ON A SUBCLASS OF ANALYTIC FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

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by

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathcal{A}$	Class of all normalized analytic functions in $\mathbb{D}$
$\mathbb{C}$	Complex plane
$\mathbb{D}$	Unit disk
$\mathcal{H}$	Class of all analytic functions in $\mathbb{D}$
$\mathcal{C}$	Class of convex functions in $\mathcal{A}$
Im	Imaginary part of a complex number
k	Koebe function, $k(z) = z/(1-z)^2$
$\mathcal{P}$	Class of normalized analytic function with positive real part
Re	Real part of a complex number
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$
$\mathcal{S}^*$	Class of starlike functions in $\mathcal{A}$
$\mathcal{U}_f$	$\left(\frac{z}{f(z)}\right)^2 f'(z)$
$\mathcal{U}$	$\left\{f \in \mathcal{A} : \left\lvert\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right\rvert < 1\right\}$
$\mathcal{U}(\lambda)$	$\left\{f \in \mathcal{A} : \left\lvert\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right\rvert < \lambda\right\}$
$\mathcal{U}(\lambda,\mu)$	$\left\{f \in \mathcal{A} : \left\lvert\left(\frac{z}{f(z)}\right)^2 f'(z) - \mu\right\rvert < \lambda\right\}$
ω	Schwarz function
$\prec$	Subordinate to
$\bar{z}$	Conjugate of complex number z
$\equiv$	equivalent

SUATU SUBKELAS FUNGSI ANALISIS YANG MEMENUHI KETAKSAMAAN PEMBEZAAN

ABSTRAK

Disertasi ini mengkaji mengkaji fungsi analisis bernilai kompleks dalam cakera unit terbuka $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Suatu kajian ringkas mengenai konsep asas dan keputusan dari teori fungsi univalen analisis telah diberikan. Untuk $\lambda \in (0,1]$, kelas yang terdiri daripada fungsi analisis ternormal f yang memenuhi syarat $|f'(z)(z/f(z))^2 - 1| < \lambda$ telah dikaji dan terbukti keunivalenannya dalam $\mathbb{D}$. Didorong oleh kelas ini, kelas fungsi analisis ternormal f yang memenuhi syarat $|f'(z)(z/f(z))^2 - \mu| < \lambda$ telah diperkenalkan. Syarat pada λ dan μ dipilih dengan sesuai untuk memastikan f adalah univalen dalam $\mathbb{D}$. Subkelas ini ditunjukkan terpelihara di bawah beberapa transformasi asas. Syarat perlu dan cukup (dari segi perwakilan kamiran) bagi fungsi f diperoleh. Beberapa keputusan penting seperti mencari anggaran pekali dan batas untuk penentu Hankel kedua dan ketiga telah diperoleh. Akhir sekali, beberapa masalah jejari telah dikaji. Keputusan yang diperoleh menyatukan hasil kajian terdahulu.

ON A SUBCLASS OF ANALYTIC FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

ABSTRACT

The present dissertation investigates complex-valued analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. A brief survey of the basic concepts and results from the classical theory of analytic univalent functions are given. For $\lambda \in (0,1]$, the class of normalized analytic functions f satisfying $|f'(z)(z/f(z))^2 - 1| < \lambda$ has been actively investigated and is shown to be univalent in $\mathbb{D}$. Motivated by this class, a class of normalized analytic functions f satisfying $|f'(z)(z/f(z))^2 - \mu| < \lambda$ is introduced. Conditions on λ and μ are chosen suitably to ensure f is univalent in $\mathbb{D}$. This family is shown to be preserved under a number of elementary transformations. The necessary and sufficient condition (in terms of integral representation) of the function f is derived. Several important results such as finding the coefficient estimate and the bound for the second and third Hankel determinant are determined. Lastly, some radius problems are investigated. Connection are made with earlier results.

CHAPTER 1

INTRODUCTION

1.1 A short history

Geometric function theory is a branch of complex analysis, which studies the geometric properties of analytic functions. The theory of univalent functions is one of the most important subjects in geometric function theory. The study of univalent functions was initiated by Koebe [21] in 1907. One of the major problems in this field had been the Bieberbach [7] conjecture dating from the year 1916, which asserts that the modulus of the nth Taylor coefficient of each normalized analytic univalent function is bounded by n. The conjecture was not completely solved until 1984 by French-American mathematician Louis de Branges [10].

1.2 Basic theories of the class of univalent functions

Let $\mathbb{C}$ be the complex plane of complex numbers. A *domain* is a nonempty connected open in $\mathbb{C}$. A domain is said to be *simply connected* if its complement is connected. Geometrically, a simply connected domain is a domain without any holes in it. A complex-valued function f of a complex variable is said to be *differentiable* at a point $z_0 \in \mathbb{C}$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists at z_0. The function f is *analytic* at z_0 if it is differentiable at every point in some neighborhood of z_0. It is a "miracle" of complex analysis that an analytic function f

has derivatives of all order at z_0 with a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!},$$

which converges in some open disk centered at z_0. The function f is analytic in a domain if it is analytic at every point of the domain.

Definition 1.1. [16] A function f on $\mathbb{C}$ is said to be *univalent* (*one-to-one*) in a domain $D \subset \mathbb{C}$ if for any $z_1, z_2 \in D$,

$$f(z_1) = f(z_2) \ \Rightarrow z_1 = z_2,$$

or equivalently

$$z_1 \neq z_2 \ \Rightarrow f(z_1) \neq f(z_2).$$

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the class of normalized ($f(0) = 0 = f'(0) - 1$) analytic functions f in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

Example 1.1. The function $k : \mathbb{D} \to \mathbb{C}$ defined by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$$

is in the class $\mathcal{S}$. Note that

$$k'(z) = \frac{1+z}{(1-z)^3}$$

and the derivative k' exists everywhere except at $z = 1$. So, k is analytic in $\mathbb{D}$. Also, the Koebe function satisfies the condition $k(0) = 0$ and $k'(0) = 1$. Hence, the Koebe function belongs to the class $\mathcal{A}$. To see that the Koebe function is univalent in $\mathbb{D}$, suppose that $k(z_1) = k(z_2)$, $z_1, z_2 \in \mathbb{D}$,

$$\frac{z_1}{(1-z_1)^2} = \frac{z_2}{(1-z_2)^2}.$$

After a simple computation, we get

$$z_1 + z_1 z_2^2 - z_2 - z_2 z_1^2 = 0$$

or

$$(z_1 - z_2)(1 - z_1 z_2) = 0.$$

Since $z_1, z_2 \in \mathbb{D}$, we have $|z_1| < 1$ and $|z_2| < 1$ and therefore $|z_1 z_2| = |z_1||z_2| < 1$. This shows that $1 - z_1 z_2 \neq 0$ in $\mathbb{D}$. Thus we must have $z_1 - z_2 = 0$, that is, $z_1 = z_2$. Therefore, the Koebe function, k is univalent in $\mathbb{D}$. ■

Example 1.2. The function $f : \mathbb{D} \to \mathbb{C}$ defined by

$$f(z) = -\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

belongs to the class $\mathcal{S}$. The argument to show f belongs to class $\mathcal{S}$ follows similarly as in Example 1.1. ■

The class $\mathcal{S}$ is preserved under a number of elementary transformations [11]:

(i) Conjugation. If $f \in \mathcal{S}$ and $g(z) = \overline{f(\bar{z})} = z + \overline{a_2}z^2 + \overline{a_3}z^3 + \cdots$, then $g \in \mathcal{S}$.

(ii) Rotation. If $f \in \mathcal{S}$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in \mathcal{S}$.

(iii) Dilation. If $f \in \mathcal{S}$ and $g(z) = r^{-1} f(rz)$, where $0 < r < 1$, then $g \in \mathcal{S}$.

(iv) Disk automorphism. If $f \in \mathcal{S}$ and

$$g(z) = \frac{f\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right)}{(1-|\alpha|^2)f'(\alpha)}, \quad |\alpha| < 1,$$

then $g \in \mathcal{S}$.

(v) Omitted-value transformation. If $f \in \mathcal{S}$ and $f(z) \neq w$, then $g = wf/(w-f) \in \mathcal{S}$.

(vi) Square-root transformation. If $f \in \mathcal{S}$ and $g(z) = \sqrt{f(z^2)}$, then $g \in \mathcal{S}$.

For every function f in $\mathcal{S}$, Bieberbach [7] proved that $|a_2| \leq 2$ and equality holds if and only if f is a rotation of the Koebe function k. This result is known as Bieberbach Theorem. He also conjectured that $|a_n| \leq n$ in 1916. This conjecture was popularly known as Bieberbach's conjecture. The conjecture had been proven for the case $n = 2,3,4,5,6$ by some researchers before Louis de Branges [10] proved the general case $|a_n| \leq n$ in 1984. This is summarized in the table below.

