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UNIVERSITI SAINS MALAYSIA

Second Semester Examination  
Academic Session 2007/2008

April 2008

**MAT 263 – Probability Theory**  
**[Teori Kebarangkalian]**

Duration : 3 hours  
[Masa : 3 jam]

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Please check that this examination paper consists of FOURTEEN pages of printed material before you begin the examination.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi EMPAT BELAS muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]*

**Instructions** : Answer **all five** [5] questions.

**Arahan** : Jawab **semua lima** [5] soalan.]

1. (a) Let  $X$  be a random variable with p.d.f.

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ c & \text{if } 0 \leq x \leq 1, \\ \frac{c}{x^2} & \text{if } 1 < x < \infty. \end{cases}$$

- (i) Find the value of  $c$ .  
 (ii) Obtain the distribution function of  $X$ .  
 (iii) Find the p.d.f of  $Y = \frac{1}{X}$ .

[30 marks]

- (b) A random variable  $X$  has a c.d.f.

$$F(x) = \begin{cases} 0 & ; c < 0 \\ \frac{c}{4} & ; 0 \leq c < 1 \\ \frac{1}{2} + \frac{c-1}{4} & ; 1 \leq c < 2 \\ \frac{11}{12} & ; 2 \leq c < 3 \\ 1 & ; 3 \leq c \end{cases}$$

- (i) Find the p.d.f. of  $X$ . Using the p.d.f., calculate  $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ .  
 (ii) Obtain the moment generating function of  $X$ .  
 (iii) Using the result in (ii), find the mean and variance of  $X$ .

[50 marks]

- (c) Prove that for any two events  $A$  and  $B$ ,

$$P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B}).$$

Hence, letting  $\{A_j\}; j = 1, 2, \dots$  be a countable sequence of events, show that

$$P\left(\bigcap_{j=1}^{\infty} A_j\right) \geq 1 - \sum_{j=1}^{\infty} P(\bar{A}_j)$$

[20 marks]

...3/-

1. (a) Katakan  $X$  adalah suatu pembolehubah rawak dengan f.k.k.

$$f(x) = \begin{cases} 0 & \text{jika } x < 0, \\ c & \text{jika } 0 \leq x \leq 1, \\ \frac{c}{x^2} & \text{jika } 1 < x < \infty. \end{cases}$$

- (i) Cari nilai  $c$ .  
 (ii) Dapatkan fungsi taburan bagi  $X$ .  
 (iii) Cari f.k.k bagi  $Y = \frac{1}{X}$

[30 markah]

- (b) Suatu pembolehubah rawak  $X$  mempunyai f.t.l.

$$F(x) = \begin{cases} 0 & ; c < 0 \\ \frac{c}{4} & ; 0 \leq c < 1 \\ \frac{1}{2} + \frac{c-1}{4} & ; 1 \leq c < 2 \\ \frac{11}{12} & ; 2 \leq c < 3 \\ 1 & ; 3 \leq c \end{cases}$$

- (i) Dapatkan f.k.k. bagi  $X$ . Dengan menggunakan f.k.k., hitung  $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ .  
 (ii) Dapatkan fungsi penjana momen bagi  $X$ .  
 (iii) Dengan keputusan yang diperolehi di (ii), dapatkan min dan varians bagi  $X$ .

[50 markah]

- (c) Bagi sebarang dua peristiwa  $A$  dan  $B$ , buktikan

$$P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B}).$$

Seterusnya, katakan  $\{A_j\}; j = 1, 2, \dots$  adalah peristiwa jujukan terhingga. Tunjukkan bahawa

$$P\left(\bigcap_{j=1}^{\infty} A_j\right) \geq 1 - \sum_{j=1}^{\infty} P(\bar{A}_j)$$

[20 markah]

