NEW TRAVELING WAVE SOLUTIONS FOR SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS BY EXTENSIONS OF BASIC (G'/G)-EXPANSION METHOD

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by

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LIST OF ABBREVIATIONS

BP	Biological population
CB	Coupled Burgers
CPDEs	Classical partial differential equations
CWBK	Coupled Whitham-Broer-Kaup
DR	Diffusion reaction
EEs	Evolution equations
FPDEs	Fractional partial differential equations
KdV-ZK	KdV-Zakharov-Kuznetsov
KG	Klein-Gordon
MKdV	Modified Korteweg-de-Vries
ODE	Ordinary differential equation
PDEs	Partial differential equations
STO	Sharma-Tasso-Olver
ZK	Zakharov-Kuznetsov
ZKBBM	Zakharov-Kuznetsov-Benjamin-Bona-Mahony

LIST OF SYMBOLS

$a_0, a_1, a_2, a_{-1}, a_{-2}, b_1, b_2$	Constants
A_1, A_2	Arbitrary constants
A, B, E, D	Real parameters
C	Speed of the wave propagation
C_1, C_2, d	Arbitrary constants
K, N, M	Nonzero constants
n	Positive integer
ξ_0, ξ_1	Integral constants
λ, μ	Real parameters

PENYELESAIAN GELOMBANG MENJALAR BAHARU UNTUK BEBERAPA PERSAMAAN PECAHAN TAK LINEAR MENGIKUT PELUASAN KAEDAH PENGEMBANGAN-(G'/G) ASAS

ABSTRAK

Disebabkan oleh kepelbagaian dan kepentingan aplikasi persamaan pembezaan pecahan tak linear dalam masalah dunia sebenar, wujud keperluan untuk membina penyelesaian analitikal yang tepat. Dengan bantuan penyelesaian analitik yang tepat, jika wujud, fenomena yang dimodelkan boleh difahami dengan lebih baik. Secara umumnya, kelas penyelesaian utama bagi persamaan evolusi (PE) tak linear adalah penyelesaian gelombang menjalar. Kaedah pengembangan (G'/G) ialah salah satu kaedah yang kerap digunakan untuk membina penyelesaian gelombang menjalar kepada beberapa PE tak linear. Objektif tesis ini adalah untuk membina dan menggunakan beberapa peluasan kaedah pengembangan (G'/G) yang sediaada untuk mendapatkan penyelesaian gelombang bentuk tertutup kepada beberapa persamaan pembezaan pecahan tak linear berdasarkan teori gelombang bersendirian dan pemetaan ansatz. Peluasan ini merangkumi: kaedah pengembangan-$(G'/G\,,1/G)$, kaedah pengembangan-(G'/G) umum dan ditambahbaik, dan kaedah pengembangan umum-(G'/G) baru. Melalui peluasan ini, telah ditemui beberapa penyelesaian dan ilustrasi penting untuk persamaan pembezaan pecahan yang dipertimbangkan. Penyelesaian yang diperoleh terdiri dari parameter tertentu, dan jika parameter memberikan nilai tertentu, beberapa penyelesaian yang diperoleh bersamaan dengan hasil yang diterbitkan dalam kesusateraan terbuka sebagai kes khas. Sebilangan penyelesaian yang diperoleh dapat dilihat dalam bentuk 3D dan 2D dengan bantuan perisian algebra komputer Maple 17. Beberapa contoh ujian diberikan untuk

menggambarkan kesahihan cadangan peluasan kaedah pengembangan-(G'/G) dan perkembangannya. Pecahan tak linear persamaan pembezaan yang dipertimbangkan adalah model populasi biologi pecahan masa, persamaan KdV-Zakharov-Kuznetsove pecahan masa, persamaan Burgers gandingan pecahan ruang-waktu, persamaan KdV terubahsuai pecahan masa, persamaan Whitham-Broer-Kaup gandingan pecahan ruang-waktu, persamaan Sharma-Tasso-Olever pecahan masa am dan persamaan Klein-Gordon pecahan ruang-waktu. Cadangan pengembangan ini telah berjaya digunakan untuk menguraikan persamaan pembezaan separa pecahan tak linear yang dinyatakan di atas. sebilangan besar corak gelombang perjalanan serta digunakan untuk mengkaji beberapa jenis persamaan evolusi tak linear lain yang wujud dalam sains dan kejuruteraan tak linear yang diminati semasa. Hasil yang diperoleh menggunakan tiga peluasan kaedah pengembangan (G'/G) yang ada dan perkembangannya dibandingkan dengan beberapa hasil yang diperoleh dalam literatur terbuka. Hasil kajian menunjukkan bahawa terdapat penyelesaian gelombang menjalar baharu yang dihasilkan termasuk soliton berkala, berpintal, memadat, memuncak, bentuk lonceng dan lain-lain.

NEW TRAVELING WAVE SOLUTIONS FOR SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS BY EXTENSIONS OF BASIC (G'/G)-EXPANSION METHOD