Table 1.1: Special cases of Bieberbach conjecture

Researchers	Result
Bieberbach [7] (1916)	$\lvert a_2 \rvert \leq 2$
Löwner [30] (1923)	$\lvert a_3 \rvert \leq 3$
Garabedian and Schiffer [15] (1955)	$\lvert a_4 \rvert \leq 4$
Pederson [49] (1968), Ozawa [48] (1969)	$\lvert a_6 \rvert \leq 6$
Pederson and Schiffer [50] (1972)	$\lvert a_5 \rvert \leq 5$
de Branges [10] (1984)	$\lvert a_n \rvert \leq n$

Nowadays, the Bieberbach conjecture is also called the de Branges theorem. The estimate $|a_2| \leq 2$ is useful in proving some of the important theorems in class $\mathcal{S}$. One of them is the Koebe 1/4- theorem which asserts that the range of every function in class $\mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$, see [11, Theorem 2.3]. Another important consequence of Bieberbach's theorem is the distortion theorem [11, Theorem 2.5] for class $\mathcal{S}$, which provides upper and lower bounds for $|f'(z)|$, i.e.,

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad r = |z| < 1.$$

For each nonzero $z \in \mathbb{D}$, equality holds when f is the Koebe function or its rotations. From the distortion theorem, one may deduce the growth theorem [11, Theorem 2.6] for class $\mathcal{S}$, that is,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad r = |z| < 1.$$

1.2.1 Functions with positive real part

Definition 1.2. [16] An analytic function p with the normalization $p(0) = 1$ in $\mathbb{D}$ is called a *function of positive real part of order* α, $0 \leq \alpha < 1$ if Re $(p(z)) > \alpha$ for all

z in $\mathbb{D}$. The set of all functions of positive real part of order α is denoted by $\mathcal{P}(\alpha)$. For $\alpha = 0$, we have $\mathcal{P}(0) = \mathcal{P}$ which is the class of function of positive real part or Carathéodory class.

Example 1.3. For $n \in \mathbb{N}$, the function

$$m_n(z) = \frac{1+z^n}{1-z^n} = \frac{2}{1-z^n} - 1, \quad z \in \mathbb{D}, \tag{1.2}$$

is in the class $\mathcal{P}$. The derivative

$$m_n'(z) = \frac{2nz^{n-1}}{(1-z^n)^2}$$

exists everywhere except at $z = 1$ and so m_n is analytic in $\mathbb{D}$. Clearly, $m_n(0) = 1$. From (1.2), we have $z^n = (m_n(z)-1)/(m_n(z)+1)$. Since $|z| < 1$, it follows that $|z|^n < 1$ and so $|m_n(z)-1| < |m_n(z)+1|$. Squaring both sides and using

$$|z \pm w|^2 = |z|^2 \pm 2\,\mathrm{Re}\,(z\overline{w}) + |w|^2, \quad z, w \in \mathbb{C}, \tag{1.3}$$

we obtain

$$|m_n(z)|^2 - 2\,\mathrm{Re}\,(m_n(z)) + 1 < |m_n|^2 + 2\,\mathrm{Re}\,(m_n(z)) + 1.$$

Hence $\mathrm{Re}\,(m_n(z)) > 0$. ■

Example 1.4. For $f \in \mathcal{P}$, it follows that $1/f \in \mathcal{P}$ since

$$\mathrm{Re}\left(\frac{1}{f(z)}\right) = \mathrm{Re}\left(\frac{\overline{f(z)}}{f(z)\overline{f(z)}}\right) = \mathrm{Re}\left(\frac{f(z)}{|f(z)|^2}\right) > 0.$$

In view of (1.2), it follows immediately that

$$M_n(z) = \frac{1-z^n}{1+z^n}, \quad n \geq 1, \quad z \in \mathbb{D}, \tag{1.4}$$

belongs to class $\mathcal{P}$. ■

Example 1.5. Consider the function

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in \mathbb{D}.$$

Clearly, $f(0) = 1$. Furthermore,

$$\begin{aligned} \operatorname{Re}\left(\frac{1}{1-z}\right) &= \operatorname{Re}\left(\frac{1}{2}\left(\frac{1+z}{1-z}+1\right)\right) \\ &= \frac{1}{2}\operatorname{Re}\left(\frac{1+z}{1-z}\right)+\frac{1}{2} \\ &> 0+\frac{1}{2} \quad \text{(by Example 1.3)} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore, the function $f(z) = 1/(1-z)$ belongs to $\mathcal{P}(1/2)$. Using the fact that $z \in \mathbb{D}$ implies $-z \in \mathbb{D}$, it is interesting to see that

$$g(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad z \in \mathbb{D}$$

also satisfies $\operatorname{Re} g(z) > 1/2$. ■

1.2.2 Subclasses of univalent functions

In the course of tackling the Bieberbach conjecture, new classes of analytic and univalent functions were defined and some nice properties of these classes were widely investigated. Examples of such classes are the classes of starlike and convex functions.

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *starlike with respect to a point* w_0 in $\mathcal{D}$ if every line joining the point w_0 to every other point w in $\mathcal{D}$ lies entirely inside $\mathcal{D}$. A domain which is starlike with respect to the origin is simply called a *starlike* domain. Geometrically, a starlike domain is a domain whose all points can be seen from the origin. A function $f \in \mathcal{A}$ is called a *starlike* function if $f(\mathbb{D})$ is a starlike domain. The subclass of $\mathcal{S}$ consisting of all starlike functions is denoted by $\mathcal{S}^*$. The analytical characterization of $f \in \mathcal{S}^*$, due to Nevalinna [33], is given as follows:

Theorem 1.1. [33] Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.5}$$

In 1946, Friedman [14] discovered that the functions in $\mathcal{S}_{\mathbb{Z}}$, that is,

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{1 \pm z + z^2} \right\} \tag{1.6}$$

are the only functions in $\mathcal{S}$ having integral coefficients (coefficients are integers) in the power series expansions of functions $f \in \mathcal{S}$. It is interesting to see that the Koebe function, $k(z) = z/(1-z)^2$ belongs to the set $\mathcal{S}_{\mathbb{Z}}$. For the nine functions f in $\mathcal{S}_{\mathbb{Z}}$, the $\operatorname{Re}(zf'(z)/f(z))$ is given in the next table:

Table 1.2: The expression $\mathrm{Re}\,(zf'(z)/f(z))$ for f in $\mathcal{S}_{\mathbb{Z}}$

Functions, $f(z)$	$\mathrm{Re}\,(zf'(z)/f(z))$
z	1
$\frac{z}{1\pm z}$	$\mathrm{Re}\left(\frac{1}{1\pm z}\right)$
$\frac{z}{(1-z)^2}$	$\mathrm{Re}\left(\frac{1+z}{1-z}\right)$
$\frac{z}{(1+z)^2}$	$\mathrm{Re}\left(\frac{1-z}{1+z}\right)$
$\frac{z}{1-z^2}$	$\mathrm{Re}\left(\frac{1+z^2}{1-z^2}\right)$
$\frac{z}{1+z^2}$	$\mathrm{Re}\left(\frac{1-z^2}{1+z^2}\right)$
$\frac{z}{1+z+z^2}$	$\mathrm{Re}\left(\frac{1-z^2}{1+z+z^2}\right)$
$\frac{z}{1-z+z^2}$	$\mathrm{Re}\left(\frac{1-z^2}{1-z+z^2}\right)$

The identity function $f(z)=z$ is a starlike function. The function $f(z)=z/(1\pm z)$ belongs to $\mathcal{S}^*$ because

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)=\mathrm{Re}\left(\frac{1}{1\pm z}\right)>\frac{1}{2}>0$$

by Example 1.5. In view of (1.2) and (1.4), it follows that the functions

$$\frac{1+z}{1-z},\quad \frac{1-z}{1+z},\quad \frac{1+z^2}{1-z^2},\quad \text{and}\quad \frac{1-z^2}{1+z^2}$$

belongs to class $\mathcal{P}$. Hence, the functions

$$\frac{z}{(1-z)^2},\quad \frac{z}{(1+z)^2},\quad \frac{z}{1-z^2},\quad \frac{z}{1+z^2}$$

belongs to class $\mathcal{S}^*$. Note that

$$\frac{1\pm z+z^2}{1-z^2}=\frac{1}{2}\left(\frac{1\pm z}{1\mp z}+\frac{1+z^2}{1-z^2}\right).$$

belongs to class $\mathcal{P}$. By (1.4), it follows that

$$\frac{1-z^2}{1 \pm z + z^2} \in \mathcal{P}.$$

Therefore, the functions

$$\frac{z}{1+z+z^2} \quad \text{and} \quad \frac{z}{1-z+z^2}$$

belongs to class $\mathcal{S}^*$. Thus, we have

$$\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*. \tag{1.7}$$

Here, some image domain of starlike functions are presented.

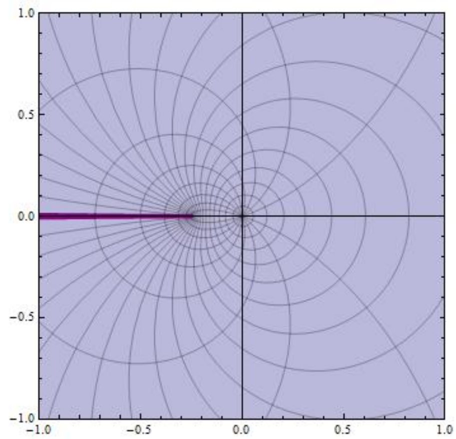

Figure 1.1: Image domain of the Koebe function

From Figure 1.1, it can be seen that the Koebe function maps the unit disk $\mathbb{D}$ univalently onto the entire complex plane minus the negative axis from $-1/4$ to infinity.

