
UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

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MGM563 – Statistical Inference
[Pentaabiran Statistik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of TEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi SEPULUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all four** [4] questions.

[Arahan : Jawab ***semua empat*** [4] soalan].

1. (a) Let X and Y be two random variables with joint probability mass function (pmf)

$$f(x, y) = \left(\frac{1}{2}\right)^{x+y}, \text{ for } x = 1, 2, \dots, y = 1, 2, \dots, \text{ and zero elsewhere.}$$

- (i) Determine the joint moment generating function (mgf) of X and Y .
 (ii) Show that $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$.

[20 marks]

- (b) Let $f(x, y) = 21x^2y^3$, $0 < x < y < 1$, and zero elsewhere, be the joint probability density function (pdf) of X and Y .

- (i) Find the conditional mean of X , given $Y = y$, for $0 < y < 1$.
 (ii) Find the distribution of $Z = E(X|Y)$.

[30 marks]

- (c) Let X and Y be two independent random variables, whose marginal pdfs are given as follows:

$$f_X(x) = 1, 0 \leq x \leq 1$$

and

$$f_Y(y) = 1, 0 \leq y \leq 1.$$

Find the pdf of $W = X + Y$.

[30 marks]

- (d) Assume that X_1, X_2, \dots, X_n denote a random sample of size n from a $N(0, 1)$ distribution and Y_1, Y_2, \dots, Y_m denote a random sample of size m from a $N(0, 1)$ distribution. If the two samples are independent, find the distribution of the following statistics:

(i) $\bar{X} - \bar{Y}$

(ii) $\frac{1}{2}(X_n - Y_m)^2$

(iii) $\frac{\sqrt{2mn}(\bar{X} - \bar{Y})}{\sqrt{m+n}(X_n - Y_m)}$

[20 marks]

1. (a) Biarkan X dan Y sebagai dua pembolehubah rawak dengan fungsi jisim kebarangkalian (fjk) $f(x,y) = \left(\frac{1}{2}\right)^{x+y}$, untuk $x = 1, 2, \dots, y = 1, 2, \dots$, dan sifar di tempat lain.

(i) Cari fungsi penjana momen (fpm) tercantum untuk X dan Y .

(ii) Tunjukkan bahawa $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$.

[20 markah]

(b) Biarkan $f(x,y) = 21x^2y^3$, $0 < x < y < 1$, dan sifar di tempat lain, sebagai fungsi ketumpatan kebarangkalian (fkk) tercantum untuk X dan Y .

(i) Cari min bersyarat X , diberi $Y = y$, untuk $0 < y < 1$.

(ii) Cari taburan untuk $Z = E(X|Y)$.

[30 markah]

(c) Biarkan X dan Y sebagai dua pembolehubah rawak tak bersandar dengan fkk sut masing-masing diberikan seperti berikut:

$$f_x(x) = 1, 0 \leq x \leq 1$$

dan

$$f_y(y) = 1, 0 \leq y \leq 1.$$

Cari fkk untuk $W = X + Y$.

[30 markah]

(d) Andaikan bahawa X_1, X_2, \dots, X_n mewakili suatu sampel rawak saiz n daripada taburan $N(0,1)$ dan Y_1, Y_2, \dots, Y_m mewakili suatu sampel rawak saiz m daripada taburan $N(0,1)$. Jika kedua-dua sampel itu adalah tak bersandar, cari taburan untuk statistik berikut:

(i) $\bar{X} - \bar{Y}$

(ii) $\frac{1}{2}(X_n - Y_m)^2$

(iii) $\frac{\sqrt{2mn}(\bar{X} - \bar{Y})}{\sqrt{m+n}(X_n - Y_m)}$

[20 markah]

2. (a) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x) = 2x$, $0 < x < 1$, and zero elsewhere.
- Find the joint pdf of Y_3 and Y_4 .
 - Find the conditional pdf of Y_3 , given $Y_4 = y_4$.
 - Find $E(Y_3 | y_4)$.

[30 marks]

- (b) Let W_n represent a random variable with mean μ and variance b/n^p , where $p > 0$, μ and b are constants (not functions of n). Prove that W_n converges in probability to μ .
Hint: Use the Chebyshev's inequality

[20 marks]

- (c) Find a formula for the method of moments estimate for the parameter θ in the following pdf:

$$f_Y(y; \theta) = \theta k^\theta \left(\frac{1}{y}\right)^{\theta+1}; y \geq k, \theta \geq 1$$

Assume that k is known and the data consist of a random sample of size n .

[20 marks]

- (d) Let X_1, X_2, \dots, X_n represent a random sample from the $\text{Be}(\theta)$ distribution. If the prior distribution of Θ is given by $g_\Theta(\theta) = I_{(0,1)}(\theta)$, find the posterior Bayes estimator of $\theta(1-\theta)$ with respect to the prior pdf $g_\Theta(\theta)$.