ABSTRACT

Due to varied and important applications of nonlinear fractional differential equations in real world problems, it is often required to construct their exact analytical solutions. With the help of exact analytical solutions, if they exist, the modelled phenomena can be better understood. Generally, an important class of solutions of nonlinear evolution equations (EEs) is their travelling wave solutions. The (G'/G)-expansion method is one of the more frequently used to construct travelling wave solutions to some nonlinear EEs. The objective of this thesis is to establish and put in use some extensions of the existing (G'/G)-expansion method to obtain closed-form wave solutions to some nonlinear fractional differential equations based on solitary wave theory and mapping ansatz. The extensions include: $(G'/G\,,1/G)$-expansion method, generalized and improved (G'/G)-expansion method and new generalized (G'/G)-expansion method. Through these extensions, it has found some significant and illustrative solutions to the considered fractional differential equations. The solutions obtained are comprised of certain parameters, and if the parameters assign particular values, some of the obtained solutions become identical to published results in the open literature as a special case. Some of the solutions obtained are visualized in 3D and 2D figures with the aid of computer algebra software Maple 17. Several test examples are given for illustrating the validity of these extensions of the (G'/G)-expansion method and its developments. The nonlinear fractional differential equations considered are: the time fractional biological population model, the time fractional KdV-Zakharov-Kuznetsov equation, the space-time fractional coupled Burgers equations, the time fractional modified KdV equation, the space-time fractional coupled Whitham-Broer-Kaup equations, the general time fractional Sharma-Tasso-Olever equation and the space-time fractional Klein-Gordon equation. The suggested extensions have successfully been applied to unravel the above stated nonlinear fractional partial differential equations. This study also reveals that the results obtained can illustrate a large number of traveling wave patterns and can be used to study other types of nonlinear evolution equations arise in nonlinear science and engineering of current interest. The obtained results, using three existing extensions of the (G'/G)-expansion method and its developments are compared with some obtained results in the open literature. The findings show that new travelling wave solutions were generated including, periodic, kink, compacton, cuspon, bell shape soliton etc.

CHAPTER 1

INTRODUCTION

1.1 Background

Fractional calculus is the generalization in the theory of integral and derivative of arbitrary order, dated back to 1695 in a discussion between L'Hôpital and Leibnitz. Fractional calculus has elicited much interest over the past few decades, and its history and development were explored in detail by Miller and Ross (1993), Samko et al. (1993) and Podlubny (1998). Nonlinear fractional partial differential equations (FPDEs) are defined as a type of equations that utilize fractional derivatives, have gained the interest to many scientists due to their applications in various fields of science and engineering. Apart from the theoretical aspects, the modelling of nonlinear FPDEs has numerous applications and has become a major topic of interest. It is worth mentioning that the first application of fractional derivative was presented in 1823 by Abel (Oldham and Spanier, 1974; Miller and Ross, 1993), who applied fractional derivative to the solution of an integral equation that arises in the formulation of the tautochrone problem. This problem deals with the determination of the shape of the curve such that the required time of descent of a mass sliding down along the curve under the action of gravity with ignoring of the friction is independent of the starting position.

The importance of nonlinear FPDEs arises in wide variety of physical problems, such as, fluid dynamics, plasma physics, solid mechanics, optical fiber and quantum field theory (Rudolf, 2000; Meerschaert and Tadjeran, 2004; Sabatier et al., 2007; Gupta, 2016; Bibi et al., 2017; Khater and Kumar, 2017; Khater et al., 2017; Mohyud-Din et al., 2017; Akbar et al., 2018; Ellahi et al., 2018; Ferdous and Hafez, 2018; Uddin et al., 2019). The analytical solutions of nonlinear FPDEs play an important role in understanding and analysing the internal mechanisms of natural phenomena (Tang et al., 2012; Zhang et al., 2012; Ahmad and Mohyud-Din, 2014; Zhang, 2015; Yaşar and Giresunlu, 2016; Yaşar et al., 2016; Gupta and Ray, 2017; Islam et al., 2019).

1.2 Motivation

Nonlinear FPDEs are assumed to be the generalized form of classical partial differential equations (CPDEs). Many experimental data highlighted that the state of a physical phenomenon does not depend only on its current state but also depends on its historical states, which can be successfully modelled by using the theory of derivatives of fractional order (Guo et al., 2012a; Hesameddini et al., 2016; Mohyud-Din et al., 2017).

Wang et al. (2008a) established a method called the (G'/G)-expansion method for obtaining travelling wave solutions to nonlinear evolution equations (EEs), which claims it can be used to solve a wide variety of nonlinear EEs. This thesis is motivated by the Wang et al.'s (2008a) leading work, named the basic (G'/G)-expansion method. In this thesis, we extend the method to examine some nonlinear FPDEs in the sense of conformable fractional derivative. The main idea of this method with conformable fractional derivative is to express the solution of nonlinear FPDEs by a polynomial in (G'/G) where $G = G(\xi)$ satisfies a second order linear ordinary differential equation (ODE) and $\xi = K\frac{x^\beta}{\beta} + N\frac{y^\gamma}{\gamma} + M\frac{z^\delta}{\delta} \pm C\frac{t^\alpha}{\alpha}$, where $0 < \alpha, \beta, \gamma, \delta \leq 1$; K, N, M are nonzero constants and C is the wave speed. The degree of the polynomial can be determined by considering the homogeneous balance between highest order derivatives and nonlinear terms that appear in nonlinear ODE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations that result from the process of using the method (Eslami, 2017; Yaslan and Girgin, 2019). Thus, it has been claimed that the (G'/G)-expansion method is direct, concise and effective (Bekir and Güner, 2013; Bekir and Güner, 2014). The second order linear ODE is known as an auxiliary equation.

Obtaining more travelling wave solutions may assist in providing more information for understanding certain complex physical phenomena. To generate more travelling wave solutions the basic (G'/G)-expansion method can be extended and this is the primary focus of this thesis.

1.3 Problem Statement

Most of the analytical methods are rather cumbersome and the exact solutions procedures become very complex as the degree of nonlinearity increases (Aslan, 2009). Furthermore, they are applicable only to certain classes of nonlinear FPDEs. Therefore, Wang et al. (2008a) established the basic (G'/G)-expansion method for obtaining travelling wave solutions to nonlinear EEs which can be used to solve wide variety of nonlinear FPDEs. Thus, in order to demonstrate the substantiality of the basic (G'/G)-expansion method, to expand its applicability, and to extend the method, this study will introduce further new extensions of the basic (G'/G)-expansion method.