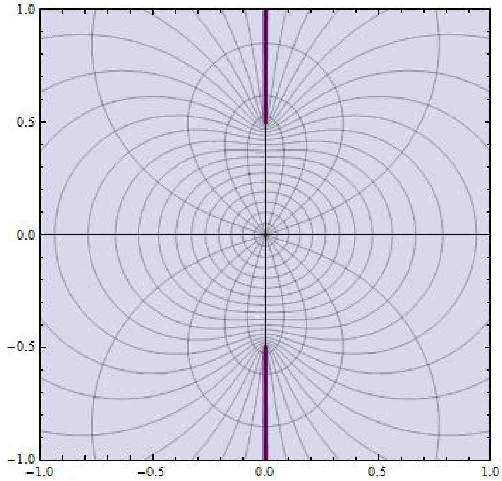

Figure 1.2: Image domain of $f(z) = z/(1-z^2)$.

From Figure 1.2, it can be seen that the f maps the unit disk $\mathbb{D}$ univalently onto the entire complex plane minus the two lines $0.5 \leq y < \infty$ and $-\infty < y \leq -0.5$.

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be *convex* if every linear segment joining any two points in $\mathcal{D}$ lies completely inside $\mathcal{D}$. In other words, the domain $\mathcal{D}$ is convex if and only if it is starlike with respect to every point in $\mathcal{D}$. A function $f \in \mathcal{A}$ is said to be *convex* if $f(\mathbb{D})$ is a convex domain. Every convex function f in $\mathbb{D}$ is evidently starlike because the convex domain $f(\mathbb{D})$ is also a starlike domain (starlike with respect to the origin). The subclass of $\mathcal{S}$ consisting of all convex functions is denoted by $\mathcal{C}$.

Theorem 1.2. [11, Theorem 2.11] Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.8}$$

Example 1.6. Consider the function $f(z) = -\ln(1-z)$. Note that $f'(z) = 1/(1-z)$

and $f''(z) = 1/(1-z)^2$. So,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1+\frac{z(1-z)}{(1-z)^2}\right) = \operatorname{Re}\left(\frac{1}{1-z}\right) > \frac{1}{2} > 0$$

by Example 1.5. Hence, the function $f(z) = -\ln(1-z)$ is convex in $\mathbb{D}$. ■

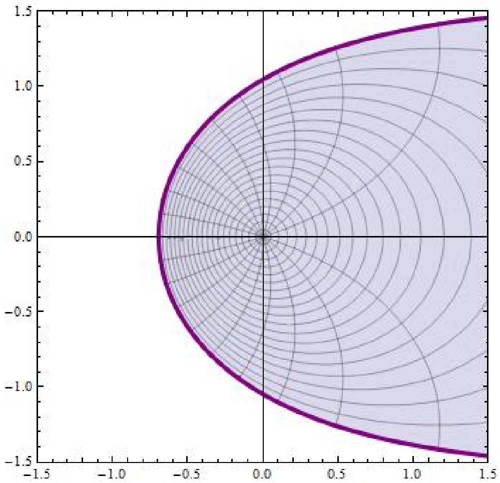

Figure 1.3: Image domain of $f(z) = -\ln(1-z)$.

Example 1.7. Consider the function

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n.$$

Note that $f'(z) = 1/(1-z)^2$ and $f''(z) = 2/(1-z)^3$. Hence,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1+\frac{2z(1-z)^2}{(1-z)^3}\right) = \operatorname{Re}\left(\frac{1+z}{1-z}\right) > 0.$$

Therefore, the function $z/(1-z)$ is convex in $\mathbb{D}$. ■

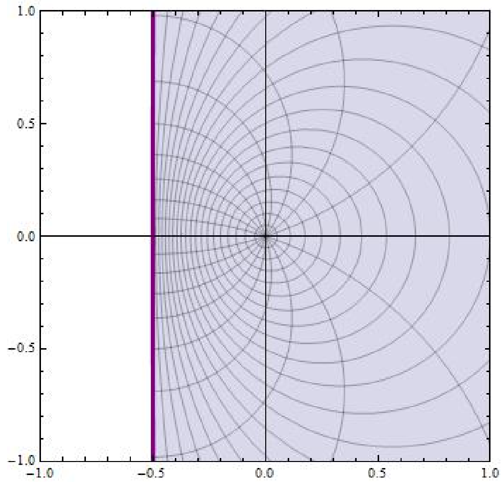

Figure 1.4: Image domain of $f(z) = z/(1-z)$.

Another interesting subclass of $\mathcal{S}$ which attracts attention many years ago is the class $\mathcal{U}(\lambda)$. Let

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad 0 < \lambda \leq 1 \right\}$$

for all $z \in \mathbb{D}$. When $\lambda = 1$, we have

$$\mathcal{U}(1) := \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1, \quad z \in \mathbb{D} \right\}. \tag{1.9}$$

Obradovic [37] introduced the notations $\mathcal{U}(1)$ and $\mathcal{U}(\lambda)$. The notation

$$\mathcal{U}(1) \equiv \mathcal{U} \tag{1.10}$$

was coined by Barnard *et al* [6]. However, before the notations $\mathcal{U}(1)$ or $\mathcal{U}$ are introduced, this class has been widely studied and several interesting results has been obtained, see [1], [34] and [47]. More details about these two classes will be discussed in Chapter 2.

The Schwarz lemma is one of the important results in complex function theory. First, the definition of Schwarz function is given.

Definition 1.3. [11] A function ω which is analytic in $\mathbb{D}$ and satisfies the properties $\omega(0) = 0$ and $|\omega(z)| < 1$ is called a *Schwarz function*. The class of all Schwarz functions is denoted by Ω.

Definition 1.4. [11] For analytic functions f and g on $\mathbb{D}$, we say that f is subordinate to g, denoted $f \prec g$, if there exists a Schwarz function ω in $\mathbb{D}$ such that $f(z) = g(\omega(z)), \quad z \in \mathbb{D}$.

Lemma 1.1. [11] (Schwarz lemma) Let $f \in \Omega$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, and $|f'(0)| \leq 1$. Equality holds if $f(z) = e^{i\theta}z$ for some real θ.

1.3 Scope of dissertation

Here is the summary of the dissertation. The dissertation is divided into four chapters, followed by references at the end.

In the first chapter, which is the introductory chapter, we review and assemble some of the general principles of theory of univalent functions which underlie the geometric function theory of a complex variable.

Chapter 2 deals with literature review. Some background of the study is given.

Chapter 3 studies a new subclass of univalent functions. Some interesting results are obtained such as a sufficient condition, integral representation, univalency condition and transformation preserving. Connections are made with previously known results.

Chapter 4 mainly focuses on the coefficient problems of the new subclass and their applications. The integral representation obtained in Chapter 3 is useful in solving the coefficient problems. Some of the applications are Hankel determinants, Zalcman functional and others.

Chapter 5 deals with a class of nonvanishing univalent function. Some radius problems are explored.

In Chapter 6, a summary of the work done in this dissertation is presented.

CHAPTER 2

LITERATURE REVIEW

2.1 Literature review

The univalence of analytic functions is an important property in geometric function theory. Sometimes, it is difficult, and in many cases impossible, to show directly that a certain function is univalent, see Definition 1.1. For this reason, many authors were trying to determine sufficient conditions of univalence. In 1972, Ozaki and Nunokawa [47, Theorem 2] proved that if $f \in \mathcal{A}$ satisfies the condition

$$\operatorname{Re}\left(\left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)}\right) > \frac{1}{2}, \tag{2.1}$$

then f is univalent in $\mathbb{D}$.

The inequality (2.1) can be written in the form

$$2\operatorname{Re}\frac{1}{U_f(z)} - 1 > 0 \tag{2.2}$$

where

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z). \tag{2.3}$$

Note that

$$\operatorname{Re}\frac{1}{U_f(z)} = \operatorname{Re}\frac{\overline{U_f(z)}}{U_f(z)\overline{U_f(z)}} = \operatorname{Re}\frac{\overline{U_f(z)}}{|U_f(z)|^2} = \operatorname{Re}\frac{U_f(z)}{|U_f(z)|^2}.$$

Therefore, (2.2) becomes

$$2\operatorname{Re}U_f(z) - \left|U_f(z)\right|^2 > 0$$

or

$$1 - (|U_f(z)|^2 - 2\operatorname{Re}U_f(z) + 1) > 0. \tag{2.4}$$

Note that

$$|U_f(z)|^2 - 2\operatorname{Re}U_f(z) + 1 = |U_f(z) - 1|^2.$$

Hence, (2.4) becomes

$$1^2 - |U_f(z) - 1|^2 > 0$$

and so

$$\left|U_f(z) - 1\right| < 1. \tag{2.5}$$

It can be shown that (2.5) implies (2.1) and so they are equivalent. Therefore, the inequality (2.5) is also a sufficient condition for $f \in \mathcal{A}$ to be univalent in $\mathbb{D}$.

It is worth noting that the inequality (2.5) was proved by Aksent'ev [1] in 1959 with different approach. A function F is said to be meromorphic in a domain D if it is analytic except for poles in D. Let $\mathcal{M}$ be the set of meromorphic functions F in $\triangle := \{\zeta \in \mathbb{C} : |\zeta| > 1\} \cup \{\infty\}$ with the following Laurent series expansion

$$F(\zeta) = \zeta + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{\zeta^n}, \quad \zeta \in \triangle.$$

Aksent'ev [1] proved that if $F \in \mathcal{M}$ satisfies the inequality

$$|F'(\zeta) - 1| < 1, \ \zeta \in \triangle, \tag{2.6}$$

then F is univalent in $\triangle$. For any $f \in \mathcal{A}$, the function

$$F(\zeta) = \frac{1}{f(1/\zeta)} = \zeta - a_2 + \frac{a_2^2 - a_3}{\zeta} + \cdots$$

is in $\mathcal{M}$. Note that

$$F'(\zeta) = \frac{f'(1/\zeta)}{\zeta^2 f^2(1/\zeta)} = \left(\frac{z}{f(z)}\right)^2 f'(z) = U_f(z), \quad z = \frac{1}{\zeta} \in \mathbb{D}, \tag{2.7}$$

by (2.3). In view of (2.6) and (2.7), the inequality (2.5) follows.

