tag{2.9}$$

where D_*^{α} is a Caputo fractional derivative whereas f is a known function, while L is a linear operator and N is a nonlinear operator. The solution y is assumed to be a known function, β is a boundary operator.

A homotopy is constructed as follows (He, 1999a, 2000):

$$(1-p)\,[D_*^{\alpha} u(t) - D_*^{\alpha} u_0(t)] + p\,[D_*^{\alpha} u(t) - f(t) + Lu(t) + Nu(t)] = 0, \tag{2.10}$$

Distributing the parentheses, we get:

$$D_*^{\alpha}u(t) - D_*^{\alpha}u_0(t) - pD_*^{\alpha}u(t) + pD_*^{\alpha}u_0(t) + p\left[D_*^{\alpha}u(t) - f(t) + Lu(t) + Nu(t)\right] = 0, \tag{2.11}$$

This in turn leads to:

$$D_*^{\alpha}u(t) = D_*^{\alpha}u_0(t) + p\left[-D_*^{\alpha}u_0(t) + f(t) - Lu(t) - Nu(t)\right], p \in [0,1], \tag{2.12}$$

where $p \in [0,1]$ is a homotopy parameter, while u_0 is the initial approximation for the solution of Eq(2.8) that satisfies the boundary condition.
Suppose the solution for Eq.(2.11) or the Eq.(2.12) can be expressed as a power series of p as:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + ... \tag{2.13}$$

The values for the sequence u_0, u_1, u_2, ... can be found by substituting Eq.(2.13) into the Eq.(2.12) and equating coefficients of p with the same power. When $p \to 1$, it gives the approximate solution for Eq.(2.8) as:

$$u(t) = \lim_{p \to 1} u = u_0 + u_1 + u_2 + u_3 + ... \tag{2.14}$$

The convergence of the series (2.14) has been proved by He (1999a, 2000). The HPM has some advantages that attracted the attention of many authors. One advantage is that it can be applied to many nonlinear problems because it does not require a small parameter in an equation (Zedan & Adrous, 2012). Besides it provides a simple way to ensure the convergence of the solution (Nave et al., 2012). Furthermore, the HPM can be freely constructed in many ways as well as in choosing the initial approximation (Mechee & Al-Juaifri, 2018). Moreover, it can be combined with many other mathematical methods such as: numerical methods, series expansion methods, and integral transform methods, which can take the full advantages of those mathematical methods. However, in some applications, when the series solution is used for the HPM method, it creates a disadvantage in computational time which limits the effectiveness of the method (Liao, 2005). For instance, an initial approximation has to be suitably selected; otherwise, infinite iterations are required that will repeat calculations of unnecessary terms. Therefore, many authors had avoided this disadvantage using modifications and integrating it with other methods like the LT.

2.8.2 Laplace Transform Homotopy Perturbation Method (LT-HPM)

The LT is a perfect technique for solving nonlinear differential equations and has played a significant role in the field of science and engineering (Tripathi & Mishra, 2016). The authors have presented the combination of LT with the HPM to be used reliably in solving different types of FDEs arising in various fields of science and engineering because it allows solving, in a simpler fashion (Abbasbandy, 2006; Khan et al., 2012; Madani et al., 2011).

To present LT-HPM, the same steps of the HPM have to be followed until Eq.(2.12); next the LT is applied on both sides of Eq.(2.12) to obtain:

$$\mathscr{L}\left[D_*^{\alpha}u(t)\right] = \mathscr{L}\left[D_*^{\alpha}u_0(t) + p(-D_*^{\alpha}u_0(t) + f(t) - Lu(t) - Nu(t))\right]. \tag{2.15}$$

Using the formula for LT, we obtain:

$$\begin{aligned} s^{\alpha}\mathscr{L}[u(t)]-\sum_{k=0}^{m-1}s^{\alpha-k-1}u^{(k)}(0) &= \mathscr{L}[D_*^{\alpha}u_0(t)+p(-D_*^{\alpha}u_0(t)+f(t) \\ &- Lu(t)-Nu(t))]. \end{aligned} \tag{2.16}$$

or

$$\begin{aligned} \mathscr{L}[u(t)] &= \frac{1}{s^{\alpha}}\left[\sum_{k=0}^{m-1}s^{\alpha-k-1}u^{(k)}(0)\right]+\frac{1}{s^{\alpha}}\mathscr{L}[D_*^{\alpha}u_0(t)+p(-D_*^{\alpha}u_0(t)+f(t) \\ &- Lu(t)-Nu(t))]. \end{aligned} \tag{2.17}$$

where $,m-1<\alpha\leq m$.