1.4 Research Gap

If the order of the reduced ODE (the ODE found from the PDE by using traveling wave variable) is equal to or less than three, it may usually possible to establish usable solutions of the resulting algebraic equations using the basic (G'/G)-

expansion approach with the use of symbolic computation software, such as Maple. Otherwise, the existence of the solution to the resulting algebraic equations cannot be guaranteed. This is due to the fact that the number of equations in the set of algebraic equations is usually greater than the number of unknowns. However, there are many problems in the real world where the order of modified ODEs is four or greater. In this case, sometimes no useful solutions of the algebraic equations can be found to deal with such problems. Thus, modification of the existing methods is required. On the other hand, each nonlinear equation has its own characteristics physically substantial rich structure. Therefore, in order to demonstrate the suitability of a method and to expand the range of applicability, further studies should be conducted.

1.5 Objectives

The main objective of this thesis is to extend some existing modifications, as for instance, the $(G'/G\,,1/G)$-expansion method, generalized and improved (G'/G)-expansion method and new generalized (G'/G)-expansion method of the basic (G'/G)-expansion approach to establish standard and inclusive travelling wave solutions to some nonlinear FPDEs in the sense of conformable fractional derivative. The secondary aim is, we will further validate three existing extensions of the basic (G'/G)-expansion method by applying these methods to a number of important nonlinear FPDEs for which they have yet to be applied. Therefore, the objectives pursued in this study are:

- To develop and apply new extensions of the basic (G'/G)-expansion method which can yield further travelling wave solutions for certain nonlinear FPDEs.

- To apply certain existing extensions of the basic (G'/G)-expansion method to nonlinear FPDEs for which they have not been applied and thus validating the methods.
- To perform a comparative study between the travelling wave solutions obtained and previous results solved by using other similar approaches.
- To classify some of the travelling wave solutions obtained into different patterns of soliton.

1.6 Methodology

The methodology of this study is as follows. Three existing extensions of the basic (G'/G)-expansion method which are, the $(G'/G\,,1/G)$-expansion method, the generalized and improved (G'/G)-expansion method and the new generalized (G'/G)-expansion method are applied to solve nonlinear FPDEs for which they have not been applied and thus validating the methods. New modifications of these existing extensions will be proposed and applied. The extensions, we have established consist of the use of different skilled assumption of the solution and compute the travelling wave solutions for diverse nonlinear FPDEs. To check that the extensions yield realistic results, the travelling wave solutions are compared with those solutions found in the literature solved by using other similar approaches. Some of the travelling wave solutions will also be classified using techniques that are well-established.

1.7 Scope of Study

This study focuses on some existing extensions of the basic (G'/G)-expansion method. In particular, $(G'/G\,,1/G)$-expansion method, generalized and improved (G'/G)-expansion method and new generalized (G'/G)-expansion method for finding travelling wave solutions of some nonlinear FPDEs which they have not been applied. Besides, new extensions of these existing extensions of the basic (G'/G)-expansion method were proposed and applied to solve certain nonlinear FPDEs. The nonlinear FPDEs include:

- The (2+1)-dimensional time-fractional biological population model, the (3+1)-dimensional time fractional KdV-Zakharov-Kuznetsove equation and the space-time fractional coupled Burgers equations (Chapter 3).
- The time-fractional modified KdV equation and the space-time fractional coupled Whitham-Broer-Kaup equations (Chapter 4).
- The (2+1)-dimensional time fractional biological population model and the (3+1)-dimensional time-fractional KdV-Zakharov-Kuznetsove equation (Chapter 5).
- The general time-fractional Sharma-Tasso-Olever equation and the space-time fractional Klein-Gordon equation (Chapter 6).
- The time fractional modified KdV equation and the space-time fractional coupled Whitham-Broer-Kaup equations (Chapter 7).
- The general time fractional Sharma-Tasso-Olever equation and the space-time fractional coupled Burgers equations (Chapter 8).

1.8 Limitations

In reality, there is no unique method that can be utilized for investigating all types of space-time fractional conformable nonlinear FPDEs. As a result, any change made to a certain method at any time allows it to develop some new solutions which are always beneficial. There are some limitations depending on the problems of all the existing implemented methods and the proposed approaches. When the order of fractional differential equations is higher (greater than four), it is difficult to find the solutions and even in some cases, solution is not found. As a result, mathematicians are frustrated with their efforts to find solutions to space-time FPDEs. Therefore, research on new methods is crucial to recovering exact solutions to the space-time conformable FPDEs.

1.9 Outline of the Thesis

A description of the chapters contained in this thesis is as follows. In Chapter 1, the background of this study, the motivation of study, the scope of study, the research objectives and the research methodology are described. In Chapter 2, some basic concepts which are required in this study will be reviewed. From Chapter 3 to Chapter 8, some important nonlinear FPDEs have been investigated by applying existing and new extensions of the basic (G'/G)-expansion method and the subsequent solutions obtained are investigated.

Chapter 9 consists of the conclusions of the thesis and a discussion of possible further works in this field.

CHAPTER 2

BASIC CONCEPTS AND LITERATURE REVIEW

2.1 Introduction

In recent times, nonlinear FPDEs have availed a lot of attention. This interest can be attributed to the advancement in the theory of fractional calculus in addition to the applications of such constructs in different disciplines such as engineering, physics, biology, etc. To improve the understanding of the mechanisms of complex nonlinear physical phenomena, and ensure their practical applications, there is a need to develop solutions to these equations. Research on travelling wave solutions of nonlinear FPDEs plays a significant role in understanding the qualitative and quantitative aspects of several phenomena and processes in mathematical physics. In this chapter, we introduce the basic concepts of fractional calculus, nonlinear FPDEs and travelling wave solutions so as to provide the necessary framework for this thesis.

