DOUBLE SAMPLING AUXILIARY INFORMATION CHART AND EXPONENTIALLY WEIGHTED MOVING AVERAGE AUXILIARY INFORMATION CHART, BOTH BASED ON VARIABLE SAMPLING INTERVAL, AND MEASUREMENT ERRORS BASED TRIPLE SAMPLING CHART

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by

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n	Sample size
μ_0	In-control mean
μ_1	Out-of-control mean
μ_X	Population mean of auxiliary variable, X
μ_Y	Population mean of study variable, Y
$\hat{\mu}_X$	Estimated population mean of auxiliary variable, X
$\hat{\mu}_Y$	Estimated population mean of study variable, Y
$\hat{\mu}^*_{Y,1i}$	Unbiased estimator of μ_Y
σ_0	In-control standard deviation
σ_M	Measurement error of the standard deviation
σ^2_X	Population variance of auxiliary variable, X
σ^2_Y	Population variance of study variable, Y
λ	Smoothing constant
γ	Measurement error ratio
δ	Size of a standardized shift
$\delta_{\min}$	Lower bound of the size of the standardized shift
$\delta_{\max}$	Upper bound of the size of the standardized shift
$\phi(\cdot)$	Probability density function for standard normal variable
$\Phi(\cdot)$	Cumulative distribution function for standard normal variable
α	Type-I error probability

β	Type-II error probability
n_1	Sample size of the first sample
n_2	Sample size of the second sample
n_3	Sample size of the third sample
h_1	Short sampling interval
h_2	Long sampling interval
h_0	Fixed sampling interval
n_0	Fixed sample size
n_k	Size of sample k
m	Number of measurements per item
X	Auxiliary variable
Y	Study variable
ρ	Correlation coefficient between the study and auxiliary variables
I	Identity matrix
1	Vector where all elements are ones
q or b	Initial state/steady state probability vector
R	Transition probability matrix with the absorbing state **h**
	Vector of sampling intervals
Q	Transition probability matrix for the transient states
P_a	Probability of declaring the process as in-control at a certain sampling stage

P_{as}	Probability of declaring the process as in-control at inspection level s of a certain sampling stage
$P(\delta)$	Probability of declaring the process as out-of-control at a certain sampling stage

LIST OF ABBREVIATIONS

AATS	Adjusted ATS
$\text{AATS}(\delta)$	Out-of-control AATS
AI	Auxiliary information
ARL_0 or ARL(0)	In-control ARL
ARL_1 or $\text{ARL}(\delta)$	Out-of-control ARL
ARL	Average run length
ASS_0	In-control ASS
ASS	Average sample size
ATS	Average time to signal
ASI	Average sampling interval
CL	Central line
DS	Double sampling
DS-AI	DS auxiliary information
EATS	Expected ATS
EAATS	Expected AATS
EARL	Expected ARL
EARL_1	Out-of-control EARL
EATS_1	Out-of-control EATS
EWMA	Exponentially weighted moving average
EWMA-AI	EWMA auxiliary information
FSI	Fixed sampling interval
FSS	Fixed sample size

FAR	False alarm rate
IC	In-control
LCL	Lower control limit
LWL	Lower warning limit
OOC	Out-of-control
RS	Run sum
RS-AI	RS auxiliary information
TS	Triple sampling
TS-ME	TS measurement error
SAS	Statistical analysis system
SPC	Statistical Process Control
SPM	Statistical Process Monitoring
SDRL	Standard deviation of the run length
SDTS	Standard deviation of the time to signal
SDRL_0	In-control SDRL
$\text{SDRL}(\delta)$	Out-of-control SDRL
$\text{SDTS}(\delta)$	Out-of-control SDTS
ssATS	Steady state ATS
ssEATS	Steady state EATS
ssATS(0)	In-control ssATS
ssATS(δ)	Out-of-control ssATS
TPM	Transition probability matrix
UCL	Upper control limit
UWL	Upper warning limit
VSI	Variable sampling interval

VSS	Variable sample size
VSSI	Variable sample size and sampling interval
VSI DS	VSI double sampling
VSI DS-AI	VSI DS auxiliary information
VSI EWMA-AI	VSI EWMA auxiliary information
VSSI-AI	VSSI auxiliary information

LIST OF APPENDICES

CARTA PENSAMPELAN BERGANDA DUA DENGAN MAKLUMAT TAMBAHAN DAN CARTA PURATA BERGERAK BERPEMBERAT EKSPONEN DENGAN MAKLUMAT TAMBAHAN, KEDUA-DUANYA BERDASARKAN SELANG PENSAMPELAN BOLEH BERUBAH, SERTA CARTA PENSAMPELAN BERGANDA TIGA DENGAN RALAT PENGUKURAN

ABSTRAK

Konsep penggunaan maklumat tambahan (AI) dalam carta kawalan semakin popular di kalangan penyelidik. Carta kawalan menggunakan teknik AI didapati lebih berkesan daripada carta kawalan tanpa teknik AI. Objektif pertama tesis ini adalah untuk membangunkan carta selang pensampelan berganda dua (DS) dengan selang pensampelan boleh berubah (VSI) menggunakan teknik AI (dipanggil carta VSI DS-AI) untuk memantau min proses. Statistik carta, reka bentuk optimum dan pelaksanaan carta VSI DS-AI dibincangkan. Masa purata untuk berisyarat keadaan mantap (ssATS) dan masa purata jangkaan untuk berisyarat keadaan mantap (ssEATS) digunakan sebagai ukuran prestasi carta VSI DS-AI yang dicadangkan. Keputusan ssATS dan ssEATS carta VSI DS-AI dibandingkan dengan keputusan setara carta-carta pensampelan berganda dua AI, saiz sampel dan selang pensampelan boleh berubah, purata bergerak berpemberat eksponen AI (EWMA-AI) dan hasil tambah larian AI (RS- AI). Perbandingan mendedahkan bahawa carta VSI DS-AI berprestasi lebih baik daripada carta-carta yang dibandingkan untuk semua saiz anjakan, kecuali carta-carta EWMA-AI dan RS-AI untuk anjakan kecil. Objektif kedua tesis ini adalah untuk mencadangkan carta VSI EWMA-AI untuk pemantauan min proses apabila saiz anjakan tepat yang memerlukan pengesanan pantas tidak dapat dinyatakan dengan

menggunakan kriteria EATS berdasarkan pendekatan rantai Markov. Statistik pencartaan, reka bentuk optimum, penilaian prestasi dan prosedur pengoperasian carta VSI EWMA-AI dibincangkan. Tambahan pula, kriteria ATS dan EATS, masing-masing berdasarkan saiz anjakan tepat dan suatu selang saiz anjakan digunakan untuk mengukur prestasi carta VSI EWMA-AI yang dicadangkan, yang mana carta yang dicadangkan ditunjukkan pada umumnya mengatasi prestasi carta EWMA-AI asas. Objektif ketiga dalam tesis ini adalah untuk membangunkan carta pensampelan ganda tiga $\bar{X}$ dengan kehadiran ralat pengukuran (dipanggil carta TS-ME) menggunakan model ralat kovariat linear untuk pemantauan min proses. Pembinaan, ciri-ciri berstatistik, reka bentuk optimum dan pelaksanaan carta TS-ME dibincangkan. Kriteria purata panjang larian (ARL), sisihan piawai panjang larian (SDRL) dan kadar amaran palsu (FAR) digunakan untuk mengukur prestasi carta TS-ME yang dicadangkan. Penemuan mendedahkan bahawa ralat pengukuran mempengaruhi prestasi carta TS-ME dengan ketara tetapi pengambilan beberapa ukuran untuk setiap *"item"* akan mengimbangi kemunduran ini.

