
UNIVERSITI SAINS MALAYSIA

Second Semester Examination
Academic Session 2007/2008

April 2008

MAT 202 – Introduction to Analysis
[Pengantar Analisis]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **all three** [3] questions.

Arahan : Jawab **semua tiga** [3] soalan.]

1. (a) (i) Define supremum and infimum for a nonempty set $S \subset \mathbb{R}$.
[\mathbb{R} = set of all real numbers]
- (ii) Find the supremum and infimum for the following sets if they exist.
- (a) $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$.
- (b) $A = \left\{ (-1)^n \frac{1}{n} : n \in \mathbb{N} \right\}$.
[\mathbb{N} = set of natural numbers]
- (b) Let S and T be nonempty sets of real numbers with T a bounded set and $S \subset T$.
- (i) Prove that $\sup S \leq \sup T$ and $\inf T \leq \inf S$.
- (ii) Give an example for the case when $\sup S = \sup T$ and $\inf S = \inf T$.
- (c) Let A, B and C be nonempty sets. Then show that
- $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
- (d) Define a countable set. Show that the set of all integers is countable.
- (e) Let $\{a_n\} = \left\{ \frac{3n+2}{n-1} : n \in \mathbb{N} \right\}$.
- (i) Determine the limit of $\{a_n\}$.
- (ii) Use the definition to verify the convergent of $\{a_n\}$.
- (iii) Find the smallest integer N such that $\left| \frac{3n+2}{n-1} - 3 \right| < 0.01$ for all $n \geq N$.

[100 marks]

1. (a) (i) Takrifkan supremum dan infimum bagi suatu set tak kosong $S \subset \mathbb{R}$.
 [\mathbb{R} = set nombor nyata]

(ii) Tentukan supremum dan infimum set berikut jika wujud

$$(a) A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}.$$

$$(b) A = \left\{ (-1)^n \frac{1}{n} : n \in \mathbb{N} \right\}.$$

[\mathbb{N} = set nombor asli]

(b) Andaikan S dan T adalah set nombor nyata dan tak kosong dengan T set terbatas dan $S \subset T$.

(i) Buktikan bahawa $\sup S \leq \sup T$ dan $\inf T \leq \inf S$.

(ii) Beri satu contoh bagi kes dimana $\sup S = \sup T$ dan $\inf S = \inf T$.

(c) Andaikan A , B dan C adalah set tak kosong. Tunjukkan bahawa

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(d) Takrifkan set terbilangkan. Tunjukkan bahawa set semua integer adalah terbilangkan.

$$(e) \text{ Andaikan } \{a_n\} = \left\{ \frac{3n+2}{n-1} : n \in \mathbb{N} \right\}.$$

(i) Tentukan had $\{a_n\}$.

(ii) Gunakan takrifan untuk menentusahkan penumpuan $\{a_n\}$.

(iii) Cari integer terkecil N supaya $\left| \frac{3n+2}{n-1} - 3 \right| < 0.01$ semua $n \geq N$.

[100 markah]

2. (a) Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and suppose that $\{x_n\}$ is a sequence such that $a_n \leq x_n \leq b_n$ for all n . If the sequence $\{a_n\}$ and $\{b_n\}$ both converges to L , then show that the sequence $\{x_n\}$ converges to L .
- (b) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be a closed interval on \mathbb{R} . Given $I_n \supset I_{n+1}$, then prove that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.
- (c) Given the sequence $\{a_n\}$ converges to ℓ . Show that every subsequence of $\{a_n\}$ also converges to ℓ . Further more determine whether or not the sequence $\{a_n\} = \{(-1)^n\}$ is convergent. Justify your answer.
- (d) Use the definition of Cauchy sequence, determine whether the sequence $\left\{ \frac{2n^2 + 1}{n^2} \right\}$ is Cauchy.