2.2 Fractional Calculus

The fractional calculus involves the integration and differentiation to arbitrary (non-integer) order. The original ideas of fractional calculus can be traced back to the end of 17th century, the time when the classical differential and integral calculus theories were developed by Newton and Leibniz (Diethelm, 2010). Exactly, it was introduced in the year of 1695, when L'Hôpital wrote a letter to Leibniz raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order (Kilbas et al., 2006). Since then, many famous mathematicians have studied this area further, creating the field which is known today as fractional calculus. Some definitions of fractional integrals and derivatives have been introduced in the next sections.

2.2.1 Fractional integrals

The fractional integrals refer to the integrals of arbitrary order (Podlubny, 1998). For a function $f(x)$, the fractional integral for the order, $\alpha > 0$, can be denoted as:

$cD_x^{-\alpha} f(x)$ or $cI_x^{\alpha} f(x)$,

wherein, c and x represent the two limits of a fractional integral operator and these are generally called as the terminals of the fractional integral (Podlubny, 1998). In 1847, Riemann obtained a formula for the fractional integration by applying the Taylor series generalization in the following manner:

$$cD_x^{-\alpha} f(x) = cI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt + \psi(x),\ \alpha \in \mathbb{R}^+, \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers, $\Gamma(\alpha)$ is a gamma function, which is one of the fundamental functions of the fractional calculus and $\psi(x)$ is a complementary function was introduced by Riemann since he did not fix the lower integration limit c (Miller and Ross, 1993). Eq. (2.1) with lower limit $c = 0$ and without a complementary function $\psi(x)$ is the most common definition of fractional integration today, called the Riemann-Liouville fractional integral. Sonin (1869) presented the Riemann-Liouville definition in his paper. Furthermore, he used the Cauchy integral formula for the integral order derivatives of the complex domain, given by (Weilbeer, 2005).

$$D^n f(z) = \frac{n!}{2\,\pi\, i} \int_C \frac{f(t)}{(t-z)^{n+1}}\, dt. \tag{2.2}$$

The Riemann-Liouville integral is defined as follows:

$$cI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt,\ \alpha \in \mathbb{R}^+. \tag{2.3}$$

Also, the Riemann-Liouville integral can be derived in another way by considering the n-fold for any function $f(x)$ as (Dold and Eckmann, 1975):

$$cI_x^n f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \ldots \int_c^{x_{n-1}} f(x_n)\, dx_n. \tag{2.4}$$

From Dirichlet's approach, the n-fold integral can be considered as a single integral

$$cI_x^n f(x) = \frac{1}{(\mathrm{n}-1)!}\int_c^x \frac{f(x_n)}{(x-x_n)^{1-n}}\, dx_n. \tag{2.5}$$

Eq. (2.5) can be thought as the general formula of Eq. (2.3) by replacing n by α and assuming $x_n = t$.

2.2.2 Fractional derivatives

The fractional derivatives can be described as the derivatives of arbitrary order. On the other hand, the integer order derivatives refer to the order of derivatives that are restricted to the positive integers. Therefore, the fractional derivatives are known as the generalized form of the integer order derivatives. There are several definitions to the fractional derivative of order $\alpha > 0$. The common definition for the fractional derivatives of a function $f(t)$ with lower limit $c = 0$ is the Riemann-Liouville definition (Klages et al., 2008).

$$cD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\xi)}{(t-\xi)^\alpha}\, d\xi,\ 0 < \alpha < 1. \tag{2.6}$$

The general form of Eq. (2.6) is written in the following manner (Diethelm, 2010)

$$cD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_c^t \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}}\, d\xi, & (n-1) < \alpha < n,\ \ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n,\ \ n \in \mathbb{N}. \end{cases} \tag{2.7}$$

Another form of fractional derivative called Jumarie modified Riemann Liouville derivative has been proposed, which is given by (Jumarie, 2006):

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\xi)-f(0)}{(t-\xi)^{\alpha}} \, d\xi, & 0 < \alpha < 1\,, \\ \left(f^n(t)\right)^{(\alpha-n)}, & n \le \alpha < n+1, \; n \ge 1. \end{cases} \tag{2.8}$$

Recently, Khalil et al. (2014) introduced a new definition of fractional derivative called conformable fractional derivative (will be explained at the end of this section), which is defined as follows:

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon\, t^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{2.9}$$

where $f: [0, \infty) \to \mathbb{R}, t > 0$ and $\alpha \in (0, 1]$.

Some important properties of the conformable fractional derivative are given in the following theorems as (Khalil et al., 2014; Abdeljawad, 2015):

Theorem 2.1: (Khalil et al., 2014; Abdeljawad, 2015) Suppose $\alpha \in (0, 1]$ and if the conformable fractional derivative of functions g, f of order α exists at $t > 0$. Then

i) $T_{\alpha}(ag + bf) = aT_{\alpha}(g) + bT_{\alpha}(f),\ \forall a, b \in \mathbb{R}$.

ii) $T_{\alpha}(t^{\varphi}) = \varphi t^{\varphi-\alpha},\ \forall \varphi \in \mathbb{R}$.

iii) $T_{\alpha}(\Upsilon) = 0,\ \forall$ constant functions $g(t) = \Upsilon$.

iv) $T_{\alpha}(gf) = gT_{\alpha}(f) + fT_{\alpha}(g)$.

v) $T_{\alpha}\left(\frac{g}{f}\right) = \frac{f\, T_{\alpha}(g) - g\, T_{\alpha}(f)}{f^2}$.

vi) If, furthermore, g is differentiable; then $T_{\alpha}(g)(t) = t^{1-\alpha} \frac{d}{dt} g(t)$.