DOUBLE SAMPLING AUXILIARY INFORMATION CHART AND EXPONENTIALLY WEIGHTED MOVING AVERAGE AUXILIARY INFORMATION CHART, BOTH BASED ON VARIABLE SAMPLING INTERVAL, AND MEASUREMENT ERRORS BASED TRIPLE SAMPLING CHART

ABSTRACT

The concept of using auxiliary information (AI) in control charts is growing in popularity among researchers. Control charts using the AI technique have been found to be more effective than control charts without the AI technique. The first objective of this thesis is to develop a variable sampling interval (VSI) double sampling (DS) chart using the AI technique (called VSI DS-AI chart) for monitoring the process mean. The charting statistics, optimal designs and implementation of the VSI DS-AI chart are discussed. The steady-state average time to signal (ssATS) and steady-state expected average time to signal (ssEATS) criteria are used as the performance measures of the proposed VSI DS-AI chart. The ssATS and ssEATS results of the VSI DS-AI chart are compared with those of the double sampling AI, variable sample size and sampling interval AI, exponentially weighted moving average AI (EWMA-AI) and run sum AI (RS-AI) charts. The comparison reveals that the VSI DS-AI chart performs better than the competing charts for all shift sizes, except the EWMA-AI and RS-AI charts for small shifts. The second objective of this thesis is to investigate the VSI EWMA-AI chart's performance in monitoring the process mean when the exact shift size which needs a swift detection cannot be specified, by using the EATS criterion, based on the Markov chain approach. The charting statistics, optimal designs, performance evaluation and operational procedure of the VSI EWMA-AI chart are discussed. Furthermore, the ATS and EATS criteria, based on an exact shift size and an interval of shift sizes, respectively, are used to measure the performance of the VSI EWMA-AI chart, where the chart is shown to generally outperform the basic EWMA-AI chart. The third objective of this thesis is to develop a triple sampling $\bar{X}$ chart in the presence of measurement errors (called TS-ME chart) using the linear covariate error model for monitoring the process mean. The construction, statistical properties, optimal designs and implementation of the TS-ME chart are discussed. The average run length (ARL), standard deviation of the run length (SDRL) and false alarm rate (FAR) criteria are used to measure the performance of the proposed TS-ME chart. The findings reveal that measurement errors significantly affect the performance of the TS-ME chart but taking multiple measurements for each item will compensate for this setback.

CHAPTER 1

INTRODUCTION

1.1 Statistical Process Control

Quality is an essential factor and a common expression when a consumer intends to purchase a product or receive a service from manufacturing and service industries. An acceptable level of quality varies from one person to another (Montgomery, 2013). Quality can simply be defined as the fitness of a product for use, the best among products and services, conformance to product specifications and a product or service surpassing the customer's expectations. The modern description of quality is "Quality is inversely proportional to variability", meaning that, the quality of a manufactured product consistently increases when the variability in a production process decreases and this leads to quality improvement (Montgomery, 2013).

To ensure continuous quality improvement in a manufacturing or service industry, we need the concept of Statistical Process Control (SPC). SPC is a combination of production stages, management plans and practices that can be fully implemented in all levels of an industry or organization. SPC is a collection of problem solving (i.e. statistical) tools to detect variation in a process, in order to enhance process performance and maintain a high level of quality in the production process so that process variation only changes within an acceptable range. SPC ensures a great understanding of a process and permits process monitoring to be conducted effectively by minimizing process variation. Process variation can be classified as common causes of variation and assignable causes of variation in SPC. In practical situations, common (or natural) causes of variation will always exist in a process and are unavoidable, while assignable (or special) causes of variation fall outside the system and can be avoided because this type of variation is caused by disturbances in the process operation. A process operating with only common causes of variation is statistically in-control (IC) or stable, while a process is out-of-control (OOC) or unstable when it is operating in the existence of assignable causes of variation. Examples of assignable causes of variation are machine faults, operator errors and defective raw materials (Montgomery, 2013; Woodall, 2000). Assignable causes of variation can be identified and eliminated by the practitioner.

The major goal of SPC is to identify assignable causes of variation faster so that appropriate corrective measures can be taken to investigate the underlying process before many defective (or nonconforming) products are produced. The important Statistical Process Monitoring (SPM) tools in SPC are scatter diagram, Pareto diagram, cause-and-effect diagram, check sheet, histogram, flowchart and control chart (Gupta and Walker, 2007). Among these SPM tools, the control chart is the most important tool in making a process decision.

1.2 An Outline of Control Charts

The idea of a control chart was originally developed by Walter A. Shewhart in 1924 when he worked in the Bell Telephone Laboratories. A control chart is a useful graphical tool to control a process to ensure the production of quality products (Montgomery, 2013). This is because a control chart maintains process stability in manufacturing over a period of time. A control chart is a time sequence plot of a quality characteristic together with decision lines (called control limits) to make an appropriate decision as to whether a production process is stable (Ryan, 2011). A control chart is a line graph that is based on some statistical distributions and it consists of a central line (CL), a lower control limit (LCL) and an upper control limit (UCL). A typical control chart with the above-mentioned CL and limits is displayed in Figure

1.1. It is assumed that no assignable causes of variation affecting the manufacturing process are present when computing the control limits. As the control chart's limits are computed based on only common causes of variation, the presence of assignable causes of variation in a process will enable the chart to issue an OOC signal (Gitlow et al., 1995). The fundamental principle of a control chart is to take samples periodically, measure the samples and plot the results on the chart to demonstrate how specific measurements change over a period of time (Montgomery, 2013). A control chart is classified as an adaptive chart if it permits at least one of its parameters (for example, sample size or sampling interval) to be varied according to the quality of the current process, while it is classified as a non-adaptive chart if all of its parameters are fixed.

Figure 1.1 A typical control chart

1.3 Research Motivations

Carot et al. (2002) combined the double sampling (DS) $\bar{X}$ chart of Daudin (1992) and the variable sampling interval (VSI) approach of Reynolds et al. (1988) to develop the VSI DS $\bar{X}$ chart for monitoring the process mean. The VSI DS $\bar{X}$ chart was shown to outperform the VSI $\bar{X}$ and DS $\bar{X}$ charts. Recently, the auxiliary information (AI) technique to enhance the precision of a control chart at hand through the use of a regression estimator was extensively investigated. For instance, Riaz (2008) proposed the AI based Shewhart $\bar{X}$ chart, which improves the performance of the standard Shewhart $\bar{X}$ chart, while Haq and Khoo (2018) introduced the DS chart with the AI technique (called the DS-AI chart), where the DS-AI chart outperforms the DS $\bar{X}$ chart. The effectiveness of the VSI DS $\bar{X}$ chart and the AI technique has motivated the development of the VSI DS-AI chart, for monitoring the process mean, in this thesis, by combining the VSI DS $\bar{X}$ and AI procedures. The new VSI DS-AI chart benefits from the salient feature of the AI technique, while retaining the effectiveness of the VSI DS $\bar{X}$ chart.