Using the inverse LT of Eq.(2.17), we get:

$$\begin{aligned} u(t) &= \mathscr{L}^{-1}[\frac{1}{s^{\alpha}}[\sum_{k=0}^{m-1}s^{\alpha-k-1}u^{(k)}(0)]+\frac{1}{s^{\alpha}}\mathscr{L}[D_*^{\alpha}u_0(t)+p(-D_*^{\alpha}u_0(t) \\ &- Lu(t)-Nu(t)+f(t))]]. \end{aligned} \tag{2.18}$$

The solution of Eq.(2.18) can be figured out as a power series of p,

$$u(t)=\sum_{n=0}^{\infty}p^n u_n. \tag{2.19}$$

Next, substituting Eq.(2.19) into Eq.(2.18), we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty}p^n u_n &= \mathscr{L}^{-1}[\frac{1}{s^{\alpha}}[\sum_{k=0}^{m-1}s^{\alpha-k-1}u^{(k)}(0)]+\frac{1}{s^{\alpha}}\mathscr{L}[D_*^{\alpha}u_0(t) \\ &+ p(-D_*^{\alpha}u_0(t)+f(t)-L\sum_{n=0}^{\infty}p^n u_n-N\sum_{n=0}^{\infty}p^n u_n)]]. \end{aligned} \tag{2.20}$$

Equating the terms with identical powers of p, we have:

$$\begin{aligned}
p^0 &: u_0(t) = \mathscr{L}^{-1}[\frac{1}{s^\alpha}[\sum_{k=0}^{m-1} s^{\alpha-k-1}u^{(k)}(0)] + \frac{1}{s^\alpha}\mathscr{L}[D_*^\alpha u_0(t)]], \quad (2.21)\\
p^1 &: u_1(t) = \mathscr{L}^{-1}[\frac{1}{s^\alpha}[\mathscr{L}(-D_*^\alpha u_0(t) + f(t) - L(u_0) - N(u_0))]],\\
p^2 &: u_2(t) = \mathscr{L}^{-1}[\frac{1}{s^\alpha}[\mathscr{L}(-L(u_1) - N(u_1))]],\\
p^3 &: u_3(t) = \mathscr{L}^{-1}[\frac{1}{s^\alpha}[\mathscr{L}(-L(u_2) - N(u_2))]],\\
&\vdots
\end{aligned}$$

Assuming that the initial approximation has the form $u(0) = \beta_0$, $u'(0) = \beta_1$, ..., $u^{(m-1)} = \beta_{m-1}$; as a result, the solution can be figured out as follows:

$$u(t) = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \dots \quad (2.22)$$

The main advantage of combining LT with the HPM is that it reduces a computational work caused by nonlinear terms because it allows solving FDEs in a simpler way. Despite of the advantage mentioned above for LT-HPM, the suitable selection of an initial approximation is still needed; in addition, infinite iterations are required, which negatively affect accuracy, convergence, and the amount of computational work required. To overcome these disadvantages, a new algorithm is introduced in Chapter 4 to improve the solution on the entire of the interest interval by replacing the initial approximation with a linear trial function, taking into account that the linear trial function contributed to reduce the amount of computational work to obtain the first order approximate solution..

2.8.3 Adomian Decomposition Method (ADM)

The ADM was first introduced by G. Adomian in the 1980s (Adomian & George, 1983) and it has been used as an effective method for solving linear and nonlinear problems which were established in science and engineering (Adomian & George, 1983; Noeiaghdam et al., 2021). To figure out how ADM works, consider the nonlinear FDE in Eq.(2.8), applying the Riemann–Liouville integral (J^{α}) to both sides of Eq. (2.8) gives:

$$u(t) = \sum_{k=0}^{m-1} \frac{D^k u(0)}{k!} t^k + J^{\alpha}[f(t) - Lu(t) - Nu(t)], \tag{2.23}$$

where

$$J^{\alpha} D_*^{\alpha} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{D^k u(0)}{k!} t^k. \tag{2.24}$$

The linear term $u(x)$ can be figured out as an infinite series of components given by:

$$u(t) = \sum_{n=0}^{\infty} u_n. \tag{2.25}$$

The nonlinear term $Nu(t)$ is represented by an infinite series of the Adomian polynomials A_n in the form:

$$Nu(t) = \sum_{n=0}^{\infty} A_n. \tag{2.26}$$

where A_n; $n \geq 0$ are defined by:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]. \tag{2.27}$$

In other words, assuming that the nonlinear function is $F(u(x))$, therefore the Adomian polynomials are given by:

$$
\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) \frac{1}{3!} u_1^3 F'''(u_0), \\
&\cdots
\end{aligned}
\tag{2.28}
$$

Upon substitution of the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials tailored to the nonlinearity Nu from Eq.(2.25) and Eq.(2.26) into Eq.(2.23), we have:

$$\sum_{n=0}^{\infty} u_n = \sum_{k=0}^{m-1} \frac{D^k u(0)}{k!} t^k + J^{\alpha}[f(t) - L \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n], \tag{2.29}$$