Theorem 2.2: (Khalil et al., 2014; Abdeljawad, 2015) Suppose $g: [0, \infty) \to \mathbb{R}$ be a function such that g is differentiable and also the conformable fractional derivative of g of order α exists. Let f be a function defined in the range of g and also differentiable. Then

$$T_{\alpha}(gof)(t) = t^{1-\alpha} f'(t) g'\left(f(t)\right).$$

The conformable fractional derivative is a good tool to model the processes of real life problems governed by fractional differential equation. This definition is simpler, efficient and also obeys some conventional properties that cannot be satisfied by the existing definitions of fractional derivative, for instance the product rule and the chain rule (Khalil et al., 2014; Abdeljawad, 2015; Liu, 2015; Eslami and Rezazadeh, 2016; Eslami, 2017).

2.3 Nonlinear Fractional Partial Differential Equations

Nonlinear FPDEs have become a useful tool for depicting the inner mechanisms of the incidents of the real world. In the last two decades, much attention to nonlinear FPDEs has been paid due to their recurrent applications in the area of physics and nonlinear science (Miller and Ross, 1993; Podlubny, 1998; Kilbas et al., 2006). Therefore, seeking analytical solutions to nonlinear FPDEs play an important role in understanding the qualitative as well as quantitative features of many nonlinear phenomena and processes in various areas, such as fluid dynamics, nonlinear optics, solid mechanics, plasma physics, and quantum field theory.

Selected nonlinear FPDEs which have been considered in this study are:

- The time fractional modified Korteweg-de-Vries (MKdV) equation.
- The general time fractional Sharma-Tasso-Olver (STO) equation.
- The space-time fractional Klein-Gordon (KG) equation.
- The systems of nonlinear FPDEs:
    - (i) The space-time fractional coupled Burgers (CB) equations.
    - (ii) The space-time fractional coupled Whitham-Broer-Kaup (CWBK) equations.
- The higher dimensional nonlinear time FPDEs:

(i) The (2+1)-dimensional time fractional biological population (BP) model.

(ii) The (3+1)-dimensional time fractional KdV-Zakharov-Kuznetsov (KdV-ZK) equation.

The nonlinear EEs are very important in several scientific and engineering fields, such as solid state physics (Eilenberger, 1981), fluid mechanics (Whitham, 1974), plasma physics (Hasegawa, 2012) etc. The upcoming sections will discuss the specific nonlinear fractional equations mentioned above.

2.3.1 The modified Korteweg-de Vries equation with time fractional derivative

In 1895, Diderik Johanes Korteweg and Gustav de Vries derived the KdV equation (Korteweg and De Vries, 1895) to describe shallow water waves of long wave length and small amplitude. The canonical KdV equation is a nonlinear dispersive equation of third-order. However, the KdV equations appear in third, fifth, seventh or higher order forms.

One of the well-known third-order KdV equations is the modified KdV (MKdV) equation. The MKdV equation with time fractional order is (Sahoo and Ray, 2016; Akbulut and Taşcan, 2017):

$$D_t^{\alpha} u + a\, u^2 u_x + b\, u_{xxx} = 0,\ \ 0 < \alpha \leq 1, \tag{2.10}$$

where α describes the order of the fractional time derivative and a, b are arbitrary constants. The MKdV equation is derived by perturbation expansions based on the assumption that the soliton width is small compared with the scale length of the plasma inhomogeneity. In this assumption, soliton maintains all of its identities, such as, amplitude, width and speed. The MKdV equation appears in applications such as electric circuits and multi-component plasmas, electromagnetic waves in size quantized films, elastic media, electrodynamics and traffic flow (Wazwaz, 2010). The MKdV equation is also used to represent physical models in various physical phenomena, such as to describe the dipole blocking, ion acoustic waves in a magnetized plasma, in the issues of atmospheric blocking phenomenon and study of coastal waves in ocean (Watanabe, 1984; Ya-Xuan et al., 2005; Xiao-Yan et al., 2006; Biswas, 2009; Johnpillai et al., 2011; Mousavian et al.,2011; Bulut, 2014). The third-order time-fractional MKdV equation has been studied by many researchers. For example, Abdulaziz et al. (2009) executed the homotopy perturbation method for this equation to establish approximate analytical solutions, whereas Song and Wang (2010) constructed approximate solutions of the same equation by using the enhanced Adomian decomposition method. Bulut et al. (2014) implemented the generalized Kudryashov method to obtain exact solutions of the same equation while Sahoo and Ray (2016) studied this equation and obtained travelling wave solutions via the basic (G'/G) method and improved (G'/G)-expansion method.

2.3.2 The general time fractional Sharma-Tasso-Olver equation

The general Sharma-Tasso-Olver (STO) equation is an important nonlinear EE which plays a crucial role both in physics and applied mathematics.

The general time fractional STO equation is of the form:

$$D_t^{\alpha}u + 3b(u_x)^2 + 3bu^2u_x + 3buu_{xx} + bu_{xxx} = 0,\ t > 0, 0 < \alpha \leq 1, \quad (2.11)$$

where b is arbitrary constant and $u = u(x,t)$ is an unrevealed function. This equation is used to investigate the fission and fusion phenomena for solitons, quantum relativistic atom theory, electromagnetic interactions and the relativistic energy-momentum relation in mathematical physics and engineering (Uddin et al., 2019). Many researchers investigated the general time-fractional STO equation using different techniques. For instance, Bibi et al. (2017) used the Khater technique to establish exact solutions to this equation. Roy et al. (2018) implemented the new generalized (G'/G)-expansion method to obtain travelling wave solutions of the same equation whilst Uddin et al. (2019) studied this equation and obtained travelling wave solutions via double $(G'/G\,,1/G)$-expansion method.

2.3.3 The space-time fractional Klein-Gordon equation

In 1926, the Klein-Gordon (KG) equation was named by the physicists Oskar Klein and Walter Gordon after they proposed that it describes relativistic electrons. The KG equation correctly describes the spinless relativistic composite particles, like the pion and plays a significant role in several real world applications, for instance, the nonlinear optics, solid-state physics and quantum field theory (Wazwaz, 2005; Wazwaz, 2008).