The exponentially weighted moving average (EWMA) chart is very sensitive in detecting small shifts (Roberts, 1959). The adoption of the VSI feature in the EWMA chart with auxiliary information (called the VSI EWMA-AI chart) introduced by Ng et al. (2020a), significantly improves the performance of the EWMA-AI chart in detecting small and moderate shifts, in terms of the average time to signal (ATS) criterion, when the exact shift size which must be detected quickly can be specified. However, in many industrial processes, the exact shift size cannot be specified, especially when practitioners have no prior knowledge about the process. The remarkable performance of the VSI EWMA-AI chart motivates an investigation to be conducted in this thesis to investigate the performance of the said chart when the exact shift size where a swift detection is required cannot be specified, by adopting the EATS criterion, based on the Markov chain approach.

To further enhance the efficiency of the DS $\bar{X}$ chart in detecting process mean shifts, He et al. (2002) suggested the TS $\bar{X}$ chart. More recently, Mim et al. (2022) pointed out that the mathematical formulae provided in He et al. (2002) for computing the average run length (ARL) of the TS $\bar{X}$ chart was incorrect, subsequently, the former presented the correct formulae for computing the ARL of the TS $\bar{X}$ chart with known and estimated process parameters. The TS $\bar{X}$ chart in He et al. (2002), Mim et al. (2022) and all existing literature were designed based on the assumption that the value of the process quality characteristic being monitored is free of measurement errors. However, in practice, measurement errors exist in the process and a control chart’s performance deteriorates when such errors occur. Linna and Woodall (2001) used the linear covariate error model to investigate the effect of measurement errors on the classical Shewhart $\bar{X}$ and S^2 charts. The effects of measurement errors on control charts were also studied by Costa and Castagliola (2011), Lee et al. (2019) and Maleki et al. (2021). Due to the severe negative effects of measurement errors on control charts, another motivation of this thesis is to study the effect of measurement errors on the TS $\bar{X}$ chart's (called the TS-ME chart) performance when measurement errors are present by adopting the linear covariate error model of Linna and Woodall (2001).

1.4 Objectives of the Thesis

The objectives of this thesis are as follows:

(i) To develop the variable sampling interval double sampling chart with auxiliary information (VSI DS-AI chart) for the process mean.

(ii) To investigate the performance of the auxiliary information based variable sampling interval exponentially weighted moving average (VSI EWMA-AI) chart for the process mean when the exact shift size cannot be specified.

(iii) To develop the triple sampling (TS) $\bar{X}$ chart in the presence of measurement errors (TS-ME chart) for the process mean.

1.5 Organization of the Thesis

This thesis consists of six chapters. It begins with an introduction in Chapter 1. In Chapter 1, the concept of SPC and control charts are briefly discussed. The research motivation, objectives and organization of the thesis are provided in the later part of this chapter.

In Chapter 2, the performance measures and literature review of related control charts, such as the Shewhart $\bar{X}$, adaptive type, AI type and measurement errors type control charts are presented.

Chapter 3 presents the proposed VSI DS-AI chart. The implementation and optimal design procedures of the VSI DS-AI chart, as well as performance comparisons with competing charts are also provided. An illustrative example is presented at the end of the chapter for showing the application of the VSI DS-AI chart in process monitoring. Finally, concluding remarks are provided to summarize the discussion in this chapter.

Chapter 4 presents the VSI EWMA-AI chart when the exact shift size cannot be specified. The implementation and optimal design procedures of the VSI EWMA-AI chart, as well as a performance comparison with the EWMA-AI chart is also provided. An illustrative example is presented at the end of the chapter for showing the application of the VSI EWMA-AI chart in process monitoring. Finally, concluding remarks are also given to summarize the discussion in this chapter.

Chapter 5 presents the developed TS-ME chart in the presence of measurement errors. A detailed discussion on the derivation of the ARL and standard deviation of the run length (SDRL) formulae of the proposed TS-ME chart is also presented together with performance assessments of the chart. Additionally, the application of the TS-ME chart is demonstrated with an illustrative example. The last section of this chapter gives some concluding remarks.

In Chapter 6, conclusions are drawn, where the main contributions and findings of this thesis, as well as recommendations for further research are summarized.

References and appendices are provided at the end of this thesis. Optimization and simulation programs written in the MATLAB and Statistical Analysis System (SAS) software, respectively, for the VSI DS-AI, VSI EWMA-AI and TS-ME charts are given in Appendices A and B. The MATLAB optimization programs for the competing charts are presented in Appendix C.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The Shewhart $\bar{X}$ chart is usually adopted in manufacturing and service industries due to its simplicity and ease of application. However, the Shewhart $\bar{X}$ chart is only efficient in detecting large process mean shifts but it is insensitive to small and moderate process mean shifts. In order to overcome the insensitivity of the Shewhart $\bar{X}$ chart towards small and moderate shifts, more efficient control charts have been developed. The control charts which are related to the proposed charts are discussed in this chapter.

The organization of this chapter is as follows: Section 2.2 explains the common performance measures of control charts. The Shewhart $\bar{X}$ chart is discussed in Section 2.3. In Section 2.4, some existing adaptive charts, such as the VSI DS $\bar{X}$ and TS $\bar{X}$ charts are presented. The AI type charts are presented in Section 2.5. These AI charts are the DS-AI, variable sample size and sampling interval AI (VSSI-AI), EWMA-AI and RS-AI charts. Lastly, a brief discussion on measurement errors type charts is provided in Section 2.6.

2.2 Performance Measures of Control Charts

The performance measure of a control chart refers to the statistical measure used to evaluate the performance of the chart. Various performance measures are employed to evaluate the efficiency of a control chart. In this thesis, some common performance measures, namely the ARL, expected ARL (EARL), SDRL, ATS, EATS, standard deviation of the time to signal (SDTS) and false alarm rate (FAR) are used in evaluating the performances of the proposed and competing charts. The performances of non-adaptive charts are evaluated using the ARL, EARL and SDRL criteria, while the ATS, EATS and SDTS criteria are adopted to evaluate the performances of both adaptive and non-adaptive charts.

2.2.1 Average Run Length

The average run length (ARL) is defined as the average number of sample points plotted on a chart before a point signals an out-of-control condition, that is, the ARL measures the speed (in terms of the average number of samples required) in detecting an out-of-control signal. The ARL is classified as an in-control ARL $\left(\text{ARL}_0\right)$ when the process is in-control or an out-of-control ARL $\left(\text{ARL}_1 \text{ or } \text{ARL}(\delta)\right)$ when the process is out-of-control (Montgomery, 2013). Here, δ denotes the standardized shift size in the process mean. ARL_0 denotes the average number of sample points required by a chart in signalling a false alarm when the process is in-control, while ARL_1 $\left(\text{or } \text{ARL}(\delta)\right)$ is the average number of sample points required in indicating an out-of-control when the process has shifted.

2.2.2 Expected Average Run Length

The ARL is adopted to evaluate the performance of a chart when the exact shift size can be specified (Castagliola et al., 2011). However, the exact shift size cannot be exactly specified in many practical situations, as practitioners do not have enough historical information about the process shift. Therefore, the expected value of the ARL (EARL) is usually used to evaluate the performance of a chart for process shifts in a certain range. The value of the in-control EARL is usually set to be similar to ARL_0.

The out-of-control EARL $\left(\mathrm{EARL}_1\right)$ is employed to determine the expected ARL required by a chart to indicate an out-of-control signal.