From (2.5), we see that $U_f(z) \neq 0$, as if $U_f(z) = 0$, then $|U_f(z) - 1| = 1$ which contradicts (2.5). So, $|U_f(z)| > 0$. Further,

$$\left|U_f(z)\right| = |U_f(z) - 1 + 1| \leq |U_f(z) - 1| + 1 < 1 + 1 = 2.$$

Since $|U_f|$ is bounded below by 0 and bounded above by 2, it follows that $|U_f|$ is bounded.

Let

$$\mathcal{U} := \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < 1, \quad z \in \mathbb{D} \right\}. \tag{2.8}$$

By the discussion above, each function in class $\mathcal{U}$ is univalent in $\mathbb{D}$. However, there are univalent functions that do not belong to class $\mathcal{U}$. In [3], the authors considered the function $f(z) = -\ln(1-z)$. Since $f'(z) = 1/(1-z)$, it follows that

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) = \frac{z^2}{\ln^2(1-z)} \frac{1}{1-z}.$$

For $z = 0.95 \in \mathbb{D}$, we have $\left|U_f(0.95) - 1\right| = 1.011 > 1$. Thus, $f(z) = -\ln(1-z)$ is not in $\mathcal{U}$. Hence, we have the inclusion

$$\mathcal{U} \subsetneq \mathcal{S}.$$

The leading example in the class $\mathcal{U}$ is the Koebe function, $k(z) = z/(1-z)^2$. Since $k'(z) = (1+z)/(1-z)^3$, it follows that

$$\begin{aligned}
\left|\left(\frac{z}{k(z)}\right)^2 k'(z) - 1\right| &= \left|\frac{(1-z)^4(1+z)}{(1-z)^3} - 1\right| \\
&= |(1-z)(1+z) - 1| \\
&= |-z^2| \\
&= |z|^2 \\
&< 1.
\end{aligned}$$

Therefore, k is in $\mathcal{U}$. Recall from (1.6), we have

$$\mathcal{S}_{\mathbb{Z}} = \left\{z, \quad \frac{z}{(1\pm z)^2}, \quad \frac{z}{1\pm z}, \quad \frac{z}{1\pm z^2}, \quad \frac{z}{1\pm z+z^2}\right\}$$

and Koebe function, $k(z) = z/(1-z)^2$ belongs to the class $\mathcal{S}_{\mathbb{Z}}$. The U_f of the remaining eight functions in $\mathcal{S}_{\mathbb{Z}}$ are given in the next table:

Table 2.1: The value of U_f and $|U_f - 1|$ for f in $\mathcal{S}_{\mathbb{Z}}$

Functions	U_f	$\lvert U_f - 1\rvert$
$z, \frac{z}{1\pm z}$	1	0
$\frac{z}{1\pm z+z^2}, \frac{z}{1+z^2}, \frac{z}{(1+z)^2}$	$1-z^2$	$\lvert -z^2\rvert$
$\frac{z}{1-z^2}$	$1+z^2$	$\lvert z^2\rvert$

From the table above, it can be seen that all the functions satisfy the condition $|U_f - 1| < 1$ and so are in $\mathcal{U}$. Hence,

$$\mathcal{S}_{\mathbb{Z}} \subset \mathcal{U}. \tag{2.9}$$

Recall from (1.7), we have $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*$. Obradovic and Ponnusamy [38] discovered the inclusion relationship as follows:

$$\mathcal{S}_{\mathbb{Z}} \subset \mathcal{U} \cap \mathcal{S}^* \subset \mathcal{S}.$$

Motivated by Friedman's results, Hiranuma and Sugawa [18] studied functions in $\mathcal{S}$ whose coefficients are half-integers. They showed that the following six functions

$$z \pm \frac{z^2}{2}, \frac{z(2\pm z)}{2(1\pm z)}, \frac{z(2\pm z)}{2(1\pm z)^2} \tag{2.10}$$

belongs to class $\mathcal{U}$. We provide the proof for the function $f(z) = z + z^2/2$ belongs to the class $\mathcal{U}$. Note that

$$\frac{z}{f(z)} = \frac{1}{1+z/2} \quad \text{and} \quad f'(z) = 1+z.$$

It follows that

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right| &= \left|\frac{1+z}{(1+z/2)^2}-1\right| \\ &= \left|\frac{-z^2/4}{(1+z/2)^2}\right| \\ &= \left(\frac{z/2}{1+z/2}\right)^2 \\ &< 1.\end{aligned}$$

For the other functions in (2.10), their U_f and $|U_f-1|$ is given below respectively:

Table 2.2: The value of U_f and $|U_f-1|$ for several function f

Functions	U_f	$\lvert U_f-1\rvert$
$z-\frac{z^2}{2}$	$\frac{1-z}{(1-z/2)^2}$	$\left(\frac{z/2}{1-z/2}\right)^2$
$\frac{z(2+z)}{2(1+z)}$	$\frac{2(z^2+2z+2)}{(2+z)^2}$	$\left(\frac{z}{2+z}\right)^2$
$\frac{z(2-z)}{2(1-z)}$	$\frac{2(z^2-2z+2)}{(2+z)^2}$	$\left(\frac{z}{2-z}\right)^2$
$\frac{z(2+z)}{2(1+z)^2}$	$\frac{4(1+z)}{(2+z)^2}$	$\left(\frac{z}{2+z}\right)^2$
$\frac{z(2-z)}{2(1-z)^2}$	$\frac{4(1-z)}{(2-z)^2}$	$\left(\frac{z}{2-z}\right)^2$

Next, we study the relationships between class $\mathcal{U}$ and other subclasses of univalent functions. Recall that the Koebe function belongs to both the subclasses $\mathcal{S}^*$ and $\mathcal{U}$, that is, $k \in \mathcal{U}\cap\mathcal{S}^*$. It is natural to ask whether $\mathcal{U}$ is included in $\mathcal{S}^*$ or $\mathcal{S}^*$ is included in $\mathcal{U}$. Consider the function

$$f(z)=\frac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3}.$$

Since

$$f'(z)=\frac{1-z^3}{\left(1+\frac{1}{2}z+\frac{1}{2}z^3\right)^2},$$

it follows that

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right| &= \left|\left(1+\frac{1}{2}z+\frac{1}{2}z^3\right)^2\left(\frac{1-z^3}{\left(1+\frac{1}{2}z+\frac{1}{2}z^3\right)^2}\right)-1\right| \\ &= |-z^3| \\ &= |z|^3 \\ &< 1.\end{aligned}$$

Therefore, $f \in \mathcal{U}$. On the other hand, we have

$$\frac{zf'(z)}{f(z)} = \frac{1-z^3}{1+\frac{1}{2}z+\frac{1}{2}z^3}.$$

At the boundary points $z_0 = (i-1)/\sqrt{2}, |z_0| = 1$, since

$$1-\left(\frac{i-1}{\sqrt{2}}\right)^3 = 1-\left(\frac{2+2i}{2\sqrt{2}}\right) = \frac{\sqrt{2}-1-i}{\sqrt{2}}$$

and

$$1+\frac{i-1}{2\sqrt{2}}+\frac{1}{2}\left(\frac{i-1}{\sqrt{2}}\right)^3 = 1+\frac{i-1}{2\sqrt{2}}+\frac{1}{2}\left(\frac{2+2i}{2\sqrt{2}}\right) = 1+\frac{i}{\sqrt{2}},$$

it follows that

$$\frac{z_0 f'(z_0)}{f_1(z_0)} = \frac{\sqrt{2}-1-i}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}+i} = \frac{1-\sqrt{2}}{3}+i\frac{1-2\sqrt{2}}{3}.$$

Since

$$\operatorname{Re}\frac{z_0 f'(z_0)}{f(z_0)} = \frac{1-\sqrt{2}}{3} \approx -0.13807 < 0,$$

it follows that there are points in $\mathbb{D}$ for which $\operatorname{Re}(zf'(z)/f(z)) < 0$ showing that the function f is not in $\mathcal{S}^*$. Therefore, $\mathcal{U} \not\subset \mathcal{S}^*$.