...4/-

2. (a) The values that can be taken on by a random variable  $X$  are 0, 1 and 2 with probabilities  $a, b$  and  $c$ , respectively. The first and second moments of  $X$  are  $\frac{9}{10}$  and  $\frac{17}{10}$ , respectively.
- (i) Determine the values of  $a, b$  and  $c$ .
  - (ii) Suppose  $X_i, i = 1, 2, 3$  are independent random variables that have a common distribution as  $X$ . If  $Y = X_1 + X_2 + X_3$ , find the probability distribution and moment generating function of  $Y$ .
- [40 marks]
- (b) The mean number of automobiles entering a mountain tunnel per 3-minute period is one. An excessive number of cars entering the tunnel during a brief period of time produces a hazardous situation.
- (i) Find the probability that the number of automobiles entering the tunnel during a 3-minute period exceeds three.
  - (ii) Assume that the tunnel is observed during ten 3-minute intervals, thus giving ten independent observations,  $Y_1, Y_2, \dots, Y_{10}$ . Find the probability that  $Y > 3$  during at least one of the ten 3-minute intervals.
  - (iii) Suppose that there is a toll plaza right before the tunnel. Find the probability that the toll operator has to wait at least 3 minutes before the first car arrives after 9.00 am.
  - (iv) Find the probability that the toll operator has to wait less than 15 minutes before the eighth car arrives after 9.00 am.

[60 marks]

2. (a) Nilai-nilai yang boleh diambil oleh suatu pembolehubah rawak  $X$  adalah 0, 1 dan 2 dengan kebarangkalian masing-masing adalah  $a$ ,  $b$  dan  $c$ . Momen pertama dan kedua bagi  $X$  adalah masing-masing  $\frac{9}{10}$  dan  $\frac{17}{10}$ .

- (i) Tentukan nilai  $a$ ,  $b$  dan  $c$ .  
(ii) Katakan  $X_i, i = 1, 2, 3$  adalah pembolehubah rawak tak bersandar dan mempunyai taburan sepunya yang sama seperti  $X$ . Jika  $Y = X_1 + X_2 + X_3$ , cari taburan kerangkalian dan fungsi penjana momen bagi  $Y$ .

[40 markah]

- (b) Min bilangan kenderaan yang memasuki sebuah terowong gunung dalam setiap satu tempoh 3 minit adalah satu. Bilangan kenderaan berlebihan yang memasuki terowong pada suatu tempoh pendek menimbulkan suatu situasi yang merbahaya.

- (i) Cari kebarangkalian bahawa bilangan kenderaan yang memasuki terowong dalam suatu tempoh 3-minit melebihi tiga.  
(ii) Andaikan terowong ini diperhatikan selama sepuluh selang 3-minit, dengan itu memberikan 10 cerapan tak bersandar  $Y_1, Y_2, \dots, Y_{10}$ . Cari kebarangkalian bahawa  $Y > 3$  pada sekurang-kurangnya satu daripada sepuluh selang 3-minit.  
(iii) Katakan terdapat suatu plaza tol sebelum terowong. Cari kebarangkalian bahawa operator tol terpaksa menunggu sekurang-kurangnya 3 minit sebelum kereta pertama tiba selepas 9.00 pagi.  
(iv) Cari kebarangkalian bahawa operator tol terpaksa menunggu kurang dari 15 minit sebelum kereta kelapan tiba selepas 9.00 pagi.

[60 markah]

3. (a) Katakan  $(X, Y)$  mempunyai f.k.k. tercantum

$$f(x, y) = xy + \frac{x^2}{2}; \quad 0 < x < 1, \quad 0 < y < 2.$$

- (i) Tunjukkan bahawa  $f(x, y)$  adalah suatu f.k.k.  
 (ii) Dapatkan  $P\left(Y < 1 \mid X < \frac{1}{2}\right)$  dan  $P(Y > X)$ .

[30 markah]

- (b) Katakan f.k.k. tercantum bagi  $X_1$  dan  $X_2$  diberi sebagai

$$\begin{aligned} f(1,1) &= \frac{1}{8} & f(2,1) &= \frac{1}{8} \\ f(1,2) &= \frac{1}{4} & f(2,2) &= \frac{1}{2} \end{aligned}$$

Adakah  $X_1$  dan  $X_2$  tak bersandar? Huraikan jawapan anda.

[30 markah]

- (c) Katakan  $X$  adalah suatu pembolehubah rawak dengan suatu taburan seragam pada selang  $(0,1)$ . Cari fungsi taburan kebarangkalian bagi  $Y = e^X$ .