2.2.3 Standard Deviation of the Run Length

When the ARL measure is employed to evaluate the performance of a control chart, the standard deviation of the run length (SDRL) is usually computed to measure the variability of the run length distribution (Saha, 2019). The in-control and out-of-control SDRLs are denoted as SDRL_0 and $\mathrm{SDRL}(\delta)$, respectively. A small SDRL value is preferable as it indicates a small variation in the run length distribution. A control chart is considered as effective in detecting process shifts if it yields smaller $\mathrm{ARL}(\delta)$ and $\mathrm{SDRL}(\delta)$ values compared with its competitors when all of the charts have the same ARL_0 value.

2.2.4 Average Time to Signal

The average time to signal (ATS) is defined as the average length of time needed by a control chart to detect an out-of-control signal (Abubakar, 2021). The ATS criterion is employed to evaluate an adaptive chart's performance, such as an adaptive chart with the variable sampling interval (VSI) technique. The ATS is a product of the ARL and the value of the fixed sampling interval (FSI), for the FSI chart. However, for an adaptive chart with the VSI feature, the ATS is determined directly by measuring the average length of the time required before a control chart signals a process shift. The ATS is categorized as an in-control ATS (ATS_0) when the process is in-control and an out-of-control ATS (ATS(δ)) when the process is out-of-control.

2.2.5 Expected Average Time to Signal

The expected average time to signal (EATS) criterion is adopted to evaluate the performance of a control chart in detecting shifts, for shift sizes in a specified interval. The EATS measures the expected ATS needed by a chart to issue an out-of-control signal. The value of the in-control EATS is usually set to be similar to the ATS_0, when the process is in-control. The out-of-control EATS (EATS_1) is used to measure the expected ATS needed by a control chart to signal a shift over the aforementioned shift interval (Saha, 2019).

2.2.6 Standard Deviation of the Time to Signal

When the ATS measure is used to evaluate the performance of a control chart, the standard deviation of the time to signal (SDTS) is usually calculated to measure the spread of the time to signal distribution (Yeong et al., 2017a). The in-control and out-of-control SDTSs are represented as SDRL_0 and $\text{SDTS}(\delta)$, respectively. A small SDTS value is preferable as it indicates a small variation in the time to signal distribution. A control chart is considered to perform well if it yields smaller $\text{ATS}(\delta)$ and $\text{SDTS}(\delta)$ values compared with its competitors when all the charts have the same ATS_0 value.

2.2.7 False Alarm Rate

The false alarm rate (FAR) is the reciprocal of the average number of samples taken until a control chart signals an out-of-control even though the process is in-control. Controlling the FAR and ARL_0 has traditionally been the major priority in process monitoring. A false alarm in the process can lead to unwanted distractions, as well as a loss of time and effort. A substantial number of false alarms may cause practitioners to entirely ignore the signals from the monitoring technique.

2.3 Shewhart $\bar{X}$ Control Chart

Let $\{X_{i1}, X_{i2}, ..., X_{in}\}$ denote the i^{th} sample of size n (≥ 1), from a process which is assumed to follow a normal distribution with the mean $\mu_0 + \delta\sigma_0$ and variance σ_0^2, i.e. $X_{ij} \sim N\left(\mu_0 + \delta\sigma_0, \sigma_0^2\right)$, where μ_0 and σ_0 represents the nominal process mean and standard deviation, respectively, for the observations $j = 1, 2, ..., n$. The process is in-control when the standardized shift size, $\delta = 0$, otherwise, the process is out-of-control. The i^{th} sample mean is computed as

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij} . \tag{2.1}$$

The sample mean $\bar{X}$ is plotted on the Shewhart $\bar{X}$ chart and the lower control limit (LCL), central line (CL) and upper control limit (UCL) of the chart are (Montgomery, 2013)

$$\text{LCL} = \mu_0 - A_{(\bar{X})}\frac{\sigma_0}{\sqrt{n}}, \tag{2.2a}$$

$$\text{CL} = \mu_0, \tag{2.2b}$$

$$\text{UCL} = \mu_0 + A_{(\bar{X})}\frac{\sigma_0}{\sqrt{n}}, \tag{2.2c}$$

where $A_{(\bar{X})}$ is the width constant parameter which is required to obtain a desired ARL_0 performance. $A_{(\bar{X})}$ is usually set as 3 so that the size of the Type-I error is 0.0027. The Shewhart $\bar{X}$ chart issues an out-of-control signal when the plotting statistic $\bar{X}$ falls beyond the limits in Equations (2.2a) and (2.2c).

The ARL of the Shewhart $\bar{X}$ chart for a standardized mean shift, $\delta = |\mu_1 - \mu_0| / \sigma_0$, where $\mu_1 = \mu_0 + \delta\sigma_0$ is the out-of-control mean, is given as (Montgomery, 2013)

$$\text{ARL}(\delta) = \frac{1}{P(\delta)}. \tag{2.3}$$

Here, $P(\delta)$ is the probability that $\bar{X}$ falls beyond the limits in Equations (2.2a) and (2.2c) and it is computed as

$$P(\delta) = 1 - \Phi\left(A_{(\bar{X})} - \delta\sqrt{n}\right) + \Phi\left(-A_{(\bar{X})} - \delta\sqrt{n}\right), \tag{2.4}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the standard normal random variable.

The EARL value of the Shewhart $\bar{X}$ chart is computed as

$$\text{EARL}(\delta_{\text{min}}, \delta_{\text{max}}) = \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} \text{ARL}(\delta) f(\delta) \text{d}\delta, \tag{2.5a}$$

where δ_{min} and δ_{max} denote the lower and upper bounds, respectively, for the shift interval $(\delta_{\text{min}}, \delta_{\text{max}})$. The expression of $\text{ARL}(\delta)$ is adopted from Equation (2.3), while $f(\delta)$ is the probability density function (pdf) of the standardized mean shift δ. It is assumed that an arbitrary value of δ in the interval $(\delta_{\text{min}}, \delta_{\text{max}})$ exists with an equal probability of occurrence (see for example, Sparks (2000) and Castagliola et al. (2011)). Consequently, δ is assumed to follow the uniform, $U(\delta_{\text{min}}, \delta_{\text{max}})$ distribution. Then Equation (2.5a) becomes

$$\text{EARL}(\delta_{\text{min}}, \delta_{\text{max}}) = \frac{1}{\delta_{\text{max}} - \delta_{\text{min}}} \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} \text{ARL}(\delta) \text{d}\delta. \tag{2.5b}$$

2.4 A Review on Adaptive Type Control Charts

An adaptive control chart allows for an adjustment of at least one of the chart's parameters, such as the sample size, sampling interval and control limits' width constant, depending on the quality of the current process. Adaptive charts are more efficient in detecting process shifts compared to non-adaptive charts. The developments on adaptive charts will be discussed in the next paragraph.