[30 marks]

2. (a) Biarkan $Y_1 < Y_2 < Y_3 < Y_4$ sebagai statistik tertib untuk suatu sampel rawak saiz $n = 4$ daripada taburan dengan fkk $f(x) = 2x$, $0 < x < 1$, dan sifar di tempat lain.
- (i) Cari fkk tercantum Y_3 dan Y_4 .
 - (ii) Cari fkk bersyarat Y_3 , diberi $Y_4 = y_4$.
 - (iii) Cari $E(Y_3 | y_4)$.

[30 markah]

- (b) Biarkan W_n mewakili suatu pembolehubah rawak dengan min μ dan varians b/n^p , yang mana $p > 0$, μ dan b adalah pemalar (bukan fungsi n). Buktikan bahawa W_n menumpu secara kebarangkalian kepada μ .
Petua: Gunakan ketaksamaan Chebyshev

[20 markah]

- (c) Cari suatu formula untuk anggaran kaedah momen bagi parameter θ dalam fkk berikut:

$$f_y(y; \theta) = \theta k^\theta \left(\frac{1}{y}\right)^{\theta+1}; y \geq k, \theta \geq 1$$

Andaikan bahawa k diketahui dan data terdiri daripada suatu sampel rawak saiz n .

[20 markah]

- (d) Biarkan X_1, X_2, \dots, X_n mewakili suatu sampel rawak daripada taburan $Be(\theta)$. Jika taburan prior bagi θ diberi oleh $g_\theta(\theta) = I_{(0,1)}(\theta)$, cari penganggar Bayes posterior untuk $\theta(1-\theta)$ terhadap fkk prior $g_\theta(\theta)$.

[30 markah]

3. (a) Let X_1, X_2, \dots, X_n be a random sample from a distribution with $f_X(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$.

(i) Find the Cramer-Rao lower bound for the variance of unbiased estimators of θ .

(ii) Is $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ a uniformly minimum variance unbiased estimator (UMVUE) of θ ?

[30 marks]

(b) Show that $\prod_{i=1}^n X_i$ is a sufficient statistic for θ if the random sample is taken from the gamma distribution with the following pdf:

$$f(x_i; \theta) = \frac{x_i^{\theta-1} e^{-x_i/6}}{\Gamma(\theta) 6^\theta}$$

[20 marks]

(c) Assume that X_1, X_2, \dots, X_n is a random sample of size n having the following pdf:

$$f(x; \theta) = \begin{cases} (\theta+1)x^\theta, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

- (i) Is $f(x; \theta)$ a member of the exponential class of pdfs? Explain.
 (ii) Find a complete sufficient statistic for θ .

[20 marks]

(d) Let X_1, X_2, \dots, X_n be independently and identically distributed (iid) random variables with common pdf

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta; \quad -\infty < \theta < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Let $Y_1 = \min(X_1, X_2, \dots, X_n)$. Show that Y_1 is a consistent estimator of θ .

[30 marks]

3. (a) Biarkan X_1, X_2, \dots, X_n sebagai suatu sampel rawak daripada taburan dengan $f_x(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$.

(i) Cari batas bawah Cramer-Rao untuk varians penganggar saksama θ .

(ii) Adakah $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ suatu penganggar saksama bervarians minimum secara seragam (PSVMS) bagi θ ?

[30 markah]

(b) Tunjukkan bahawa $\prod_{i=1}^n X_i$ ialah suatu statistik cukup untuk θ jika sampel rawak diambil daripada taburan gama dengan fkk berikut:

$$f(x_i; \theta) = \frac{x_i^{\theta-1} e^{-x_i/6}}{\Gamma(\theta) 6^\theta}$$

[20 markah]

(c) Andaikan bahawa X_1, X_2, \dots, X_n ialah suatu sampel rawak saiz n dengan fkk berikut:

$$f(x; \theta) = \begin{cases} (\theta+1)x^\theta, & 0 < x < 1 \\ 0, & \text{di tempat lain} \end{cases}$$

(i) Adakah $f(x; \theta)$ fkk ahli kelas eksponen? Jelaskan.

(ii) Cari suatu statistik cukup lengkap bagi θ .

[20 markah]

(d) Biarkan X_1, X_2, \dots, X_n sebagai pembolehubah rawak yang tak bersandar dan bertaburan secara secaman dengan fkk sepunya

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta; \quad -\infty < \theta < \infty \\ 0, & \text{di tempat lain} \end{cases}$$

Biarkan $Y_1 = \min(X_1, X_2, \dots, X_n)$. Tunjukkan bahawa Y_1 ialah penganggar konsisten bagi θ .

[30 markah]

4. (a) Let \bar{X} denote the mean of a random sample of size 25 from a gamma, $G(\alpha, \beta)$ distribution with the following pdf:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & 0 < x < \infty \\ 0 & , \text{ elsewhere} \end{cases}$$

Here, $E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$. Solve the following questions if $\alpha = 4$ and $\beta > 0$.

- (i) Use the central limit theorem to find an approximate pivotal quantity.
- (ii) Use the approximate pivotal quantity in no. (i) to find an 95.4% approximate confidence interval for β .