Recall from page 11, every convex function is starlike in $\mathbb{D}$. The convex function $f(z) = -\ln(1-z)$ is starlike but not in $\mathcal{U}$ (from page 19). So $\mathcal{S}^* \not\subset \mathcal{U}$ and $\mathcal{C} \not\subset \mathcal{U}$. Also, $\mathcal{U} \not\subset \mathcal{C}$ as demonstrated by the Koebe function. However,

$$\mathcal{U} \cap \mathcal{C} = \left\{ z, \frac{z}{1-z}, \frac{z}{1+z} \right\}.$$

In 1995, Obradović [34] pointed out that the function

$$f_1(z) = \frac{z}{1 + \frac{1}{2}iz + \frac{1}{2}\lambda e^{i\beta} z^3},$$

where $\frac{\sqrt{10}-\sqrt{2}}{2} < \lambda \leq 1$ and $\arcsin \frac{2-\lambda^2}{\sqrt{2}\lambda} - \frac{\pi}{4} < \beta < \frac{3\pi}{4} - \arcsin \frac{2-\lambda^2}{\sqrt{2}\lambda}$ satisfies

$$\left| \left(\frac{z}{f_1(z)} \right)^2 f_1'(z) - 1 \right| < \lambda, \tag{2.11}$$

but

$$\operatorname{Re}\left(\frac{z f_1'(z)}{f_1(z)} \right) = \frac{\sqrt{2}k}{|1 + \frac{1}{2}i + \frac{1}{2}ke^{i\beta}|^2} \left[\frac{2-k^2}{\sqrt{2}k} - \sin\left(\frac{\pi}{4} + \beta \right) \right] < 0.$$

Therefore, $f_1 \notin \mathcal{S}^*$. On the other hand, he showed that if $f \in \mathcal{A}$ satisfies the inequality

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad 0 < \lambda < 1, \tag{2.12}$$

and

$$\left| \arg \frac{z}{f(z)} \right| \leq \arctan \frac{\sqrt{1-\lambda^2}}{\lambda},$$

then $f \in \mathcal{S}^*$.

This motivates Obradović [34] to introduce the class

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad 0 < \lambda \le 1 \right\}, \tag{2.13}$$

Clearly, $\mathcal{U} := \mathcal{U}(1)$. Since $\mathcal{U}(\lambda) \subset \mathcal{U}$ for $\lambda \in (0,1]$, it follows that functions in $\mathcal{U}(\lambda)$ are univalent in $\mathbb{D}$. Hence, the inclusion

$$\mathcal{U}(\lambda) \subset \mathcal{U} \subsetneq \mathcal{S}, \quad 0 < \lambda \le 1$$

holds.

Example 2.1. The function

$$k_\lambda(z) = \frac{z}{(1-z)(1-\lambda z)} = z + (1+\lambda)z^2 + \cdots \tag{2.14}$$

in the class $\mathcal{U}(\lambda)$. Note that

$$\frac{z}{k_\lambda(z)} = (1-z)(1-\lambda z) \quad \text{and} \quad k'_\lambda(z) = \frac{1-\lambda z^2}{[(1-z)(1-\lambda z)]^2}.$$

It follows that

$$\left| \left(\frac{z}{k_\lambda(z)} \right)^2 k'_\lambda(z) - 1 \right| = \left|1 - \lambda z^2 - 1\right| = \left| -\lambda z^2 \right| < \lambda.$$

Therefore, $k_\lambda \in \mathcal{U}(\lambda)$. It is obvious that $k_1(z) = k(z)$, the Koebe function. ■

The relationships of $\mathcal{S}$ and the subclasses of $\mathcal{S}$ discussed above are presented in the following diagram:

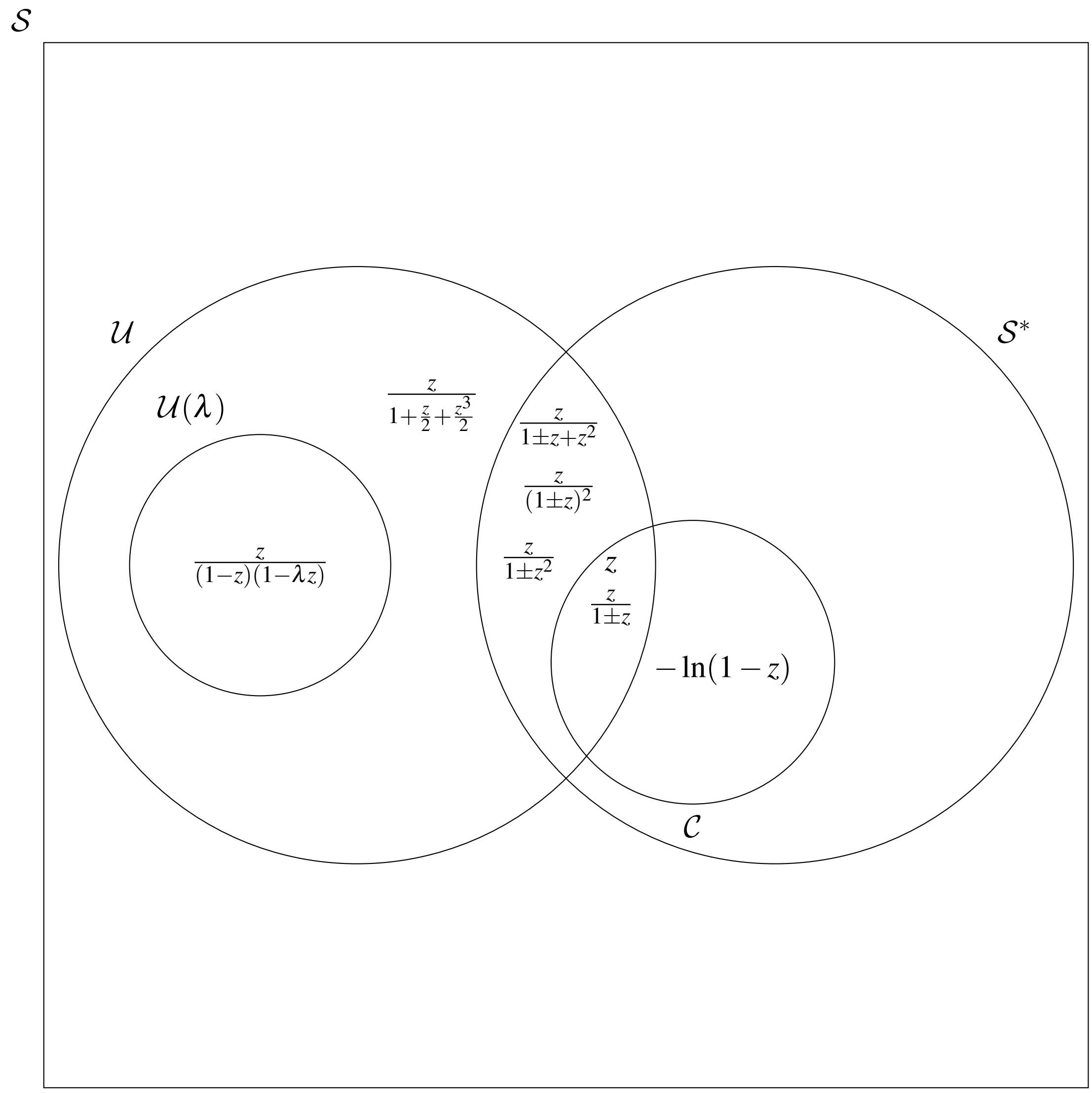

Figure 2.1: Relationships of $\mathcal{S}$ and the subclasses of $\mathcal{S}$

Recall from page 4, the class $\mathcal{S}$ is preserved under several transformations such as dilation, conjugation, rotation, omitted-value transform , and square-root transform. Obradović, Ponnusamy and Wirths [45] proved the class $\mathcal{U}(\lambda)$ is preserved under dilation, conjugation, rotation, and omitted-value transform but not preserved under square-root transform.

The problem of determining bounds for the second coefficients of functions in $\mathcal{U}(\lambda)$ was solved by Vasudevarao and Yanagihara [56, Theorem 2.6] in 2013. They proved that $|a_2| \leq 1+\lambda$ and equality holds for function in (2.14). Obradović, Ponnusamy and Wirths [45, Theorem 1] presented a simpler proof. Beside that, they also proved that $f \in \mathcal{U}(\lambda)$ satisfies the subordination

$$\frac{z}{f(z)} \prec 1+(1+\lambda)z+\lambda z^2, \quad z \in \mathbb{D}, \tag{2.15}$$

see [45, Theorem 4]. In particular, if $f \in \mathcal{U} := \mathcal{U}(1)$, then (2.15) becomes

$$\frac{z}{f(z)} \prec (1+z)^2, \quad z \in \mathbb{D},$$

which was obtained by Obradovic [34, Theorem 1].

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class $\mathcal{U}(\lambda)$, Obradović, Ponnusamy and Wirths [45, Conjecture 1] propose the conjecture

$$|a_n| \leq \sum_{k=0}^{n-1} \lambda^k = 1+\lambda+\lambda^2+\lambda^3+\cdots+\lambda^{n-1}, \quad n \geq 2, \tag{2.16}$$

They proved the conjecture in (2.16) is true for $n=3$ and $n=4$, that is, $|a_3| \leq 1+\lambda+\lambda^2$ and $|a_4| \leq 1+\lambda+\lambda^2+\lambda^3$, see [46, Theorem 3]. Recently, Li, Ponnusamy and Wirths [29] provides a counterexample to show that the inequality $|a_3| \leq 1+\lambda+\lambda^2$ is invalid. For more works on coefficients for the class $\mathcal{U}(\lambda)$, see [9], [32], [42] and [52].

In 1998, Obradovic [35] considered the function $f \in \mathcal{A}$ satisfies the inequality

$$\left| \left(\frac{z}{f(z)} \right)^{1+\mu} f'(z) - 1 \right| < \lambda, \quad z \in \mathbb{D} \tag{2.17}$$

where $0 < \mu < 1$ and $0 < \lambda < 1$. Note that by taking $\mu = 1 = \lambda$ in (2.17) yield (2.8).

So, this is a generalization of class $\mathcal{U}$. He [36] obtained several interesting results such as integral representation and the relationship between other family of univalent functions, see [36, Theorem 1] and [36, Theorem 3].

