UNIVERSITI SAINS MALAYSIA

Final Examination
Academic Session 2007/2008

April 2008

JIM 213 – Differential Equations I
[Persamaan Pembezaan I]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains NINE printed pages before you begin the examination.

Answer ALL questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEMBILAN muka surat yang bercetak sebelum anda memulakan peperiksaan ini.

Jawab SEMUA soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggeris.

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.]

1. (a) Find the general solutions of the following differential equations:

(i) $x\frac{dy}{dx} - y = 0$

(ii) $x\frac{dy}{dx} - y = 1$.

(60 marks)

(b) Show that the differential equation

$$\left(x^2 + xy\right)\frac{dy}{dx} = xy - y^2$$

is homogeneous. Hence, find its solution subject to the boundary condition $y(1) = 1$.

(40 marks)

2. Given the differential equation

$$yy'' + \left(y'\right)^2 = 0 \text{ for } t > 0.$$

(a) Determine the order of the equation and state whether the equation is linear or non linear.

(20 marks)

(b) Verify that $y_1(t) = 1$ and $y_2(t) = t^{\frac{1}{2}}$ are solutions of the differential equation.

(35 marks)

(c) Show that

$$C_1 + C_2 t^{\frac{1}{2}}$$

is not, in general a solution of the equation. Explain why this result does not contradict the theorem of Principle of Superposition.

(45 marks)

3. (a) Based on the method of undetermined coefficients, suggest the correct form for the particular solution of the differential equation,

$$y'' + 4y = 3\sin 2t.$$

(40 marks)

(b) Verify that $y_1 = x$ and $y_2 = x\,\ell nx$ form a fundamental set of solutions of the differential equation

$$x^2y'' - xy' + y = 0.$$

Hence, using the method of variation parameters, find the general solution of the differential equation

$$x^2y'' - xy' + y = \ell nx.$$

(60 marks)

4. Consider the system of homogenous linear differential equations

$$\frac{dx}{dt} = y + z$$

$$\frac{dy}{dt} = x + z$$

$$\frac{dz}{dt} = x + y.$$

(a) Write the system of equations in the form

$$\frac{dX}{dt} = AX,$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and identify the matrix A.

(25 marks)

(b) Determine the eigenvalues of A and their corresponding eigenvectors.

(45 marks)

(c) Find the general solution of the system.

(30 marks)

5. A function $f(t)$ is defined by

$$f(t)=\begin{cases} 3, & 0 \le t < 4 \\ 2t-3, & t \ge 4. \end{cases}$$

(a) Graph the function f(t) from t = 0 to t = 7.

(10 marks)

(b) Write the function $f(t)$ in term of Heaviside functions $U_a(t)$ where $U_a(t)$ is defined to be

$$U_a(t)=\begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

(10 marks)

(c) Find the Laplace transform of f(t).

(30 marks)

(d) Using the method of Laplace Transform, find the solution of the initial value problem

$$y'' + y' = f(t)$$

where f(t) is given as above.

[You may use the following partial fractions

$$\frac{1}{s(s^2+s)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1},$$

$$\frac{1}{s\ (s^2+s)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s+1}$$

and the results in Table 1].

(50 marks)

f(t)	F(s)
1	$\frac{1}{s}$
$t^n,\ n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos wt$	$\frac{s}{s^2+w^2}$
$\sin\ wt$	$\frac{w}{s^2+w^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$

Table 1: Laplace Transform

Important properties of Laplace Transform

1. $\mathscr{L}\{c_1 f(t)+c_2 g(t)\} = c_1\ \mathscr{L}\{f(t)\} + c_2\mathscr{L}\{g(t)\}$
2. $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$
3. $\mathscr{L}\{f(t-a).H(t-a)\} = e^{-as}F(s)$
4. $\mathscr{L}\{f^n(t)\} = s^n F(s) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$
5. $\mathscr{L}\{f^n(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

1. (a) Cari penyelesaian am bagi persamaan pembezaan berikut:

(i) $x\dfrac{dy}{dx} - y = 0$

(ii) $x\dfrac{dy}{dx} - y = 1$.