[20 marks]

- (b) Let X have a normal distribution with mean θ and standard deviation 5000. Consider the simple hypothesis $H_0 : \theta = 30000$ and the alternative composite hypothesis $H_1 : \theta > 30000$. We shall observe n independent values of X , say x_1, x_2, \dots, x_n , and we shall reject H_0 if and only if $\bar{x} \geq c$.

- (i) Find the power function $\pi(\theta)$ of the test.
- (ii) Determine n and c so that $\pi(\theta)$ has the values $\pi(30000) = 0.01$ and $\pi(35000) = 0.98$.

[30 marks]

- (c) Let X_1, X_2, \dots, X_n denote a random sample of size n from a Bernoulli distribution with pmf $f(x; \theta) = \theta^x (1 - \theta)^{1-x}$, $x = 0, 1$, and zero elsewhere. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$. Find the uniformly most powerful (UMP) test for testing the above hypotheses.

[20 marks]

- (d) Let X_1, X_2, \dots, X_{10} be a random sample from an exponential pdf $f(x; \lambda) = \lambda e^{-\lambda x}$, $x > 0$, and zero elsewhere. Find the generalized likelihood-ratio test for testing $H_0 : \lambda = \lambda_0$ vs. $H_1 : \lambda \neq \lambda_0$. Explain how is the critical value determined if the size of the Type-I error, α is 0.05.

[30 marks]

4. (a) Biarkan \bar{X} mewakili min suatu sampel rawak saiz 25 daripada taburan gama, $G(\alpha, \beta)$ dengan fkk berikut:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & 0 < x < \infty \\ 0 & , \text{ di tempat lain} \end{cases}$$

Di sini, $E(X) = \alpha\beta$ dan $Var(X) = \alpha\beta^2$. Selesaikan soalan berikut jika $\alpha = 4$ dan $\beta > 0$.

- (i) Gunakan teorem had memusat untuk mencari suatu kuantiti pangsaan hampiran.
- (ii) Gunakan kuantiti pangsaan hampiran dalam no. (i) untuk mencari selang keyakinan hampiran 95.4% untuk β .

[20 markah]

- (b) Biarkan X mempunyai taburan normal dengan min θ dan sisihan piawai 5000. Pertimbangkan hipotesis ringkas $H_0 : \theta = 30000$ dan hipotesis gubahan alternatif $H_1 : \theta > 30000$. Kita akan memperhatikan n nilai tak bersandar untuk X , katakan x_1, x_2, \dots, x_n , dan kita akan menolak H_0 jika dan hanya jika $\bar{x} \geq c$.

- (i) Cari fungsi kuasa $\pi(\theta)$ untuk ujian tersebut.
- (ii) Tentukan nilai-nilai n dan c supaya $\pi(30000) = 0.01$ dan $\pi(35000) = 0.98$.

[30 markah]

- (c) Biarkan X_1, X_2, \dots, X_n mewakili suatu sampel rawak saiz n daripada taburan Bernoulli dengan fkk $f(x; \theta) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$, dan sifar di tempat lain. Pertimbangkan pengujian $H_0 : \theta = \theta_0$ lawan $H_1 : \theta > \theta_0$. Cari ujian paling berkuasa secara seragam (PBS) bagi menguji hipotesis di atas.

[20 markah]

- (d) Biarkan X_1, X_2, \dots, X_{10} sebagai suatu sampel rawak daripada fkk eksponen $f(x; \lambda) = \lambda e^{-\lambda x}$, $x > 0$, dan sifar di tempat lain. Cari ujian nisbah kebolehjadian teritlak bagi menguji $H_0 : \lambda = \lambda_0$ lawan $H_1 : \lambda \neq \lambda_0$. Jelaskan bagaimana nilai genting dicari jika saiz ralat Jenis-I, α ialah 0.05.

[30 markah]

Taburan	Fungsi Ketumpatan	Min	Varians	Fungsi Penjana Momen
Seragam Diskrit	$f(x) = \frac{1}{N} I_{(1,2,\dots,N)}(x)$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\sum_{j=1}^N \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{(0,1)}(x)$	p	pq	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{(0,1,\dots,n)}(x)$	np	npq	$(q + pe^t)^n$
Geometri	$f(x) = pq^x I_{(0,1,\dots)}(x)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}, qe^t < 1$
Poisson	$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} I_{(0,1,\dots)}(x)$	λ	λ	$\exp\{\lambda(e^t - 1)\}$
Seragam	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}, t \neq 0$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-(x-\mu)^2/2\sigma^2\} I_{(-\infty,\infty)}(x)$	μ	σ^2	$\exp\{i\mu t + (\sigma t)^2/2\}$
Eksponen	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}, t < \lambda$
Gama	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} I_{(0,\infty)}(x)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha, t < \lambda$
Khi Kuasa Dua	$f(x) = \left(\frac{1}{2}\right)^{r/2} \frac{1}{\Gamma(r/2)} e^{-x/2} x^{(r/2)-1} I_{(0,\infty)}(x)$	r	$2r$	$\left(\frac{1}{1-2t}\right)^{r/2}, t < \frac{1}{2}$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$	