Motivated by the discussion throughout this chapter, we are ready to define another generalization of the class $\mathcal{U}$ in Chapter 3.

CHAPTER 3

A SUBCLASS OF UNIVALENT FUNCTIONS

3.1 Subclass of univalent functions

Based on the discussion in Chapter 2, we are ready to define the subclass $\mathcal{U}(\lambda,\mu)$.

Definition 3.1. Let $0<\lambda\le 1$ and $\mu\in\mathbb{C}$ such that $|1-\mu|<\lambda\le 1$ and $\lambda+|1-\mu|\le 1$. Then a function $f\in\mathcal{A}$ is in $\mathcal{U}(\lambda,\mu)$ if

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-\mu\right|<\lambda,\quad z\in\mathbb{D}. \tag{3.1}$$

The class $\mathcal{U}(\lambda,\mu)$ is nonempty because $\mathcal{U}(\lambda):=\mathcal{U}(\lambda,1)$ and $\mathcal{U}:=\mathcal{U}(1,1)$. Now, we discuss the two conditions imposed in Definition 3.1. The first condition $|1-\mu|<\lambda\le 1$ can be obtained by taking $z=0$ in (3.1) and using $0<\lambda\le 1$. The second condition $\lambda+|1-\mu|\le 1$ will guarantee each function in $\mathcal{U}(\lambda,\mu)$ is univalent in $\mathbb{D}$. To see this, by (3.1) and $\lambda+|1-\mu|\le 1$, we have

$$\begin{aligned}
\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right| &= \left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1+\mu-\mu\right| \\
&\le \left|\left(\frac{z}{f(z)}\right)^2 f'(z)-\mu\right|+|\mu-1| \\
&< \lambda+(1-\lambda) \\
&= 1.
\end{aligned}$$

In view of (2.5), we conclude that f in the class $\mathcal{U}(\lambda,\mu)$ is univalent in $\mathbb{D}$. So, the inclusion

$$\mathcal{U}(\lambda,\mu) \subset \mathcal{S}$$

holds.

It is worth mentioning the properties of μ. By $|1-\mu| < \lambda$ and $\lambda + |1-\mu| \leq 1$, we have

$$|1-\mu| \leq 1-\lambda < 1-|1-\mu|.$$

Therefore, $|\mu-1| < 1/2$, that is, μ lie in the disk of center 1 and radius $1/2$. Next, the inequality $|\mu| \geq \lambda$ also holds true. Assume $|\mu| < \lambda$. We have $|\mu| < 1-|1-\mu|$ from $\lambda + |1-\mu| \leq 1$. But,

$$1 = |1| = |1-\mu+\mu| \leq |1-\mu| + |\mu| < 1,$$

a contradiction. Hence $|\mu| \geq \lambda$.

Several examples of function belongs to the class $\mathcal{U}(\lambda,\mu)$ are presented below.

Example 3.1. Let $0 < \lambda \leq 1$ and $\mu \in \mathbb{C}$ satisfy $|1-\mu| < \lambda$ and $\lambda + |1-\mu| \leq 1$. Consider the function $f(z) = z$ and so

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - \mu \right| = |1-\mu| < \lambda.$$

Therefore, the function $f(z) = z$ is in $\mathcal{U}(\lambda,\mu)$. ■

Example 3.2. Let $0 < \lambda \leq 1$ and $\mu \in \mathbb{C}$ satisfy $|1-\mu| < \lambda$ and $\lambda + |1-\mu| \leq 1$.

Consider the function

$$f(z) = \frac{z}{1 \pm z} \quad \text{and so} \quad f'(z) = \frac{1}{(1 \pm z)^2}.$$

Then

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - \mu \right| = |1-\mu| < \lambda.$$

Therefore, the function $f(z) = z/(1 \pm z)$ is in $\mathcal{U}(\lambda, \mu)$. ■

Example 3.3. Let $0 < \lambda \leq 1, \mu \in \mathbb{C}$, $\psi = \lambda - |1-\mu| > 0$ and $\lambda + |1-\mu| \leq 1$. Consider the function

$$k_\psi(z) = \frac{z}{(1-z)(1-\psi z)}, \quad z \in \mathbb{D}.$$

Then

$$\begin{aligned}
\left| \left(\frac{z}{k_\psi(z)} \right)^2 k_\psi'(z) - \mu \right| &= \left| \frac{z}{k_\psi(z)} - z \left(\frac{z}{k_\psi(z)} \right)' - \mu \right| \\
&= \left| 1 - \mu - \psi z^2 \right| \\
&= \left| 1 - \mu - (\lambda - |1-\mu|) z^2 \right| \\
&\leq |1-\mu| + \left| \lambda - |1-\mu| \right|. \\
&= |1-\mu| + \lambda - |1-\mu| \\
&= \lambda.
\end{aligned}$$

Therefore, k_ψ is in $\mathcal{U}(\lambda, \mu)$. ■

First, let us recall a lemma which would be useful later.

Lemma 3.1. [54, Lemma 1.3] Let w be meromorphic in $\mathbb{D}$ and $w(0) = 0$. Then if for a certain $z_0 \in \mathbb{D}$, the inequality $|w(z)| \leq |w(z_0)|$ holds for $|z| \leq |z_0|$, it follows that $z_0 w'(z_0)/w(z_0) \geq 1$.

3.2 Sufficient condition and integral representation

We begin by providing a sufficient condition for analytic functions to be in $\mathcal{U}(\lambda,\mu)$.

Theorem 3.1. For $\lambda > 0$ and $\mu \in \mathbb{C}$ such that $\lambda > |1-\mu|$, if $f \in \mathcal{A}$ satisfies

$$\left|2\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{zf''(z)}{f'(z)}\right| < \frac{\lambda-|1-\mu|}{1+\lambda-|1-\mu|}, \quad z \in \mathbb{D}, \tag{3.2}$$

then $f \in \mathcal{U}(\lambda,\mu)$.

Proof. Define $w : \mathbb{D} \to \mathbb{C}$ by

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = 1+(\lambda-|1-\mu|)w(z). \tag{3.3}$$

Note that

$$\begin{aligned}\left(\frac{z}{f(z)}\right)^2 f'(z) &= -z\left(\frac{-zf'(z)}{f^2(z)}\right)\\ &= -z\left(\frac{f(z)-zf'(z)}{f^2(z)}\right)+\frac{z}{f(z)}\\ &= -z\left(\frac{z}{f(z)}\right)'+\frac{z}{f(z)}.\end{aligned} \tag{3.4}$$

and so (3.3) becomes

$$\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' = 1+(\lambda-|1-\mu|)w(z). \tag{3.5}$$

Differentiating (3.5) with respect to z yield

$$-z\left(\frac{z}{f(z)}\right)'' = (\lambda - |1-\mu|)w'(z).$$

and so $w'(0) = 0$. The function w is meromorphic in $\mathbb{D}$ satisfies $w(0) = 0$. From (3.3), we have

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - \mu\right| &= \left|1-\mu+\big(\lambda-|1-\mu|\big)w(z)\right| \\ &\leq |1-\mu| + \left|\lambda-|1-\mu|\right||w(z)| \\ &= |1-\mu| + (\lambda-|1-\mu|)|w(z)|. \qquad (3.6)\end{aligned}$$

We show that $|w(z)| < 1$. Suppose that there exists $z_0 \in \mathbb{D}$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1.$$

From Lemma 3.1, we have $z_0 w'(z_0) \geq w(z_0)$. Logarithmic differentiation of (3.3) yields

$$2\left(1-\frac{zf'(z)}{f(z)}\right) + \frac{zf''(z)}{f'(z)} = \frac{(\lambda-|1-\mu|)zw'(z)}{1+(\lambda-|1-\mu|)w(z)}. \qquad (3.7)$$

So,

$$\left|2\left(1-\frac{z_0 f'(z_0)}{f(z_0)}\right) + \frac{z_0 f''(z_0)}{f'(z_0)}\right| \geq \left|\frac{(\lambda-|1-\mu|)w(z_0)}{1+(\lambda-|1-\mu|)w(z_0)}\right| = \frac{\lambda-|1-\mu|}{1+\lambda-|1-\mu|},$$

which contradicts (3.2). Therefore, $|w(z)| < 1$ holds for all $z \in \mathbb{D}$. Hence, from (3.6), we have

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-\mu\right| < |1-\mu|+(\lambda-|1-\mu|)=\lambda.$$

Therefore, the function f is in $\mathcal{U}(\lambda,\mu)$. $\square$

Taking $\mu=1$ in Theorem 3.1, we have the result proved by Frasin and Darus [13].