Reynolds et al. (1988) proposed an adaptive $\bar{X}$ chart with the VSI feature, known as the VSI $\bar{X}$ chart. Subsequently, Costa (1994) and Prabhu et al. (1994) developed an adaptive $\bar{X}$ chart with variable sample size (called VSS $\bar{X}$ chart) and a combined adaptive sample size and sampling interval $\bar{X}$ chart (called VSSI $\bar{X}$ chart), respectively. When adopting the VSI approach, the sampling interval is not assumed to be constant, instead, the sampling interval, h_i, may change from one sample to another. Two possible sampling intervals $h_i \in (h_1, h_2)$ are used at regular intervals of time, with $h_2 > h_1$, where h_1 and h_2 denote the short and long sampling intervals, respectively. The VSI technique allows the next sample to be taken after a short sampling interval (h_1) if the present sample point plots in the warning region of the chart, and the next sample to be taken after a long sampling interval (h_2) if the present sample point plots in the chart's control region (Reynolds et al., 1988). Lee et al. (2015) incorporated the VSI chart and the conforming run length (CRL) chart to propose the VSI synthetic chart for the process mean. Chew et al. (2015) presented the VSI RS chart and investigated its performance using the Markov chain procedure. Liu et al. (2015) suggested an adaptive phase-II non-parametric EWMA VSI chart. Amdouni et al. (2017) proposed the VSI coefficient of variation (CV) chart for short production runs. Yeong et al. (2017a) introduced the VSI EWMA chart for monitoring the process CV, while Yeong et al. (2017b) suggested the CV chart with measurement error. The VSI EWMA cumulative counts of conforming chart (called the VSI EWMA CCC chart) using the Markov chain approach was presented by Abubakar (2021) and Abubakar et al. (2021).

In the DS $\bar{X}$ chart, the need to take only the first sample or both first and second samples depends on the quality of the current process. Thus, the in-control and out-of-control decisions of a process on the DS $\bar{X}$ chart is based on the information provided by either the first sample or both the first and second samples. The DS $\bar{X}$ chart is an adaptive chart that improves the performance of the Shewhart $\bar{X}$ chart and it was pioneered by Croasdale (1974) and further improved by Daudin (1992). Irianto and Shinozaki (1998) introduced an optimal DS $\bar{X}$ chart that improves the sensitivity of the DS $\bar{X}$ charts by Croasdale (1974) and Daudin (1992) in detecting small mean shifts.

The economic and economic-statistical designs of the DS $\bar{X}$ chart was presented by Torng et al. (2009a) and Torng et al. (2009b), respectively. Khoo et al. (2010) developed the synthetic DS $\bar{X}$ chart, by incorporating the DS $\bar{X}$ chart into the CRL chart, where the proposed chart was found to outperform both the synthetic $\bar{X}$ and DS $\bar{X}$ charts. The performances of the DS and variable parameters charts in the presence of correlation were compared by Costa and Machado (2011). Khoo et al. (2013) investigated the effects of parameter estimation on the optimal DS $\bar{X}$ chart. Teoh et al. (2015) examined the effect of parameter estimation on the DS $\bar{X}$ chart based on the median run length criterion. Costa and Machado (2015) studied the steady-state behavior of the synthetic and side-sensitive synthetic DS charts in detecting shifts in the process mean. The side-sensitive group runs DS chart to detect shifts in the process mean was suggested by Khoo et al. (2015). Other recent works on the DS charts were reported in Teoh et al. (2016), Castagliola et al. (2017), Saha et al. (2017a), You (2018), Saha (2019), Haq and Khoo (2019), Malela-Majika and Rapoo (2019), Umar et al. (2020) and Maleki et al. (2021).

The DS charts were also combined with the VSI technique to further improve the sensitivity of the former towards process shifts. Carot et al. (2002) incorporated the DS $\bar{X}$ chart of Daudin (1992) with the VSI $\bar{X}$ approach of Reynolds et al. (1988) to propose the VSI DS $\bar{X}$ chart for monitoring the process mean. On similar lines, Lee et al. (2012a) adopted the idea of Carot et al. (2002) to suggest the VSI DS *S* chart for detecting shifts in the process standard deviation, where *S* denotes the sample standard deviation. Lee et al. (2012b) suggested the economic design of the VSI DS $\bar{X}$ chart to extend the work of Carot et al. (2002) by considering the cost of implementing the chart. Lee (2013) developed a VSI DS chart for jointly monitoring the process mean and standard deviation. Lee and Khoo (2017) combined the DS and VSI *np* charts for detecting increases in the number of non-conforming units in a process.

To further enhance the efficiency of the DS $\bar{X}$ chart in detecting process mean shifts, the TS $\bar{X}$ chart was developed by He et al. (2002). The TS $\bar{X}$ chart provides an additional chance to take the third sample if the first and second samples do not provide enough information for decision making about the actual status of the process. The TS $\bar{X}$ chart surpasses the DS $\bar{X}$ chart in minimizing the in-control average sample size (ASS_0). More recent researches on the TS $\bar{X}$ charts are discussed below. The economic statistical design of the DS $\bar{X}$ and TS $\bar{X}$ charts was introduced by Iziy et al. (2017). Oprime et al. (2019) presented the Shewhart $\bar{X}$ chart with asymmetric limits and TS, as well as considered the effect of estimating the statistical parameters in the analysis. More recently, Mim et al. (2022) pointed out the error in the mathematical formulae presented in He et al. (2002) for computing the ARL of the TS $\bar{X}$ chart and subsequently, provided the correct formulae for computing the ARL of the TS $\bar{X}$ chart with known and estimated process parameters.

2.4.1 Variable Sampling Interval Double Sampling $\bar{X}$ Chart

The VSI DS $\bar{X}$ chart of Carot et al. (2002), which significantly improves the performance of the DS $\bar{X}$ chart of Daudin (1992), in detecting process mean shifts, will be discussed in this section.

The VSI DS $\bar{X}$ chart considers a single quality characteristic of interest, which follows a normal distribution with the mean μ and variance σ_0^2. The process mean is in-control when $\mu_1 = \mu_0$, while it is out-of-control when $\mu_1 = \mu_0 \pm \delta\sigma_0$, for $\delta > 0$. The control charting statistics of the VSI DS $\bar{X}$ chart at sampling stage i are $Z_{1,i} = \sqrt{n_1}\left(\bar{X}_{1,i} - \mu_0\right)/\sigma_0$ and $Z_i = \sqrt{n_1 + n_2}\left(\bar{Y}_i - \mu_0\right)/\sigma_0$, where $\bar{Y}_i = \left(n_1\bar{X}_{1,i} + n_2\bar{X}_{2,i}\right)/\left(n_1 + n_2\right)$. Here, $Z_{1,i}$ and Z_i represent the standardized statistics based on the first sample and combined samples, respectively, while $\bar{Y}_i$ denotes the combined sample means. Note that $\bar{X}_{1,i}$ and $\bar{X}_{2,i}$ are the sample means for the first and second samples, with sizes n_1 and n_2, respectively.

The VSI DS $\bar{X}$ chart has eight parameters, i.e., $n_1, n_2, h_1, h_2, w_{(\text{VSIDS})}$, $L_{(\text{VSIDS})}$, $L_{1(\text{VSIDS})}$ and $L_{2(\text{VSIDS})}$. Here, h_1 and h_2 are the short and long sampling intervals, respectively, satisfying $h_1 < h_2$. A graphical representation of the VSI DS $\bar{X}$ chart is given in Figure 2.1, and the chart's step-by-step implementation procedure is explained below.