The KG equation with space-time fractional derivative is (Shallal et al., 2018)

$$D_t^{2\alpha}u - D_x^{2\alpha}u - \omega u - eu^3 = 0,\ \ 0 < \alpha \leq 1, \tag{2.12}$$

where ω and e are nonzero constants and $u = u(x,t)$ is an unknown function. Many researchers used a variety of approaches to look into the space-time fractional KG equation. For illustration, Yaşar and Giresunlu (2016) used the $(G'/G\,,1/G)$-expansion method to establish exact solutions of this equation whilst Shallal et al. (2018) functioning the modified extended tanh method for obtaining travelling wave solutions for the same equation.

2.3.4 The systems of nonlinear FPDEs

Many phenomena cannot be easily described by a single nonlinear FPDEs. To describe them accurately, we have to construct and study systems of nonlinear FPDEs. Moreover, coupled systems of fractional order are amongst the strongest tools of modern mathematics as they play a key role in developing differential models for high complication systems. So, the study of coupled systems of fractional order are significant because this kind of systems appears in many scientific applications and physical phenomena (Ahmad and Nieto, 2009; Su, 2009; Wang et al., 2010; Sun et al., 2012). In this thesis, we will put in use the space-time fractional coupled Burgers (CB) equations (Bekir and Güner, 2014; Islam and Akbar, 2018) given by:

$$D_t^{\alpha}u - D_x^{2\alpha}u + 2uD_x^{\alpha}u + pD_x^{\alpha}(uv) = 0, \tag{2.13}$$

$$D_t^{\alpha}v - D_x^{2\alpha}v + 2vD_x^{\alpha}v + qD_x^{\alpha}(uv) = 0, \tag{2.14}$$

where $0 < \alpha \leq 1, u = u(x,t)$ and $v = v(x,t)$.

The study to CB equations is very significant, since the system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity (Nee and Duan, 1998). The constants p and q depend on the system parameters such as the Stokes velocity of particles due to gravity and the Brownian diffusivity. The CB equations with space and time-fractional derivative have been studied by many researchers. As for instance, Zhao et al. (2012) executed the extended fractional sub-equation method for this system of equations to establish analytical solutions. Bekir and Güner (2014) constructed exact solutions of the same system by using the basic (G'/G) method, whereas Islam and Akbar (2018) studied this system of equations and obtained exact wave solutions via the new generalized (G'/G)-expansion method.

We will also investigate the coupled Whitham-Broer-Kaup (CWBK) equations of space-time fractional order (Bekir and Güner, 2013):

$$D_t^{\alpha}u + uD_x^{\alpha}u + D_x^{\alpha}v + pD_x^{2\alpha}u = 0, \tag{2.15}$$

$$D_t^{\alpha}v + D_x^{\alpha}(uv) - pD_x^{2\alpha}v + qD_x^{3\alpha}u = 0, \tag{2.16}$$

where $0 < \alpha \leq 1$ represents the order of the fractional space-time derivative.

In this coupled system, the field of horizontal velocity is represented by $u = u(x,t), v = v(x,t)$, which is the height that deviate from equilibrium position of liquid and the constants p, q are represented in different diffusion power (Kupershmidt, 1985). As it is known, the CWBK equations, originally introduced by Whitham (1967), Broer (1975) and Kaup (1975) describe the propagation of shallow water waves with different dispersion relations. Furthermore, shallow water in porous medium, which is used to absorb wave energy and prevent tsunami is also described by the CWBK equations of fractional order (Wang et al., 2017). Many researchers studied this system by using different methods. For example, Guo et al. (2012a) utilized an improved fractional sub-equation method to solve this system. Lu (2012a) solved the same system by the Bäcklund transformation while Bekir and Güner (2013) investigated this system via the basic (G'/G)-expansion method.

2.3.5 The Higher-dimensional Nonlinear FPDEs

The higher dimensional nonlinear FPDEs are widely used to describe natural complex phenomena in various field of the real world, especially plasma physics, quantum field theory, nonlinear optics and many others (Sahoo and Ray, 2015). Many engineering phenomenon are being modelled by applying high-dimensional FPDEs, see for example (Flik et al., 1992; Wheatcraft and Meerschaert, 2008; Ghazizadeh and

Maerefat, 2010). Due to important applications of higher-dimensional fractional equations in science and engineering, it is essential to obtain their travelling wave solutions.

The time fractional biological population (BP) model is one of the higher dimensional equations (Bekir and Güner, 2013)

$$D_t^{\alpha}u - (u^2)_{xx} - (u^2)_{yy} - h(u^2 - r) = 0, \;\; 0 < \alpha \leq 1, \tag{2.17}$$

where h and r are constants, u represents the population density and $h(u^2 - r)$ represents the population supply due to births and deaths. A BP model is a mathematical model, which helps us to understand the dynamical procedure of population changes and provides valuable predictions. Most of the earth's processes affect human life. Procedures in population modelling have significantly enhanced our understanding of biology and the natural world. Diverse researchers have investigated the (2+1)-dimensional BP model of time-fractional order by different methods. For instance, El-Sayed et al. (2009) implemented the Adomian decomposition method to investigate this model. Zhang and Zhang (2011) assessed the same model via the fractional sub-equation method. Lu (2012a) studied the same model by using the Bäcklund transformation of fractional Riccati equation whilst Bekir and Güner (2013) investigated this model via the basic (G'/G)-expansion method.

In this thesis, we also study, the (3+1)-dimensional time fractional KdV-Zakharov-Kuznetsov (KdV-ZK) equation

$$D_t^{\alpha}u + auu_x + u_{xxx} + b\left(u_{yyx} + u_{zzx}\right) = 0, \tag{2.18}$$

where $0 < \alpha \leq 1$ and a, b are arbitrary constants.