[40 markah]

4. (a) Suppose that

$$f(x, y) = \begin{cases} \frac{3}{16}(4 - 2x - y) & ; x > 0, y > 0, 2x - y < 4 \\ 0 & ; \text{otherwise.} \end{cases}$$

Find

- (i)  $E(Y|x)$ ,
- (ii)  $Var(Y|x)$ ,
- (iii)  $\rho$ .

[40 marks]

- (b) (i) Show how the Chebyshev's inequality was obtained.
- (ii) If  $X$  is a random variable with  $E(X) = 3$  and  $E(X^2) = 13$ , use Chebyshev's inequality to find  $P(-2 < X < 8)$ .
- (iii) Suppose that  $\bar{X}$  is the mean of a random sample of size 6 from a distribution in (ii). Use the Central Limit Theorem to find  $P(-2 < \bar{X} < 8)$ .

[30 marks]

- (c) Suppose that  $X_1, \dots, X_k$  are i.i.d. geometric random variables.
  - (i) What is the distribution of  $Y = \sum_{i=1}^k X_i$ ? Justify your answer.
  - (ii) What is the mean and variance of  $Y$ ?

[30 marks]

4. (a) Andaikan bahawa

$$f(x, y) = \begin{cases} \frac{3}{16}(4 - 2x - y) & ; x > 0, y > 0, 2x - y < 4 \\ 0 & ; \text{di tempat lain.} \end{cases}$$

Cari

- (i)  $E(Y|x)$ ,
- (ii)  $Var(Y|x)$ ,
- (iii)  $\rho$ .

[40 markah]

- (b) (i) Tunjukkan bagaimana ketaksamaan Chebyshev diperolehi.  
 (ii) Jika  $X$  adalah suatu pembolehubah rawak dengan  $E(X) = 3$  dan  $E(X^2) = 13$ , gunakan ketaksamaan Chebyshev untuk mendapatkan  $P(-2 < X < 8)$ .  
 (iii) Katakan bahawa  $\bar{X}$  adalah min bagi suatu sampel rawak bersaiz 6 dari suatu taburan di (ii). Dengan menggunakan Teorem Had Memusat, cari  $P(-2 < \bar{X} < 8)$ .

[30 markah]

- (c) Katakan  $X_1, \dots, X_k$  adalah pembolehubah rawak geometrik i.i.d..  
 (i) Apakah taburan bagi  $Y = \sum_{i=1}^k X_i$ ? Huraikan jawapan anda.  
 (ii) Apakah min dan varians bagi  $Y$ ?

[30 markah]

5. (a) If  $X$  is a Poisson,  $P(\lambda)$  and the conditional distribution of  $Y$ , given  $X = x$  is a Binomial,  $B(n, p)$ , show that  $Y$  is a  $P(\lambda p)$  random variable.  
[30 marks]

- (b) Suppose  $X_1, \dots, X_8$  and  $Y_1, \dots, Y_5$  are random samples from independent normal distributions  $N(3, 8)$  and  $N(3, 15)$ , respectively.

(i) Find  $P(\bar{X} > \bar{Y} + 1)$

(ii) Find a constant  $c$  such that  $P\left(\frac{\bar{X} - 2}{S} < c\right) = 0.90$  where  $S$  is a sample standard deviation of  $X$ .

(iii) Find a constant  $c$  such that  $P(S < c) = 0.90$ , where  $S$  is as in (ii).

[40 marks]

- (c) Let  $X$  and  $Y$  be i.i.d. random variables with common p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ; \quad -\infty < x < \infty$$

(i) Find the joint p.d.f. of  $U = \left(\frac{X+Y}{2}\right)$  and  $V = \left(\frac{X-Y}{2}\right)^2$

(ii) Are  $U$  and  $V$  independent? Justify your answer.

[30 marks]

5. (a) Jika  $X$  adalah tertabur Poisson,  $P(\lambda)$  dan taburan bersyarat bagi  $Y$ , diberi  $X = x$  adalah tertabur Binomial,  $B(n, p)$ , tunjukkan bahawa  $Y$  adalah suatu pemboleh ubah rawak  $P(\lambda p)$ .

[30 markah]

- (b) Katakan  $X_1, \dots, X_8$  dan  $Y_1, \dots, Y_5$  adalah sampel-sampel rawak masing-masing dari taburan normal tak bersandar  $N(3, 8)$  dan  $N(3, 15)$ .