(60 markah)

(b) Tunjukkan persamaan pembezaan

$$\left(x^2 + xy\right)\frac{dy}{dx} = xy - y^2$$

adalah homogen. Dengan itu, cari penyelesaiannya tertakluk kepada syarat sempadan $y(1) = 1$.

(40 markah)

2. Diberi persamaan pembezaan

$$yy'' + \left(y'\right)^2 = 0 \text{ for } t > 0.$$

(a) Tentukan peringkat persamaan dan nyatakan sama ada persamaan tersebut linear atau bukan linear.

(20 markah)

(b) Tentusahkan bahawa $y_1(t) = 1$ and $y_2(t) = t^{\frac{1}{2}}$ adalah penyelesaian bagi persamaan pembezaan.

(35 markah)

(c) Tunjukkan bahawa

$$C_1 + C_2 t^{\frac{1}{2}}$$

<u>bukan</u> penyelesaian am bagi persamaan. Terangkan kenapa keputusan tersebut <u>tidak</u> bercanggah dengan teorem Prinsip Superposisi.

(45 markah)

3. (a) Berdasarkan kaedah koefisien belum tentu, cadangkan suatu bentuk sesuai bagi penyelesaian pelengkap untuk persamaan pembezaan,

$$y'' + 4y = 3\sin 2t\,.$$

(40 markah)

(b) Tentusahkan bahawa $y_1 = x$ and $y_2 = x\,\ell nx$ membentuk set penyelesaian asasi bagi persamaan pembezaan

$$x^2 y'' - xy' + y = 0.$$

Dengan yang demikian, menggunakan kaedah variasi parameter, cari penyelesaian am bagi persamaan pembezaan

$$x^2 y'' - xy' + y = \ell nx\,.$$

(60 markah)

4. Pertimbangkan sistem persamaan pembezaan linear homogen

$$\frac{dx}{dt} = y + z$$

$$\frac{dy}{dt} = x + z$$

$$\frac{dz}{dt} = x + y\,.$$

(a) Tuliskan sistem persamaan tersebut dalam bentuk

$$\frac{dX}{dt} = AX,$$

dimana $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ dan kenalpasti matriks A.

(25 markah)

(b) Tentukan nilaieigen A dan cari vektoreigen yang bersepadan.

(45 markah)

(c) Cari penyelesaian am bagi sistem berkenaan.

(30 markah)

5. Fungsi $f(t)$ ditakrifkan oleh

$$f(t)=\begin{cases} 3, & 0\le t<4 \\ 2t-3, & t\ge 4. \end{cases}$$

(a) Grafkan fungsi f(t) dari t = 0 ke t = 7.

(10 markah)

(b) Tuliskan fungsi $f(t)$ dalam sebutan fungsi Heaviside $U_a(t)$ dimana $U_a(t)$ adalah ditakrifkan oleh

$$U_a(t)=\begin{cases} 0, & 0\le t<a \\ 1, & t\ge a. \end{cases}$$

(10 markah)

(c) Cari jelmaan Laplace bagi f(t).

(30 markah)

(d) Dengan menggunakan kaedah jelmaan Laplace, cari penyelesaian bagi masalah nilai awal

$$y''+y'=f(t)$$

dimana f(t) seperti diberi di atas.

[Anda boleh menggunakan pecahan separa berikut

$$\frac{1}{s\left(s^2+s\right)}=-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1},$$

$$\frac{1}{s\,\left(s^2+s\right)}=-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s+1}$$

dan keputusan dalam Jadual 1].

(50 markah)

f(t)	F(s)
1	$\frac{1}{s}$
t^n, $n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos wt$	$\frac{s}{s^2+w^2}$
$\sin wt$	$\frac{w}{s^2+w^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$

Jadual 1: Jelmaan Laplace

Sifat penting Jelmaan Laplace.

1. $\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = c_1\, \mathscr{L}\{f(t)\} + c_2\, \mathscr{L}\{g(t)\}$
2. $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$
3. $\mathscr{L}\{f(t-a).H(t-a)\} = e^{-as} F(s)$
4. $\mathscr{L}\{f^n(t)\} = s^n F(s) - s^{n-2} f'(0) - \ldots - f^{n-1}(0)$
5. $\mathscr{L}\{f^n(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

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