An example to illustrate Theorem 3.1 is given below:

Example 3.4. Consider the function $f:\mathbb{D}\to\mathbb{C}$ given by

$$f(z)=\frac{z}{1-(1-s)z-sz^2}, \quad s=\frac{a}{2+3a}, \quad a=\lambda-|1-\mu|. \tag{3.8}$$

Since

$$f'(z)=\frac{1+sz^2}{(1-(1-s)z-sz^2)^2}$$

and

$$\frac{f''(z)}{f'(z)}=\frac{2sz}{1+sz^2}+\frac{2(1-s)+4sz}{1-(1-s)z-sz^2},$$

it follows that

$$\begin{aligned}\left|2\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{zf''(z)}{f'(z)}\right| &= \left|2\left(1-\frac{1+sz^2}{1-(1-s)z-sz^2}\right)+\frac{2sz^2}{1+sz^2}+\frac{2(1-s)z+4sz^2}{1-(1-s)z-sz^2}\right| \\ &= \left|\frac{2sz^2}{1+sz^2}\right|.\end{aligned}$$

In view of (3.2), if

$$\left|\frac{2sz^2}{1+sz^2}\right| < \frac{\lambda-|1-\mu|}{1+\lambda-|1-\mu|}=\frac{a}{1+a},$$

then f in $\mathcal{U}(\lambda,\mu)$. Let $g(z) = 2sz^2/(1+sz^2)$. For $z = e^{i\theta} = \cos\theta + i\sin\theta$,

$$|g(e^{i\theta})| = \left|\frac{2se^{2i\theta}}{1+se^{2i\theta}}\right| = \frac{|2s|}{\sqrt{1+s^2+2s\cos 2\theta}}.$$

The maximum value of $|g(e^{i\theta})|$ is attained at $\theta = \pi/2$ or $z = e^{i\pi/2} = i$. At $z = i$, we obtain

$$\left|\frac{2sz^2}{1+sz^2}\right| = \frac{2s}{1-s} = \frac{2\left(\frac{a}{2+3a}\right)}{1-\frac{a}{2+3a}} = \frac{2a}{2+2a} = \frac{a}{1+a}.$$

Therefore, the function f in (3.8) belongs to $\mathcal{U}(\lambda,\mu)$.

Alternatively, the function f in (3.8) can be shown to be in the class $\mathcal{U}(\lambda,\mu)$ by proving directly from (3.1). Here

$$\frac{z}{f(z)} = 1-(1-s)z-sz^2 \quad \text{and} \quad f'(z) = \frac{1+sz^2}{(1-(1-s)z-sz^2)^2}.$$

Since

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-\mu\right| &= |1-\mu+sz^2| \\ &\le |1-\mu|+s \\ &= (\lambda-a)+\frac{a}{2+3a} \\ &= \lambda-\frac{a(3a+1)}{3a+2} \\ &< \lambda,\end{aligned}$$

it follows that f in $\mathcal{U}(\lambda,\mu)$. ■

By using the Koebe function, it can be shown that the condition (3.2) is not necessary. Recall that the Koebe function $k(z) = z/(1-z)^2 \in \mathcal{U}$ and $\mathcal{U} := \mathcal{U}(1,1)$. Here, $\lambda = 1 = \mu$ and so the right hand side of (3.2) is equal to $1/2$. Note that

$$k'(z) = \frac{1+z}{(1-z)^3} \quad \text{and} \quad k''(z) = \frac{2z+4}{(1-z)^4},$$

and so

$$\begin{aligned}\left|2\left(1+\frac{zk'(z)}{k(z)}\right)+\frac{zk''(z)}{k'(z)}\right| &= \left|\frac{-4z}{1-z}+\frac{z(2z+4)}{1-z^2}\right| \\ &= \left|-\frac{2z^2}{1-z^2}\right| \\ &= \left|\frac{2z^2}{1-z^2}\right|.\end{aligned}$$

When $z = 1/2$, we have

$$\frac{2z^2}{1-z^2} = \frac{2(1/2)^2}{1-(1/2)^2} = \frac{2}{3}.$$

Clearly, $2/3 \not< 1/2$. Therefore, the condition (3.2) is not necessary.

The next result gives another sufficient condition for $f \in \mathcal{U}(\lambda,\mu)$, but at the same time it is also a necessary condition for $f \in \mathcal{U}(\lambda,\mu)$.

Theorem 3.2. For $\mu \in \mathbb{C}$, let $|1-\mu| < \lambda \leq 1$ and $\lambda + |1-\mu| \leq 1$. An analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $\mathcal{U}(\lambda,\mu)$ if and only if

$$\frac{z}{f(z)} = 1 - a_2 z - \lambda(1-|a|^2)z\int_0^z \frac{\phi(t)}{t^2(1+\overline{a}\phi(t))}\,dt, \quad a = \frac{1-\mu}{\lambda}, \tag{3.9}$$

where $|a| < 1$ and ϕ is analytic in $\mathbb{D}$ with $\phi(0) = 0 = \phi'(0)$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Proof. Suppose $f \in \mathcal{U}(\lambda,\mu)$. Let $w(z) = \left[(z/f(z))^2 f'(z) - \mu\right] \Big/ \lambda$. Then $|w(z)| < 1$

by (3.1) and we have

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = \mu + \lambda w(z), \tag{3.10}$$

for some $w \in \mathcal{B} := \{w \in \mathcal{H} : |w(z)| < 1, z \in \mathbb{D}\}$. Let set

$$p(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

Then (3.10) is equivalent to

$$p(z) - zp'(z) = \mu + \lambda w(z) \tag{3.11}$$

or

$$(1-\mu) - \sum_{n=1}^{\infty} (n-1) b_n z^n = \lambda w(z).$$

So, $w(0) = (1-\mu)/\lambda$ and $w'(0) = 0$. Let $a = w(0) = (1-\mu)/\lambda$. Then $|a| = |1-\mu|/\lambda < 1$, that is, $a \in \mathbb{D}$, and the function

$$\phi(z) = \frac{w(z) - a}{1 - \overline{a}w(z)}$$

satisfies $\phi(0) = 0$ and $|\phi(z)| < 1$. Furthermore, since

$$\phi'(z) = \frac{(1-\overline{a}w(z))w'(z) + (w(z)-a)(\overline{a}w'(z))}{(1-\overline{a}w(z))^2},$$

it follows that $\phi'(0) = 0$. On the other hand, since

$$\left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = -z^2 \left(\frac{1}{f(z)} - \frac{1}{z}\right)',$$

it follows from (3.10) that

$$z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'=1-\left(\frac{z}{f(z)}\right)^2 f'(z)=1-\mu-\lambda w(z)=-\frac{\lambda(1-|a|^2)\phi(z)}{1+\overline{a}\phi(z)}$$

or

$$\left(\frac{1}{f(z)}-\frac{1}{z}\right)'=-\frac{\lambda(1-|a|^2)\phi(z)}{z^2\big(1+\overline{a}\phi(z)\big)}.$$

Integrate both sides from 0 to z yields

$$\int_0^z\left(\frac{1}{f(t)}-\frac{1}{t}\right)'dt=-\lambda(1-|a|^2)\int_0^z\frac{\phi(t)}{t^2\big(1+\overline{a}\phi(t)\big)}\,dt$$

or

$$\left(\frac{1}{f(z)}-\frac{1}{z}\right)-\left(\frac{1}{f(t)}-\frac{1}{t}\right)\Bigg|_{t=0}=-\lambda(1-|a|^2)\int_0^z\frac{\phi(t)}{t^2\big(1+\overline{a}\phi(t)\big)}\,dt. \tag{3.12}$$

For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{A}$, we have

$$\begin{aligned}\frac{z}{f(z)}&=\frac{1}{1-(-\sum_{n=2}^{\infty}a_nz^{n-1})}\\&=1-\left(\sum_{n=2}^{\infty}a_nz^{n-1}\right)+\left(\sum_{n=2}^{\infty}a_nz^{n-1}\right)^2-\cdots\\&=1-(a_2z+a_3z^2+\ldots)+(a_2^2z^2+\ldots)+\cdots\\&=1-a_2z+(a_2^2-a_3)z^2+\cdots.\end{aligned}$$

It follows that

$$\frac{1}{f(z)}=\frac{1}{z}-a_2+(a_2^2-a_3)z+\cdots.$$

and so

$$\left.\left(\frac{1}{f(t)}-\frac{1}{t}\right)\right|_{t=0}=-a_2.$$

It follows from (3.12)

$$\frac{1}{f(z)}-\frac{1}{z}=-a_2-\lambda(1-|a|^2)\int_0^z\frac{\phi(t)}{t^2\big(1+\overline{a}\phi(t)\big)}\,dt.$$

Multiply both sides by z yields (3.9).

Conversely, suppose (3.9) holds. Then

$$\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'-\mu=1-\mu+\lambda(1-|a|^2)\frac{\phi(z)}{1+\overline{a}\phi(z)}=\frac{\lambda(\phi(z)+a)}{1+\overline{a}\phi(z)}.$$

Since $|\phi(z)|<1$ and $|a|<1$, it follows that

$$\left|\frac{\phi(z)+a}{1+\overline{a}\phi(z)}\right|<1,$$

which leads to

$$\left|\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'-\mu\right|=\lambda\left|\frac{\phi(z)+a}{1+\overline{a}\phi(z)}\right|<\lambda.$$

Then, by using (3.4) and (3.1), we can conclude that f is in $\mathcal{U}(\lambda,\mu)$. $\square$

From Theorem 3.2, the function ϕ satisfies $\phi(0)=0=\phi'(0)$ and $|\phi(z)|<1$ for $z\in\mathbb{D}$. Since $\phi(0)=0=\phi'(0)$, the function ϕ can be written in the form $\phi(z)=z^2g(z)$ where g is analytic in $\mathbb{D}$ and $g(0)\neq 0$. Let $h(z)=\phi(z)/z$ and so $h(z)=zg(z)$. We have $h(0)=0$. Since $\phi(0)=0$ and $|\phi(z)|<1$, by Schwarz's lemma (Lemma 1.1), we have

$|\phi(z)| \leq |z|$ or $|h(z)| \leq 1$. Since $h(0)=0$ and $|h(z)| \leq 1$, by applying Schwarz's lemma again, we obtain $|h(z)| \leq |z|$ or $|\phi(z)| \leq |z|^2$. Setting $w(z) = \phi(z)/z^2$ in (3.9), we get another integral form of $f \in \mathcal{U}(\lambda, \mu)$, that is,

$$\frac{z}{f(z)} = 1 - a_2 z - \lambda(1-|a|^2) z \int_0^z \frac{w(t)}{1+\overline{a}t^2 w(t)}\, dt, \quad a = \frac{1-\mu}{\lambda}, \tag{3.13}$$

where w is analytic in $\mathbb{D}$ satisfying $|w(z)| \leq 1$ for all $z \in \mathbb{D}$. The representation (3.13) is needed to prove the results in Chapter 4.