It is well known that the KdV equation arises as a model for one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media as well as the evolution of weakly nonlinear ion acoustic waves in plasmas (El-Tantawy and Moslem, 2014). There are several weakly two-dimensional variations on the KdV equation. The Zakharov-Kuznetsov (ZK) equation is one of two well-studied canonical two-dimensional extensions of the KdV equation (Kadomtsev and Petviashvili, 1970). In the recent past, Guo et al (2012b) derived the (3+1)-dimensional variable coefficient cylindrical KdV equation describing the nonlinear propagation of dust acoustic waves. By considering this, the (3+1)-dimensional KdV-ZK equation is derived for a plasma comprised of cool and hot electrons and a species of fluid ions (Mace and Hellberg, 2001). Diverse researchers studied this model by using different methods. For example, Sahoo and Ray (2015) utilized the improved fractional sub-equation method to examine this equation. Kaplan and Bekir (2016) studied the same equation by the $\exp\left(-\phi(\xi)\right)$ method, whilst Unsal et al. (2017) investigated this equation via the basic (G'/G) method.

2.4 Travelling Wave Solutions

Travelling wave is a wave in which the medium moves in the direction of propagation of the wave. A special type of travelling wave solutions of nonlinear FPDEs is classified as solitary wave solutions. A travelling wave solution is a solution of permanent form moving with a constant velocity. Travelling wave solutions are able to describe various type of phenomena in nature, as for instance, vibrations, solitons and propagation with a finite speed. Thus, they can be utilized to give more insight into the physical aspects of certain problems. To understand the occurrence of travelling wave solutions, mathematical dealings have been used to describe the travelling wave function in the form of $u(x,t) = f(x - Ct)$ where C is the speed of the wave propagation. The travelling wave solutions are usually constructed by reducing the nonlinear EEs to the associated ODEs (Wazwaz, 2010; Gepreel and Omran, 2012; Chen and Jiang, 2018).

For generating travelling wave solutions, we need to convert nonlinear EEs into ODEs with the travelling wave transformation. The travelling wave transformation is represented by the form (Wang et al., 2008a):

$$u(x,t) = u(\xi), \;\; \xi = x \pm Ct,$$

where C describes the speed of the wave propagation and $u(x,t)$ represents the wave disturbance moving in the negative or positive x-direction (*i.e.*, if $\xi = x - Ct$, the wave moves in the positive x-direction, whereas the wave moves in the negative x-direction for $\xi = x + Ct$) (Wazwaz, 2010).

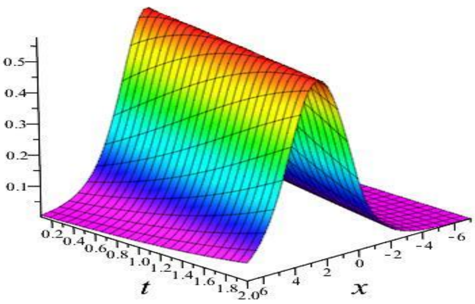

Figure 2.1 Travelling wave of $\left|v_{3_1}(x,t)\right|$ for $\lambda = -1, \mu = 1, K = 1,$ $A_2 = -0.5, \alpha = 0.9$ with $-7 \le x \le 7$ and $0.1 \le t \le 2$, $\xi = K\frac{x^\alpha}{\alpha} - C\frac{t^\alpha}{\alpha}$

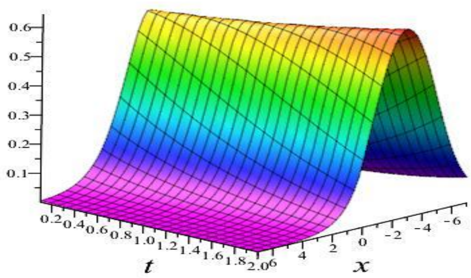

Figure 2.2 Travelling wave of $\left|v_{3_1}(x,t)\right|$ for $\lambda = -1, \mu = 1, K = 1,$ $A_2 = -0.5, \alpha = 0.9$ with $-7 \le x \le 7$ and $0.1 \le t \le 2$, $\xi = K\frac{x^\alpha}{\alpha} + C\frac{t^\alpha}{\alpha}$

Fig. 2.1 and Fig. 2.2 show the solution of the CB equations with space-time fractional derivative by considering solution v_{3_1} which is described in subsection 3.5.1 of section 3.5 of this thesis. In addition, in Fig. 2.1, the travelling wave moves in the positive x-direction, because $\xi = K\frac{x^\alpha}{\alpha} - C\frac{t^\alpha}{\alpha}$ and Fig. 2.2 shows that the travelling wave moves in the negative x-direction due to $\xi = K\frac{x^\alpha}{\alpha} + C\frac{t^\alpha}{\alpha}$.

2.5 Classification of Travelling Wave Solutions

There are many types of travelling wave solutions and we will introduce the types in this section.

2.5.1 Solitary waves and solitons

In 1834, John Scott Russell was the first who observe the solitary waves and he empirically derived the relation (Scott-Russell, 1844):

$$C^2 = g(l + s),$$

where C is the solitary wave speed, s is the maximum amplitude above the water surface, l is the finite depth of the ocean and g is the acceleration of gravity.

The solitary wave can be defined as waves which are stable and can travel over very large distance with constant shape neither decreasing in amplitude nor breaking waves in water. The speed of the wave depends on the height of the wave (Zabusky and Kruskal, 1965). Solitary waves do not follow superposition rule. For example: when a taller (faster) wave overtakes a shorter (slower) wave, they do not combine and come together. They appear to exchange places with the faster wave jump through to a slower one.

As per elastic scattering property (Wazwaz, 2010) soliton is a form of solitary waves. Even after colliding with each other they tend to keep their original form and speed. They are seen in various physical phenomena. A soliton can be described in Fig. 2.3 as a bell-shaped sech^2 in which soliton solution is characterized by infinite tails or infinite wings. The solution $\frac{1}{2}\operatorname{sech}^2\left(x + y - \frac{7}{150}\right)$, from Chapter 3 (the (3+1)-dimensional time fractional KdV-ZK) is displayed in Fig. 2.3 with $-3 \leq x, y \leq 3$.