[100 marks]

2. (a) Andaikan $\{a_n\}$ dan $\{b_n\}$ adalah jujukan menumpu dan $\{x_n\}$ adalah suatu jujukan supaya $a_n \leq x_n \leq b_n$ untuk semua n . Jika jujukan $\{a_n\}$ dan $\{b_n\}$ keduanya menumpu ke L , tunjukkan bahawa jujukan $\{x_n\}$ menumpu ke L .
- (b) Untuk setiap $n \in \mathbb{N}$, biarkan $I_n = [a_n, b_n]$ adalah selang tertutup pada \mathbb{R} . Diberi $I_n \supset I_{n+1}$, buktikan bahawa $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.
- (c) Diberi jujukan $\{a_n\}$ menumpu ke ℓ . Tunjukkan bahawa setiap subjujukan bagi $\{a_n\}$ juga menumpu ke ℓ . Selanjutnya, tentukan samada jujukan $\{a_n\} = \{(-1)^n\}$ menumpu atau tidak. Jelaskan jawapan anda.
- (d) Gunakan takrifkan jujukan Cauchy, tentukan samada jujukan $\left\{ \frac{2n^2 + 1}{n^2} \right\}$ adalah Cauchy.

[100 markah]

3. (a) Given $\tau = \{(-n, n), n \in \mathbb{N}\}$, an open covering of a set $A \subseteq \mathbb{R}$, where A is a compact set. Show that A is bounded.

(b) Given a set $A = (0, 1)$ and collection

$$\mathcal{G} = \left\{ \left(\frac{1}{3}, 1 \right), \left(\frac{1}{4}, \frac{1}{2} \right), \left(\frac{1}{5}, \frac{1}{3} \right), \dots, \left(\frac{1}{n}, \frac{1}{n-2} \right), \dots \right\}.$$

- (i) Determine whether \mathcal{G} is an open covering for A . Give your reasons.
 (ii) Use the definition of compactness (in terms of open covering) to show that A is not compact.

(c) Given a set $A = (0, 1) \cap \mathbb{Q}$. [\mathbb{Q} = set of all rational numbers]

Find the interior points, accumulation points/limit point and the isolated points. Furthermore, determine whether A is open or closed or neither?

(d) Given a function $f(x) = \frac{x-1}{x+2}$. Using the definition, show that this function is uniformly continuous on $[0, \infty)$.

(e) Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on \mathbb{R} by $f_n(x) = x^n$. Determine for what values of x on the given set, the sequence of functions converges point wise. Next find the point wise limit of the sequence $\{f_n\}$.

[100 marks]

3. (a) Diberi $\tau = \{(-n, n), n \in \mathbb{N}\}$, adalah tudung terbuka bagi set $A \subseteq \mathbb{R}$, dimana A adalah set padat. Tunjukkan bahawa A terbatas.

(b) Diberi set $A = (0, 1)$ dan pungutan

$$\mathcal{G} = \left\{ \left(\frac{1}{3}, 1 \right), \left(\frac{1}{4}, \frac{1}{2} \right), \left(\frac{1}{5}, \frac{1}{3} \right), \dots, \left(\frac{1}{n}, \frac{1}{n-2} \right), \dots \right\}.$$

- (i) Tentukan samada \mathcal{G} adalah suatu tudung terbuka bagi A . Berikan alasan.
 (ii) Gunakan takrifkan kepadatan set (dalam sebutan tudung terbuka) untuk menunjukkan bahawa A tidak padat.

(c) Diberi set $A = (0, 1) \cap \mathcal{Q}$ [\mathcal{Q} = set semua nombor nisbah]

Cari titik pedalaman, titik had/titik tumpukan dan titik terpencil.

Seterusnya, tentukan samada A adalah terbuka, tertutup atau bukan keduanya.

(d) Diberi fungsi $f(x) = \frac{x-1}{x+2}$. Menggunakan takrifkan, tunjukkan bahawa fungsi ini selanjar secara seragam pada $[0, \infty)$.

(e) Diberi suatu jujukan fungsi $\{f_n\}_{n=1}^{\infty}$ ditakrifkan sebagai $f_n(x) = x^n$ pada \mathbb{R} . Tentukan apakah nilai x di atas set yang diberi, jujukan fungsi adalah menumpu titik demi titik. Seterusnya, cari had titik demi titik bagi jujukan $\{f_n\}$.

[100 markah]