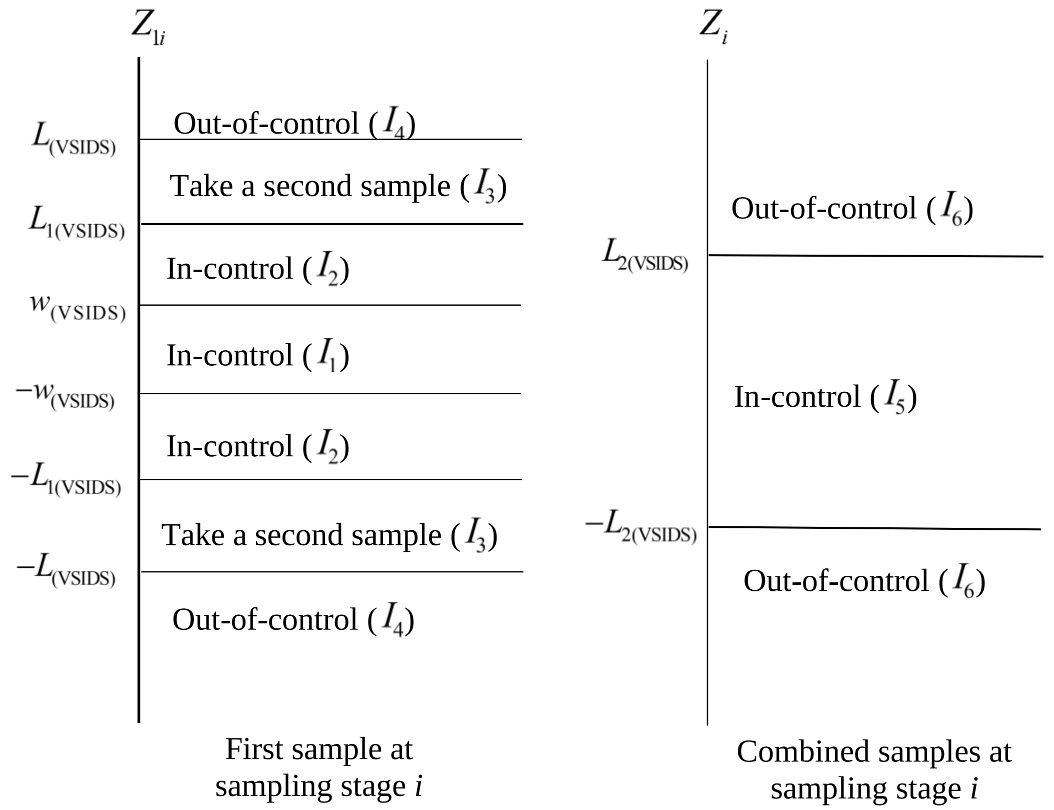

Figure 2.1 A graphical display of the VSI DS $\bar{X}$ chart

Step 1: At sampling stage *i*, take an initial sample of size n_1, then compute $\bar{X}_{1i} = \frac{1}{n_1}\sum_{j=1}^{n_1} X_{1ij}$, followed by $Z_{1i} = \sqrt{n_1}\left(\bar{X}_{1i} - \mu_0\right)/\sigma_0$.

Step 2: If $Z_{1i} \in I_1 = \left[-w_{(\text{VSIDS})}, w_{(\text{VSIDS})}\right]$, the process at sampling stage *i* is declared as in-control. Then return to Step 1 but with the sampling interval between sampling stages *i* and *i* + 1 being h_2.

Step 3: If $Z_{1i} \in I_2 = \left[-L_{1(\text{VSIDS})}, -w_{(\text{VSIDS})}\right) \cup \left(w_{(\text{VSIDS})}, L_{1(\text{VSIDS})}\right]$, the process at sampling stage *i* is still in-control. Then return to Step 1 but with the sampling interval between sampling stages *i* and *i* + 1 being h_1.

Step 4: If $Z_{1i} \in I_4 = \left(-\infty, -L_{(\text{VSIDS})}\right) \cup \left(L_{(\text{VSIDS})}, +\infty\right)$, the process at sampling stage *i* is declared as out-of-control. Then proceed to Step 7.

Step 5: If $Z_{1i} \in I_3 = \left[-L_{(\text{VSIDS})}, -L_{1(\text{VSIDS})}\right) \cup \left(L_{1(\text{VSIDS})}, L_{(\text{VSIDS})}\right]$, a second sample of size n_2 is taken immediately from the same population as that of the first sample. Then compute $\bar{Y}_i = \left(n_1 \bar{X}_{1i} + n_2 \bar{X}_{2i}\right) / \left(n_1 + n_2\right)$, followed by $Z_i = \sqrt{n_1 + n_2}\left(\bar{Y}_i - \mu_0\right) / \sigma_0$.

Step 6: If $Z_i \in I_5 = \left[-L_{2(\text{VSIDS})}, L_{2(\text{VSIDS})}\right]$, the process at sampling stage *i* is concluded as in-control and the implementation procedure returns to Step 1 but with the sampling interval between sampling stages *i* and *i* + 1 being h_1. Otherwise, the process at sampling stage *i* is out-of-control and the procedure proceeds to Step 7.

Step 7: An out-of-control signal is issued and corrective actions are taken to investigate the underlying process and remove the assignable causes. Then, return to Step 1.

The adjusted ATS (AATS) (for the steady state process) and average sampling interval (ASI) of the VSI DS $\bar{X}$ chart are computed as follows (Carot et al., 2002):

$$\text{AATS}(\delta) = \frac{\left(\frac{h_2^2}{2}\left(\frac{\Pr\left(I_1 \middle| \delta = 0\right)}{\Pr\left(A \middle| \delta = 0\right)}\right) + \frac{h_1^2}{2}\left(1 - \frac{\Pr\left(I_1 \middle| \delta = 0\right)}{\Pr\left(A \middle| \delta = 0\right)}\right)\right)}{\left(\frac{h_2 \Pr\left(I_1 \middle| \delta = 0\right)}{\Pr\left(A \middle| \delta = 0\right)} + h_1\left(1 - \frac{\Pr\left(I_1 \middle| \delta = 0\right)}{\Pr\left(A \middle| \delta = 0\right)}\right)\right)} + \left(\frac{1}{1 - \Pr\left(A \middle| \delta\right)} - 1\right) \times \left[\frac{h_2 \Pr\left(I_1 \middle| \delta\right)}{\Pr\left(A \middle| \delta\right)} + h_1\left(1 - \frac{\Pr\left(I_1 \middle| \delta\right)}{\Pr\left(A \middle| \delta\right)}\right)\right] \quad (2.6)$$

and

$$\text{ASI} = \left[h_2 \left(\frac{\Pr\left(I_1|\delta\right)}{\Pr\left(A|\delta\right)} \right) + h_1 \left(1 - \left(\frac{\Pr\left(I_1|\delta\right)}{\Pr\left(A|\delta\right)} \right) \right) \right]. \tag{2.7}$$