Taking $\mu = 1$ in Theorem 3.2, the integral representation for the class $\mathcal{U}(\lambda)$ was discussed in [44].

Corollary 3.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{U}(\lambda, \mu)$ and $a = (1-\mu)/\lambda$ with $|a| < 1$, then

$$\left| \frac{z}{f(z)} - 1 + a_2 z \right| \leq \begin{cases} \frac{\lambda(1-|a|^2)|z|}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right), & \text{if } a \neq 0, \\ \lambda |z|^2, & \text{if } a = 0, \end{cases} \tag{3.14}$$

where $a_2 = f''(0)/2$. Equality is attained by the function

$$f(z) = \begin{cases} \dfrac{z}{1 - a_2 z + \frac{\lambda(1-|a|^2)}{2b} z \log\left(\frac{1+bz}{1-bz}\right)}, & b = \sqrt{\overline{a}}, \quad \text{if } a \neq 0, \\ \dfrac{z}{1 - a_2 z + \lambda z^2}, & \text{if } a = 0. \end{cases} \tag{3.15}$$

Proof. Since $|\phi(z)| \leq |z|^2$, we have $|1 + \overline{a}\phi(z)| \geq 1 - |a||z|^2$. Hence, from (3.9), we get

$$\left|\frac{z}{f(z)}-1+a_2z\right| = \left|-\lambda(1-|a|^2)z\int_0^1 \frac{\phi(tz)}{t^2z^2(1+\overline{a}\phi(tz))}zdt\right|$$
$$\leq \lambda(1-|a|^2)|z|^2\int_0^1 \frac{dt}{1-|a|t^2|z|^2}.$$

Note that for $a \neq 0$, we have

$$\frac{1}{1-|a|t^2|z|^2} = \frac{1}{2}\left(\frac{1}{1+\sqrt{|a|}t|z|}+\frac{1}{1-\sqrt{|a|}t|z|}\right)$$
$$= \frac{1}{2\sqrt{|a|}|z|}\left(\frac{\sqrt{|a|}|z|}{1+\sqrt{|a|}t|z|}-\frac{\sqrt{|a|}|z|}{1-\sqrt{|a|}t|z|}\right).$$

Hence,

$$\int_0^1 \frac{dt}{1-|a|t^2|z|^2} = \frac{1}{2\sqrt{|a|}|z|}\int_0^1\left(\frac{\sqrt{|a|}|z|}{1+\sqrt{|a|}t|z|}-\frac{\sqrt{|a|}|z|}{1-\sqrt{|a|}t|z|}\right)dt$$
$$= \frac{1}{2\sqrt{|a|}|z|}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right).$$

Therefore,

$$\int_0^1 \frac{dt}{1-|a|t^2|z|^2} = \begin{cases} \frac{1}{2\sqrt{|a|}|z|}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right), & \text{if } a \neq 0, \\ 1, & \text{if } a = 0. \end{cases}$$

Hence,

$$\left|\frac{z}{f(z)}-1+a_2z\right| \leq \begin{cases} \dfrac{\lambda(1-|a|^2)|z|}{2\sqrt{|a|}}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right), & \text{if } a \neq 0, \\ \lambda|z|^2, & \text{if } a = 0. \end{cases}$$

To show the equality, consider $\phi(z) = -z^2$ in (3.9) and we have

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda(1-|a|^2)z\int_0^z \frac{1}{1-\overline{a}t^2}\,dt. \tag{3.16}$$

For $a \neq 0$, we get

$$\frac{z}{f(z)} = 1 - a_2 z + \frac{\lambda(1-|a|^2)}{2b} z\log\left(\frac{1+bz}{1-bz}\right), \quad b = \sqrt{\overline{a}},$$

and setting $a = 0$ in (3.16) yields

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z^2. \qquad \square$$

Corollary 3.2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{U}(\lambda,\mu)$ and $a = (1-\mu)/\lambda$ with $|a| < 1$, then

$$\left|\frac{z}{f(z)} - 1\right| \leq |z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases} \tag{3.17}$$

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) \geq 1 - |z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases} \tag{3.18}$$

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) \leq 1 + |z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}}\log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases} \tag{3.19}$$

and

$$
|f(z)| \geq \begin{cases} \dfrac{|z|}{1+|z|\left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right]}, & \text{if } a \neq 0, \\ \dfrac{|z|}{1+|z|(\lambda|z|+|a_2|)}, & \text{if } a = 0, \end{cases} \tag{3.20}
$$

where $a_2 = f''(0)/2$.

Proof. Note that

$$
\left|\frac{z}{f(z)} - 1\right| = \left|\frac{z}{f(z)} - 1 + a_2 z - a_2 z\right| \leq \left|\frac{z}{f(z)} - 1 + a_2 z\right| + |a_2 z|.
$$

Using (3.14), we have

$$
\begin{aligned}
\left|\frac{z}{f(z)} - 1 + a_2 z\right| + |a_2 z| &\leq |a_2 z| + \begin{cases} \frac{\lambda(1-|a|^2)|z|}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right), & \text{if } a \neq 0, \\ \lambda|z|^2, & \text{if } a = 0, \end{cases} \\
&= |z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases}
\end{aligned}
$$

which is (3.17). For $\alpha \in \mathbb{C}$, the inequality $|\alpha| \leq w$ implies $-w \leq \operatorname{Re} \alpha \leq w$. Using this relation and (3.17), we have

$$
\operatorname{Re}\left(\frac{z}{f(z)}\right) - 1 \leq |z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases}
$$

or

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) b \leq 1+|z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases}$$

and

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) - 1 \geq -|z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases}$$

or

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) \geq 1-|z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0, \end{cases}$$

Using (3.17),we have

$$\begin{aligned} \left|\frac{z}{f(z)}\right| &= \left|\frac{z}{f(z)} - 1 + 1\right| \\ &\leq \left|\frac{z}{f(z)} - 1\right| + 1 \\ &\leq 1+|z| \begin{cases} \left[|a_2| + \frac{\lambda(1-|a|^2)}{2\sqrt{|a|}} \log\left(\frac{1+\sqrt{|a|}|z|}{1-\sqrt{|a|}|z|}\right)\right], & \text{if } a \neq 0, \\ \lambda|z| + |a_2|, & \text{if } a = 0. \end{cases} \end{aligned}$$

Therefore, (3.20) follows. □

3.3 Transformation preserving and univalency condition

The class $\mathcal{S}$ is invariant under several transformations such as rotation, dilation, conjugation, omitted value transform, disk automorphism and square root transform. As a subclass of $\mathcal{S}$, now we also investigate the transformations that are preserved by function in $\mathcal{U}(\lambda,\mu)$.

Theorem 3.3. The class $\mathcal{U}(\lambda,\mu)$ is preserved under rotation, dilation, conjugation, omitted value transform but not under the square root transform.

Proof. Let $f \in \mathcal{U}(\lambda,\mu)$. Let $h(z) = e^{-i\theta} f(e^{i\theta}z), z \in \mathbb{D}$. Since $h'(z) = f'(e^{i\theta}z)$, it follows that

$$\left(\frac{z}{h(z)}\right)^2 h'(z) - \mu = \left(\frac{e^{i\theta}z}{f(e^{i\theta}z)}\right)^2 f'(e^{i\theta}z) - \mu,$$

Since $z \in \mathbb{D}$ implies $e^{i\theta}z \in \mathbb{D}$, we have

$$\left|\left(\frac{e^{i\theta}z}{f(e^{i\theta}z)}\right)^2 f'(e^{i\theta}z) - \mu\right| < \lambda,$$

and so h in $\mathcal{U}(\lambda,\mu)$. Let $j(z) = r^{-1}f(rz)$, $0 < r < 1$. Since $j'(z) = f'(rz)$, it follows that

$$\left(\frac{z}{j(z)}\right)^2 j'(z) - \mu = \left(\frac{rz}{f(rz)}\right)^2 f'(rz) - \mu$$

Since $z \in \mathbb{D}$ implies $rz \in \mathbb{D}$, we have

$$\left|\left(\frac{rz}{f(rz)}\right)^2 f'(rz) - \mu\right| < \lambda,$$

and so j in $\mathcal{U}(\lambda,\mu)$. Let $\phi(z) = \overline{f(\bar{z})}$. Note that

$$\phi'(z) = \left(\overline{f(\bar{z})}\right)' = \left(z + \sum_{n=2}^{\infty} \overline{a_n} z^n\right)' = 1 + \sum_{n=2}^{\infty} n\overline{a_n} z^{n-1} = \overline{f'(\bar{z})}$$

and so

$$\left(\frac{z}{\phi(z)}\right)^2 \phi'(z) - \mu = \overline{\left(\frac{\bar{z}}{f(\bar{z})}\right)^2 f'(\bar{z})} - \mu.$$