(i) Cari  $P(\bar{X} > \bar{Y} + 1)$

(ii) Cari nilai  $c$  supaya  $P\left(\frac{\bar{X} - 2}{S} < c\right) = 0.90$  yang mana  $S$  adalah suatu sisihan piawai sampel bagi  $X$ .

(iii) Dapatkan nilai  $c$  supaya  $P(S < c) = 0.90$  yang mana  $S$  adalah sama seperti di (ii).

[40 markah]

- (c) Katakan  $X$  dan  $Y$  adalah pemboleh ubah rawak i.i.d. dengan f.k.k. sepunya

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ; \quad -\infty < x < \infty$$

(i) Dapatkan f.k.k. tercantum bagi  $U = \left(\frac{X+Y}{2}\right)$  dan  $V = \left(\frac{X-Y}{2}\right)^2$

(ii) Adakah  $U$  dan  $V$  tak bersandar? Huraikan jawapan anda.

[30 markah]

<i>DISCRETE DISTRIBUTIONS</i>	
Bernoulli	$f(x) = p^x(1-p)^{1-x}, \quad x = 0,1$ $M(t) = 1-p+pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial	$f(x) = \frac{n!}{x!(n-x)!} p^x(1-p)^{n-x}, \quad x = 0,1,2,\dots,n$ $M(t) = 1-p+pe^t$ $\mu = np, \quad \sigma^2 = np(1-p)$
Geometric	$f(x) = (1-p)^x p, \quad x = 0,1,2,\dots$ $M(t) = \frac{p}{1-(1-p)e^t}, \quad t < -\ln(1-p)$ $\mu = \frac{1-p}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$
Negative Binomial	$f(x) = \frac{(x+r-1)!}{x!(r-1)!} p^r(1-p)^x, \quad x = 0,1,2,\dots$ $M(t) = \frac{p^r}{[1-(1-p)e^t]^r}, \quad t < -\ln(1-p)$ $\mu = \frac{r(1-p)}{p}, \quad \sigma^2 = \frac{r(1-p)}{p^2}$
Poisson	$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\dots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Hipergeometric	$f(x) = \frac{\binom{n_1}{x} \binom{n_2}{r-x}}{\binom{n}{r}}, \quad x \leq r, x \leq n_1, r-x \leq n_2,$ $\mu = \frac{rn_1}{n}, \quad \sigma^2 = \frac{rn_1 n_2 (n-r)}{n^2 (n-1)}$

<i>CONTINUOUS DISTRIBUTION</i>	
Uniform	$f(x) = \frac{1}{b-a}, a \leq x \leq b$ $M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, t \neq 0, M(0) = 1$ $\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$
Exponential	$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, 0 \leq x < \infty$ $M(t) = \frac{1}{1-\theta t}, t < 1/\theta$ $\mu = \theta, \sigma^2 = \theta^2$
Gamma	$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}, 0 \leq x < \infty$ $M(t) = \frac{1}{(1-\theta t)^\alpha}, t < 1/\theta$ $\mu = \alpha\theta, \sigma^2 = \alpha\theta^2$
Chi Square	$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} r^{r/2-1} e^{-x/2}, 0 \leq x < \infty$ $M(t) = \frac{1}{(1-2t)^{r/2}}, t < \frac{1}{2}$ $\mu = r, \sigma^2 = 2r$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$ $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ $E(X) = \mu, \text{Var}(X) = \sigma^2$
Beta	$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, 0 < x < 1$ $\mu = \frac{\alpha}{\alpha + \beta}, \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$

<i>FORMULAE</i>	
1.	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
2.	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
3.	$\sum_{x=0}^n \binom{n}{x} b^x a^{n-x} = (a+b)^n$
4.	$\sum_{x=0}^n \binom{n_1}{x} \binom{r-n_1}{r-x} = \binom{n}{r}$
5.	$\sum_{k=0}^{\infty} \binom{n+k-1}{k} w^k = (1-w)^{-n},  w  < 1$
6.	$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \Gamma(\alpha) = (\alpha-1)!$
7.	$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$
8.	$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$