Note that

$$\Pr\left(I_1|\delta\right) = \Phi\left(w_{(\text{VSIDS})} - \delta\sqrt{n_1}\right) - \Phi\left(-w_{(\text{VSIDS})} - \delta\sqrt{n_1}\right) \tag{2.8}$$

and

$$\Pr\left(A|\delta\right) = \Pr\left(A_1|\delta\right) + \Pr\left(A_2|\delta\right), \tag{2.9}$$

where $\Pr\left(A_1|\delta\right)$ and $\Pr\left(A_2|\delta\right)$ denote the probabilities of declaring the process as in-control at the first sample and the combined samples, respectively (Carot et al., 2002). Thus,

$$\begin{aligned} \Pr\left(A|\delta\right) &= \Pr\left(Z_{1i} \in I_1 \cup I_2\right) + \Pr\left(Z_{1i} \in I_3 \text{ and } Z_i \in I_5\right) \\ &= \Phi\left(L_{1(\text{VSIDS})} - \delta\sqrt{n_1}\right) - \Phi\left(-L_{1(\text{VSIDS})} - \delta\sqrt{n_1}\right) + \\ &\int_{z \in I_3^*} \left\{ \Phi\left(e_2 L_{2(\text{VSIDS})} - d_2 e_2 \delta - z\sqrt{\frac{n_1}{n_2}} \right) - \Phi\left(-e_2 L_{2(\text{VSIDS})} - d_2 e_2 \delta - z\sqrt{\frac{n_1}{n_2}} \right) \right\} \phi\left(z\right) \mathrm{dz}, \end{aligned} \tag{2.10}$$

where $d_2 = \sqrt{n_1 + n_2}$, $e_2 = \sqrt{\frac{n_1 + n_2}{n_2}}$, $I_3^* = \left[-L_{(\text{VSIDS})} - \delta\sqrt{n_1}, -L_{1(\text{VSIDS})} - \delta\sqrt{n_1}\right] \cup \left[L_{1(\text{VSIDS})} - \delta\sqrt{n_1}, L_{(\text{VSIDS})} - \delta\sqrt{n_1}\right]$, while $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal random variable, respectively.

The average sample size (ASS) of the VSI DS $\bar{X}$ chart at sampling stage *i* is obtained as

$$\text{ASS} = n_1 + n_2 \Pr\left(I_3|\delta\right), \tag{2.11}$$

where $\Pr\left(I_3 \middle| \delta\right)$ represents the probability of Z_{1i} falling in I_3 $\left(\text{i.e. } \Pr\left(Z_{1i} \in I_3\right)\right)$ and is given as

$$\Pr\left(I_3 \middle| \delta\right) = \Phi\left(L_{(\text{VSIDS})} - \delta\sqrt{n_1}\right) - \Phi\left(L_{1(\text{VSIDS})} - \delta\sqrt{n_1}\right) + \Phi\left(-L_{1(\text{VSIDS})} - \delta\sqrt{n_1}\right) - \Phi\left(-L_{(\text{VSIDS})} - \delta\sqrt{n_1}\right). \tag{2.12}$$

The performance of the VSI DS $\bar{X}$ chart can also be assessed through the expected AATS (EAATS) criterion given in Equation (2.13).

$$\text{EAATS}\left(\delta_{\min}, \delta_{\max}\right) = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \text{AATS}\left(\delta\right) \text{d}\delta, \tag{2.13}$$

where $\text{AATS}\left(\delta\right)$ is given in Equation (2.6) and $\left(\delta_{\min}, \delta_{\max}\right)$ represents the standardized shift range with $\delta_{\min}$ and $\delta_{\max}$ as the minimum and maximum shifts, respectively.

2.4.2 Triple Sampling $\bar{X}$ Chart

The TS $\bar{X}$ chart was originally developed by He et al. (2002) and recently revised by Mim et al. (2022). The TS $\bar{X}$ chart, which enhances the efficiency of the DS $\bar{X}$ chart of Daudin (1992) provides an additional chance of taking the third sample if the first and second samples do not provide enough information for decision making about the status of the process being monitored.

The TS $\bar{X}$ chart of He et al. (2002) and the revised TS $\bar{X}$ chart of Mim et al. (2022) assume that the underlying process follows a normal distribution with an in-control process mean μ_0 and an in-control process standard deviation σ_0. The out-of-control process mean is represented as $\mu_1 = \mu_0 + \delta\sigma_0$, where δ is the size of a standardized mean shift. The process is out-of-control when $\delta > 0$, while it is in-control when $\delta = 0$. The TS $\bar{X}$ chart has three levels of inspections (see Figure 2.2) and the chart's step-by-step operational procedure is given henceforth. After determining the optimal parameters n_1, n_2, n_3, $L_{(\mathrm{TS})}$, $L_{1(\mathrm{TS})}$, $L_{2(\mathrm{TS})}$, $L_{3(\mathrm{TS})}$ and $L_{4(\mathrm{TS})}$, the TS $\bar{X}$ chart works as follows:

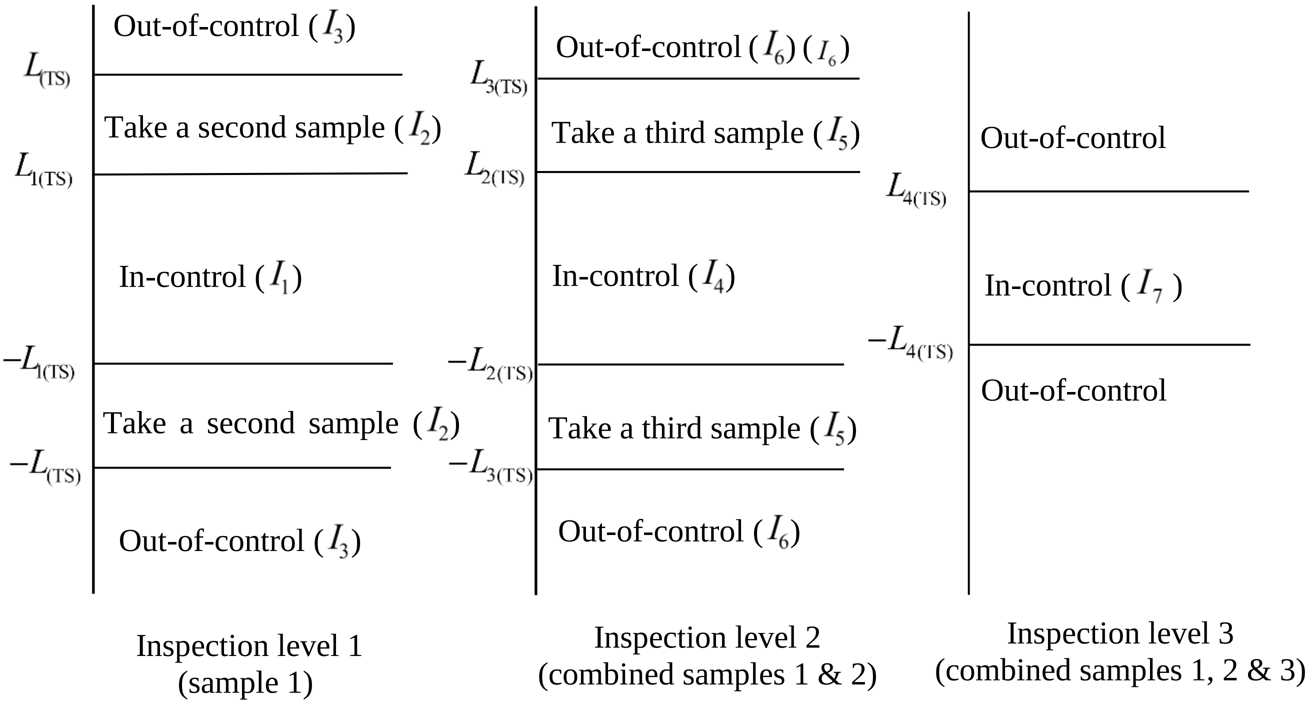

Figure 2.2 A graphical display of the TS $\bar{X}$ chart

Step 1: At sampling stage *i*, take an initial sample of size n_1 (at inspection level 1) and compute the sample mean $\bar{X}_{1i} = \frac{1}{n_1}\sum_{j=1}^{n_1} X_{1ij}$. Let $\bar{Y}_{1i} = \bar{X}_{1i}$, then compute V_{1i} $= \sqrt{n_1}\left(\bar{Y}_{1i} - \mu_0\right)/\sigma_0$. If $V_{1i} \in I_1 = \left[-L_{1(\mathrm{TS})}, L_{1(\mathrm{TS})}\right]$, the process is in-control. The process is out-of-control if $V_{1i} \in I_3 = \left(-\infty, -L_{(\mathrm{TS})}\right) \cup \left(L_{(\mathrm{TS})}, +\infty\right)$. However, if $V_{1i} \in I_2 = \left[-L_{(\mathrm{TS})}, -L_{1(\mathrm{TS})}\right) \cup \left(L_{1(\mathrm{TS})}, L_{(\mathrm{TS})}\right]$, take a second sample of size n_2 (at inspection level 2) and compute its mean $\bar{X}_{2i} = \frac{1}{n_2}\sum_{j=1}^{n_2} X_{2ij}$. Then compute the combined sample means of the first and second samples as $\bar{Y}_{2i} = \left(n_1\bar{X}_{1i} + n_2\bar{X}_{2i}\right) / \left(n_1 + n_2\right)$, followed by $V_{2i} = \sqrt{n_1 + n_2}\left(\bar{Y}_{2i} - \mu_0\right) / \sigma_0$.

Step 2: If $V_{2i} \in I_4 = \left[-L_{2(\mathrm{TS})}, L_{2(\mathrm{TS})}\right]$, the process is in-control. Nevertheless, if $V_{2i} \in I_6 = \left(-\infty, -L_{3(\mathrm{TS})}\right) \cup \left(L_{3(\mathrm{TS})}, +\infty\right)$, the process is out-of-control. Moreover, if $V_{2i} \in I_5 = \left[-L_{3(\mathrm{TS})}, -L_{2(\mathrm{TS})}\right) \cup \left(L_{2(\mathrm{TS})}, L_{3(\mathrm{TS})}\right]$, a third sample of size n_3 is taken at inspection level 3. Then compute the mean of the third sample as $\bar{X}_{3i} = \frac{1}{n_3}\sum_{j=1}^{n_3} X_{3ij}$. Subsequently, compute the combined sample means of the first, second and third samples as $\bar{Y}_{3i} = \left(n_1\bar{X}_{1i} + n_2\bar{X}_{2i} + n_3\bar{X}_{3i}\right) / \left(n_1 + n_2 + n_3\right)$, followed by $V_{3i} = \sqrt{n_1 + n_2 + n_3}\left(\bar{Y}_{3i} - \mu_0\right) / \sigma_0$.

Step 3: If $V_{3i} \in I_7 = \left[-L_{4(\mathrm{TS})}, L_{4(\mathrm{TS})}\right]$, the process is in-control. Otherwise, the process is out-of-control.

Let P_a denote the probability of declaring the process as in-control at sampling stage *i*, while P_{as} denote the probability of declaring the process as in-control at inspection level *s* (= 1, 2 or 3) of sampling stage *i*. Then

$$P_a = P_{a1} + P_{a2} + P_{a3}, \tag{2.14}$$

where P_{a1}, P_{a2} and P_{a3} are the probabilities of declaring the process as in-control at inspection levels 1, 2 and 3, respectively (He et al., 2002; Mim et al., 2022). Note that (Daudin, 1992; He et al., 2002; Mim et al., 2022)

$$P_a = \Pr\left(V_{1i} \in I_1\right) + \Pr\left(V_{2i} \in I_4 \text{ and } V_{1i} \in I_2\right) + \Pr\left[V_{3i} \in I_7 \text{ and } \left(V_{2i} \in I_5, V_{1i} \in I_2\right)\right]$$

$$= \Phi\left(L_{1(\mathrm{TS})} - \delta\sqrt{n_1}\right) - \Phi\left(-L_{1(\mathrm{TS})} - \delta\sqrt{n_1}\right) +$$

$$\int_{z_1 \in I_2^*} \left[\Phi\left(e_2 L_{2(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1 \right) - \Phi\left(-e_2 L_{2(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1 \right) \right] \phi(z_1)\,\mathrm{d}z_1 +$$

$$\int_{z_1 \in I_2^*} \int_{z_2 \in I_5^*} \left[\Phi\left(e_3 L_{4(\mathrm{TS})} - d_3 e_3 \delta - \sqrt{\frac{n_1}{n_3}} z_1 - \sqrt{\frac{n_2}{n_3}} z_2 \right) - \right.$$

$$\left. \Phi\left(-e_3 L_{4(\mathrm{TS})} - d_3 e_3 \delta - \sqrt{\frac{n_1}{n_3}} z_1 - \sqrt{\frac{n_2}{n_3}} z_2 \right) \right] \phi(z_1)\phi(z_2)\,\mathrm{d}z_2\mathrm{d}z_1, \quad (2.15)$$

where $z_1 = \sqrt{n_1}\left(\bar{x}_{1i} - (\mu_0 + \delta\sigma_0)\right)/\sigma_0$, $z_2 = \sqrt{n_2}\left(\bar{x}_{2i} - (\mu_0 + \delta\sigma_0)\right)/\sigma_0$, $d_2 = \sqrt{n_1 + n_2}$, $d_3 = \sqrt{n_1 + n_2 + n_3}$, $e_2 = \sqrt{\frac{n_1 + n_2}{n_2}}$, $e_3 = \sqrt{\frac{n_1 + n_2 + n_3}{n_3}}$, $I_2^* = \left[-L_{(\mathrm{TS})} - \delta\sqrt{n_1}, -L_{1(\mathrm{TS})} - \delta\sqrt{n_1}\right] \cup \left[L_{1(\mathrm{TS})} - \delta\sqrt{n_1}, L_{(\mathrm{TS})} - \delta\sqrt{n_1}\right]$ and $I_5^* = \left[-e_2 L_{3(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1, -e_2 L_{2(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1\right] \cup \left[e_2 L_{2(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1, e_2 L_{3(\mathrm{TS})} - d_2 e_2 \delta - \sqrt{\frac{n_1}{n_2}} z_1\right]$, while $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal pdf and cdf, respectively. The proofs that $V_{1i} \in I_2$ is equivalent to $Z_{1i} = \sqrt{n_1}\left(\bar{X}_{1i} - (\mu_0 + \delta\sigma_0)\right)/\sigma_0 \in I_2^*$ and $V_{2i} \in I_5$ is equivalent to $Z_{2i} = \sqrt{n_2}\left(\bar{X}_{2i} - (\mu_0 + \delta\sigma_0)\right)/\sigma_0 \in I_5^*$ are shown in Mim et al. (2022). Furthermore, since Z_{1i} and Z_{2i} are standard normal random variables, their pdfs can be expressed as $f_{Z_{1i}}(z_1) = \phi(z_1)$ and $f_{Z_{2i}}(z_2) = \phi(z_2)$.

The ASS of the TS $\bar{X}$ chart at sampling stage i is

$$\text{ASS} = n_1 + n_2 \Pr(V_{1i} \in I_2) + n_3 \Pr\left(V_{2i} \in I_5 \middle| V_{1i} \in I_2\right). \quad (2.16)$$

Note that

$$\Pr(V_{1i} \in I_2) = \Pr\left(Z_{1i} \in I_2^*\right)$$