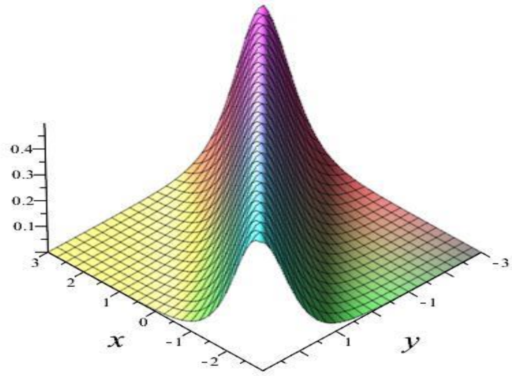

Figure 2.3 Bell-shape soliton

2.5.2 Periodic Solitons

Periodic soliton is one sort of travelling wave (Wazwaz, 2010). Trigonometric functions are periodic. The shape of periodic soliton is presented in Fig. 2.4 for $\left|u_{4_1}(x, y, t)\right|$ which will be further discussed in Chapter 3 (the (2+1)-dimensional time fractional BP) of this thesis.

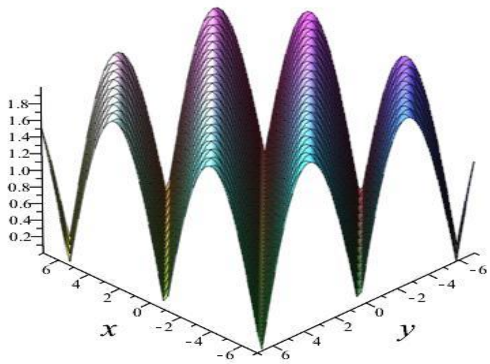

Figure 2.4 Periodic soliton

2.5.3 Kink Waves

Kink waves are a type of travelling waves, which go up or down from one asymptotic position to another. (Wazwaz, 2006; Wazwaz, 2010). Kink waves shown in Fig. 2.5 of $u_{1_2}(x, t)$, which will be further discussed in section 4.3 of this thesis.

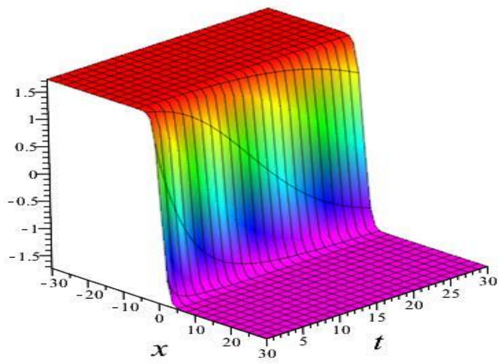

Figure 2.5 Kink wave

2.5.4 Cuspon

Cuspons are also another form of solitons where solutions represent cusps at their crests. Cuspons are classified as periodic cuspon and cuspon with exponential decay (Parkes and Vakhnenko, 2005). Cuspon is depicted in Fig. 2.6 of $|u_3(x, y, z, t)|$. Solution u_3 will be further discussed in section 3.4 of this thesis.

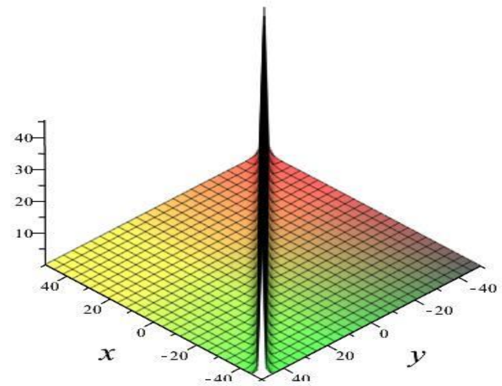

Figure 2.6 Cuspon

2.5.5 Compacton

Compactons are also solitons with finite wave length and solitary waves with compact support (Rosenau and Hyman, 1993). Compactons are solitons characterized by the absence of the infinite tails or wings where width narrows as the amplitude increases and the width of the compacton is independent of the amplitude. Compacton is described in Fig. 2.7 of $u_{1_2}(x, y, z, t)$. Solution u_{1_2} has been taken for compacton from section 5.4 of this thesis.

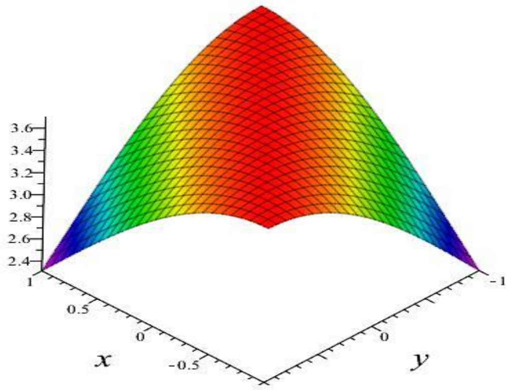

Figure 2.7 Compacton

2.6 Literature Review

In the last few years, analytical methods were proposed to obtain exact solutions of nonlinear FPDEs. In this chapter, we review the literature on the basic (G'/G)-expansion method and its applications to some nonlinear FPDEs. Also, we discuss some of the important extensions of this method.

2.6.1 Analytical Methods

Numerous physical and engineering problems are modelled by differential equations. In many cases, the solutions of these problems are difficult to obtain due to their nonlinear arrangement. Thus, the numerical methods are introduced to find the approximate solutions of nonlinear differential equations. Nevertheless, numerical solutions do not explicitly depict the nature of the physical systems and are inadequate in determining the general properties of certain system of equations. Due to these major reasons, a wide range of analytical methods have been established for solving the